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On Approximate Solutions of the Generalized Radical Cubic Functional Equation in Quasi-β-Banach Spaces

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Abstract. In this paper, we prove the generalized Hyers-Ulam-Rassias stability of the generalized radical cubic functional equation

$$f\left(\sqrt[3]{ax^3+by^3}\right) = af(x)+bf(y),$$

where $a, b \in \mathbb{R}_+$ are fixed positive real numbers, by using direct method in quasi-β-Banach spaces. Moreover, we use subadditive functions to investigate stability of the generalized radical cubic functional equations in (β, p)-Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations was initiated by Ulam in 1940 [29] arising from concerning the stability of group homomorphisms. These questions form the object of the stability theory. If the answers are affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [18] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in Banach spaces. In 1978, Rassias [24] provided a generalization of Hyers's theorem for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of Rassias's results was developed by Găvruţa [14] by replacing the unbounded Cauchy difference by a general control function. For more information on that subject and further references we refer to a survey paper [6] and to a recent monograph on Ulam stability [7].

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Throughout this paper, let $\mathbb{N}, \mathbb{R}, \mathbb{R}_+$ and $\mathbb{C}$ be the set of natural numbers, the set of real numbers, the set of positive real numbers and the set of complex numbers, respectively. We consider some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|^{\beta}\|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a p-norm on X if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for some $0 < p \leq 1$ and for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space. We refer for more details on quasi-β-normed spaces and (β, p)-Banach spaces to [4, 16, 21, 26–28].

In 2009, Rassias and Kim [25] generalized the results obtained for Jensen type mappings and established new theorems about the Hyers-Ulam stability for general additive functional equations in quasi-β-Banach spaces. In the same year, Gordji and Parviz [17] established the Hyers-Ulam-Rassias stability of the quadratic functional equation

$$f(\sqrt{x^2 + y^2}) = f(x) + f(y), \tag{1.1}$$

in Banach spaces. Later, Kim et al. [22] introduced and solved the generalized radical quadratic functional equation and the generalized radical quartic functional equation:

$$f(\sqrt{ax^2 + by^2}) = af(x) + bf(y),$$
$$f(\sqrt{ax^2 + by^2}) + f(\sqrt{|ax^2 - by^2|}) = 2a^2 f(x) + 2b^2 f(y),$$

where $a, b \in \mathbb{R}_+$ are fixed. Moreover, they proved some results in 2-normed spaces and then the stability by subadditive and subquadratic functions in p-2-Banach spaces for these functional equations. In 2012, Kim, Cho and Gordji [23] investigated the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1) and

$$f(\sqrt{x^2 + y^2}) + f(\sqrt{|x^2 - y^2|}) = 2f(x) + 2f(y), \tag{1.2}$$

in quasi-β-Banach spaces and discussed the stability by using subadditive and subquadratic functions for the functional equations (1.1) and (1.2) in (β, p)-Banach spaces.

In 2002, Jun and Kim [19] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.3}$$

It is easy to see that the function $f(x)=cx^3$ is a solution of the functional equation (1.3). The equation (1.3) is called a cubic functional equation and every solution of the equation (1.3) is said to be a cubic function. Moreover, they established the general solution and the generalized Hyers-Ulam stability for the equation (1.3) in the spirit of Găvruţa [14]. Recently, Alizadeh and Ghazanfari [2] introduced the radical cubic functional equation

$$f(\sqrt[3]{x^3+y^3})=f(x)+f(y), \tag{1.4}$$

and showed that if f is a function from $\mathbb{R}$ into a linear space X satisfying the functional equation (1.4), then f satisfies the functional equation (1.3). They used a direct method to prove the Hyers-Ulam stability of the functional equation (1.4) in quasi-β-Banach spaces and established the stability by using contractively subadditive mappings and expansively subquadratic mappings for functional equation 1.4 in (β,p)-Banach spaces. Furthermore, we refer to [1–3, 9–13, 15, 20] for stability results of radical functional equations in various spaces and to [5, 8] for recent monograph on Ulam stability.

The purpose of this paper, we prove the generalized Hyers-Ulam-Rassias stability of the generalized radical cubic functional equation (shortly in GRCE)

$$f\left(\sqrt[3]{ax^3+by^3}\right)=af(x)+bf(y), \tag{1.5}$$

where $a,b\in\mathbb{R}_+$ are fixed, by using direct method in quasi-β-Banach spaces and establish the stability results by using contractively subadditive mapping and expansively subquadratic mappings for the functional equation (1.5) in (β,p)-Banach spaces.

2. Stability of the Functional Equation (1.5) in Quasi-β-Banach Spaces

In this section, we prove the generalized Hyers-Ulam-Rassias stability of the generalized radical cubic functional equation (1.5).

Let X be a normed space and $\phi:\mathbb{R}^2\to\mathbb{R}_+\cup\{0\}$ be a function. A function $f:\mathbb{R}\to X$ is called a *ϕ-approximately generalized radical cubic function* if

$$\left\|f\left(\sqrt[3]{ax^3+by^3}\right)-af(x)-bf(y)\right\|\le\phi(x,y), \tag{2.1}$$

for all $x,y\in\mathbb{R}$, where $a,b\in\mathbb{R}_+$ are fixed.

Theorem 2.1. *Let X be a quasi-β-Banach space and $f : \mathbb{R} \to X$ be a ϕ-approximately generalized radical cubic function with $a + b \neq 1$. If a function $\phi : \mathbb{R}^2 \to \mathbb{R}_+ \cup \{0\}$ satisfies*

$$\Phi(x) := \sum_{j=0}^{\infty} \left(\frac{K}{(a+b)^{\beta}} \right)^{j} \phi\left((a+b)^{\frac{j}{3}}\, x, (a+b)^{\frac{j}{3}}\, x \right) < \infty, \tag{2.2}$$

and

$$\lim_{n\to\infty} \left(\frac{1}{a+b} \right)^{\beta n} \phi\left((a+b)^{\frac{n}{3}}\, x, (a+b)^{\frac{n}{3}}\, y \right) = 0, \tag{2.3}$$

for all $x, y \in \mathbb{R}$, then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) *and the inequality*

$$\|f(x) - F(x)\| \leq \frac{K}{(a+b)^{\beta}} \Phi(x), \tag{2.4}$$

for all $x \in \mathbb{R}$.

Proof. Setting $y = x$ in (2.1), we get

$$\left\| f(x) - \frac{1}{a+b} f\left((a+b)^{\frac{1}{3}}\, x \right) \right\| \leq \frac{1}{(a+b)^{\beta}} \phi(x, x), \tag{2.5}$$

for all $x \in \mathbb{R}$. Replacing x by $(a+b)^{\frac{m}{3}} x$ in (2.5), we obtain that

$$\begin{aligned} &\left\| \frac{1}{(a+b)^m} f\left((a+b)^{\frac{m}{3}}\, x \right) - \frac{1}{(a+b)^{m+1}} f\left((a+b)^{\frac{m+1}{3}}\, x \right) \right\| \\ &\qquad \leq \frac{1}{(a+b)^{\beta(m+1)}} \phi\left((a+b)^{\frac{m}{3}} x, (a+b)^{\frac{m}{3}} x \right), \end{aligned}$$

for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Then, by an iterative process, we get

$$\begin{aligned} &\left\| f(x) - \frac{1}{(a+b)^m} f\left((a+b)^{\frac{m}{3}}\, x \right) \right\| \\ &\qquad \leq \frac{K}{(a+b)^{\beta}} \sum_{j=0}^{m-1} \left(\frac{K}{(a+b)^{\beta}} \right)^{j} \phi\left((a+b)^{\frac{j}{3}} x, (a+b)^{\frac{j}{3}} x \right), \end{aligned} \tag{2.6}$$

for all $x \in \mathbb{R}$. From (2.6) and for any $l, m \in \mathbb{N}$ with $m > l \geq 0$, we have

$$\begin{aligned} &\left\| \frac{1}{(a+b)^l} f\left((a+b)^{\frac{l}{3}}\, x \right) - \frac{1}{(a+b)^m} f\left((a+b)^{\frac{m}{3}}\, x \right) \right\| \\ &\qquad \leq \frac{K}{(a+b)^{\beta}} \sum_{j=l}^{\infty} \left(\frac{K}{(a+b)^{\beta}} \right)^{j} \phi\left((a+b)^{\frac{j}{3}} x, (a+b)^{\frac{j}{3}} x \right), \end{aligned}$$

for all $x \in \mathbb{R}$. By (2.2) and taking the limit $l \to \infty$ in the above inequality, the sequence $\left\{ \frac{1}{(a+b)^n} f\left((a+b)^{\frac{n}{3}} x \right) \right\}_{n=1}^{\infty}$ is a Cauchy sequence in quasi-β-Banach space X. So, it converges in X. We define a function $F:\mathbb{R}\to X$ by

$$F(x)=\lim_{n\to\infty}\frac{1}{(a+b)^n}f\left((a+b)^{\frac{n}{3}}x\right),$$

for all $x\in\mathbb{R}$. Next, we consider

$$\begin{aligned}&\left\|F(\sqrt[3]{ax^3+by^3})-aF(x)-bF(y)\right\|\\&\qquad\le\lim_{n\to\infty}\frac{1}{(a+b)^{\beta n}}\phi\left((a+b)^{\frac{n}{3}}x,(a+b)^{\frac{n}{3}}y\right)=0,\end{aligned}$$

for all $x,y\in\mathbb{R}$. Therefore $F(\sqrt[3]{ax^3+by^3})=aF(x)+bF(y)$, i.e. F satisfies the functional equation (1.5) on $\mathbb{R}$.

Next, we assume that there exists another mapping $G:\mathbb{R}\to X$ which satisfies the functional equation (1.5) and (2.4). Since G satisfies (1.5), we have $G(x)=\frac{1}{(a+b)^n}G\left((a+b)^{\frac{n}{3}}x\right)$ for all $x\in\mathbb{R}$ and for all $n\in\mathbb{N}$. Similarly, we also have $F(x)=\frac{1}{(a+b)^n}F\left((a+b)^{\frac{n}{3}}x\right)$ for all $x\in\mathbb{R}$ and for all $n\in\mathbb{N}$. Next, for any $n\in\mathbb{N}$, we consider

(2.7)

$$\begin{aligned}&\|F(x)-G(x)\|\\&=\left\|\frac{1}{(a+b)^n}F\left((a+b)^{\frac{n}{3}}x\right)-\frac{1}{(a+b)^n}G\left((a+b)^{\frac{n}{3}}x\right)\right\|\\&=\frac{1}{(a+b)^{\beta n}}\left\|F\left((a+b)^{\frac{n}{3}}x\right)-f\left((a+b)^{\frac{n}{3}}x\right)+f\left((a+b)^{\frac{n}{3}}x\right)-G\left((a+b)^{\frac{n}{3}}x\right)\right\|\\&\le\frac{1}{(a+b)^{\beta n}}K\left(\left\|F\left((a+b)^{\frac{n}{3}}x\right)-f\left((a+b)^{\frac{n}{3}}x\right)\right\|\right.\\&\quad\left.+\left\|f\left((a+b)^{\frac{n}{3}}x\right)-G\left((a+b)^{\frac{n}{3}}x\right)\right\|\right)\\&\le K\frac{1}{(a+b)^{\beta n}}\frac{2K}{(a+b)^{\beta}}\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^j\phi\left((a+b)^{\frac{j+n}{3}}x,(a+b)^{\frac{j+n}{3}}x\right)\\&\le\frac{2K^2}{(a+b)^{\beta}}\left(\frac{K}{(a+b)^{\beta}}\right)^n\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^j\phi\left((a+b)^{\frac{j+n}{3}}x,(a+b)^{\frac{j+n}{3}}x\right)\\&\le\frac{2K^2}{(a+b)^{\beta}}\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^{j+n}\phi\left((a+b)^{\frac{j+n}{3}}x,(a+b)^{\frac{j+n}{3}}x\right)\\&\le\frac{2K^2}{(a+b)^{\beta}}\sum_{j=n}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^{j}\phi\left((a+b)^{\frac{j}{3}}x,(a+b)^{\frac{j}{3}}x\right),\end{aligned}$$

for all $x\in\mathbb{R}$. Taking the limit $n\to\infty$ in (2.7), we get that

$$\lim_{n\to\infty}\|F(x)-G(x)\|$$

$$\leq \frac{2K^2}{(a+b)^\beta} \lim_{n\to\infty} \sum_{j=n}^{\infty} \left(\frac{K}{(a+b)^\beta}\right)^j \phi\left((a+b)^{\frac{j}{3}}x, (a+b)^{\frac{j}{3}}x\right)$$
$$= 0,$$

for all $x \in \mathbb{R}$. We obtain that $\|F(x) - G(x)\| = 0$, so $F(x) = G(x)$ for all $x \in \mathbb{R}$. Therefore, F is unique. □

Lemma 2.2. *Let $a, b \in \mathbb{R}_+$, $p, q \in \mathbb{R}_+ \cup \{0\}$, $0 < \beta \leq 1$ and $K \geq 1$. If $a + b \neq 1$ and $p, q < \beta - \log_{(a+b)} K$, then $K(a+b)^{\frac{p+q-3\beta}{3}} < 1$.*

Proof. Since $p, q < \beta - \log_{(a+b)} K$, we get

$$p + q < 2(\beta - \log_{(a+b)} K) < 3(\beta - \log_{(a+b)} K),$$

that is,

$$\log_{(a+b)} K + \frac{p+q-3\beta}{3} < 0.$$

Therefore, we obtain that

$$\begin{aligned} 0 &> \log_{(a+b)} K + \frac{p+q-3\beta}{3} \\ &= \log_{(a+b)} K + \frac{p+q-3\beta}{3} \log_{(a+b)}(a+b) \\ &= \log_{(a+b)} K + \log_{(a+b)}(a+b)^{\frac{p+q-3\beta}{3}} \\ &= \log_{(a+b)} \left(K(a+b)^{\frac{p+q-3\beta}{3}}\right). \end{aligned}$$

Hence we have

$$K(a+b)^{\frac{p+q-3\beta}{3}} < 1.$$ □

Corollary 2.3. *Let X be a quasi-β-Banach space, $p, q \in \mathbb{R}_+ \cup \{0\}$, $\varepsilon \geq 0$ and $f : \mathbb{R} \to X$ be a function satisfying the following inequality*

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y)\right\| \leq \varepsilon |x|^p |y|^q,$$

for all $x, y \in \mathbb{R}$ where $p, q < \beta - \log_{(a+b)} K$. Then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the following inequality

$$\|f(x) - F(x)\| \leq \frac{\varepsilon K}{(a+b)^\beta} \cdot \frac{|x|^{p+q}}{1 - (a+b)^{\frac{p+q}{3} - \beta}},$$

for all $x \in \mathbb{R}$.

Proof. The result follows from Theorem 2.1 by taking $\phi(x,y)=\varepsilon|x|^p|y|^q$ for all $x,y\in\mathbb{R}$. We have

$$\begin{aligned}\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^{j}&\phi\left((a+b)^{\frac{j}{3}}x,(a+b)^{\frac{j}{3}}x\right)\\ &=\varepsilon|x|^{p+q}\left[1+K(a+b)^{\frac{p+q-3\beta}{3}}+\left(K(a+b)^{\frac{p+q-3\beta}{3}}\right)^{2}\right.\\ &\quad\left.+\left(K(a+b)^{\frac{p+q-3\beta}{3}}\right)^{3}+\cdots\right],\end{aligned}$$

for all $x\in\mathbb{R}$. By Lemma 2.2, we have $K(a+b)^{\frac{p+q-3\beta}{3}}<1$. Therefore

$$\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^{j}\phi\left((a+b)^{\frac{j}{3}}x,(a+b)^{\frac{j}{3}}x\right)=\frac{\varepsilon|x|^{p+q}}{1-K(a+b)^{\frac{p+q-3\beta}{3}}}<\infty,$$

for all $x\in\mathbb{R}$. Next, we obtain

$$\lim_{n\to\infty}\frac{1}{(a+b)^{\beta n}}\phi\left((a+b)^{\frac{n}{3}}x,(a+b)^{\frac{n}{3}}y\right)=\varepsilon|x|^p|y|^q\lim_{n\to\infty}\left((a+b)^{\frac{p+q-3\beta}{3}}\right)^{n},$$

for all $x,y\in\mathbb{R}$. Since $(a+b)^{\frac{p+q-3\beta}{3}}<\frac{1}{K}<1$, we have

$$\lim_{n\to\infty}\frac{1}{(a+b)^{\beta n}}\phi\left((a+b)^{\frac{n}{3}}x,(a+b)^{\frac{n}{3}}y\right)=0,$$

for all $x,y\in\mathbb{R}$. By Theorem 2.1, there exists a unique mapping $F:\mathbb{R}\to X$ satisfying the functional equation (1.5) and the following inequality

$$\begin{aligned}\|f(x)-F(x)\|\le&\frac{K}{(a+b)^{\beta}}\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^{\beta}}\right)^{j}\phi\left((a+b)^{\frac{j}{3}}x,(a+b)^{\frac{j}{3}}x\right)\\ =&\frac{\varepsilon K}{(a+b)^{\beta}}\cdot\frac{|x|^{p+q}}{1-K(a+b)^{\frac{p+q-3\beta}{3}}},\end{aligned}$$

for all $x\in\mathbb{R}$. □

Corollary 2.4. *Let X be a quasi-β-Banach space, $p,q\in\mathbb{R}_+\cup\{0\}$, $\varepsilon\ge 0$ and $f:\mathbb{R}\to X$ be a function satisfying the following inequality*

$$\left\|f\left(\sqrt[3]{ax^3+by^3}\right)-af(x)-bf(y)\right\|\le\varepsilon(|x|^p+|y|^q),$$

for all $x,y\in\mathbb{R}$ where $p,q<\beta-\log_{(a+b)}K$. Then there exists a unique mapping $F:\mathbb{R}\to X$ satisfying the functional equation (1.5) and the following inequality

$$\|f(x)-F(x)\|\le\frac{\varepsilon K}{(a+b)^{\beta}}\cdot\left(\frac{|x|^p}{1-K(a+b)^{\frac{p}{3}-\beta}}+\frac{|x|^q}{1-K(a+b)^{\frac{q}{3}-\beta}}\right),$$

for all $x \in \mathbb{R}$.

Proof. The result follows from Theorem 2.1 by taking $\phi(x,y) = \varepsilon(|x|^p + |y|^q)$ for all $x, y \in \mathbb{R}$. We have

$$\begin{aligned}
&\sum_{j=0}^{\infty} \left(\frac{K}{(a+b)^{\beta}}\right)^j \phi\left((a+b)^{\frac{j}{3}} x, (a+b)^{\frac{j}{3}} x\right) \\
&\quad = \sum_{j=0}^{\infty} \left(\frac{K}{(a+b)^{\beta}}\right)^j \varepsilon\left(\left|(a+b)^{\frac{j}{3}} x\right|^p + \left|(a+b)^{\frac{j}{3}} x\right|^q\right) \\
&\quad = \varepsilon\Bigg[|x|^p\left(1 + K(a+b)^{\frac{p-3\beta}{3}} + \left(K(a+b)^{\frac{p-3\beta}{3}}\right)^2 + \left(K(a+b)^{\frac{p-3\beta}{3}}\right)^3 + \cdots\right) \\
&\qquad + |x|^q\left(1 + K(a+b)^{\frac{q-3\beta}{3}} + \left(K(a+b)^{\frac{q-3\beta}{3}}\right)^2 + \left(K(a+b)^{\frac{q-3\beta}{3}}\right)^3 + \cdots\right)\Bigg],
\end{aligned}$$

for all $x \in \mathbb{R}$. By Lemma 2.2, we have $K(a+b)^{\frac{p-3\beta}{3}} < 1$ and $K(a+b)^{\frac{q-3\beta}{3}} < 1$. Therefore, we obtain that

$$\begin{aligned}
&\sum_{j=0}^{\infty} \left(\frac{K}{(a+b)^{\beta}}\right)^j \phi\left((a+b)^{\frac{j}{3}} x, (a+b)^{\frac{j}{3}} x\right) \\
&\quad = \varepsilon\left(\frac{|x|^p}{1 - K(a+b)^{\frac{p}{3}-\beta}} + \frac{|x|^q}{1 - K(a+b)^{\frac{q}{3}-\beta}}\right) \\
&\quad < \infty,
\end{aligned}$$

for all $x \in \mathbb{R}$. Next, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{(a+b)^{\beta n}} \phi\left((a+b)^{\frac{n}{3}} x, (a+b)^{\frac{n}{3}} y\right) \\
&\quad = \lim_{n\to\infty} \frac{1}{(a+b)^{\beta n}} \varepsilon\left(\left|(a+b)^{\frac{n}{3}} x\right|^p + \left|(a+b)^{\frac{n}{3}} y\right|^q\right) \\
&\quad = \varepsilon\left(|x|^p \lim_{n\to\infty}\left((a+b)^{\frac{p-3\beta}{3}}\right)^n + |y|^q \lim_{n\to\infty}\left((a+b)^{\frac{q-3\beta}{3}}\right)^n\right),
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since $(a+b)^{\frac{p-3\beta}{3}} < \frac{1}{K} < 1$ and $(a+b)^{\frac{q-3\beta}{3}} < \frac{1}{K} < 1$, we have

$$\lim_{n\to\infty} \frac{1}{(a+b)^{\beta n}} \phi\left((a+b)^{\frac{n}{3}} x, (a+b)^{\frac{n}{3}} y\right) = 0,$$

for all $x, y \in \mathbb{R}$. By Theorem 2.1, there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the following inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \frac{K}{(a+b)^{\beta}} \sum_{j=0}^{\infty} \left(\frac{K}{(a+b)^{\beta}}\right)^j \phi\left((a+b)^{\frac{j}{3}} x, (a+b)^{\frac{j}{3}} x\right) \\
&= \frac{\varepsilon K}{(a+b)^{\beta}} \cdot \left(\frac{|x|^p}{1 - K(a+b)^{\frac{p}{3}-\beta}} + \frac{|x|^q}{1 - K(a+b)^{\frac{q}{3}-\beta}}\right),
\end{aligned}$$

for all $x \in \mathbb{R}$. □

Theorem 2.5. *Let X be a quasi-β-Banach space and $f:\mathbb{R}\to X$ be a ϕ-approximately generalized radical cubic function with $a+b\neq 1$. If a function $\phi:\mathbb{R}^2\to\mathbb{R}_+\cup\{0\}$ satisfies*

$$\Phi(x):=\sum_{j=1}^{\infty}\left(K(a+b)^{\beta}\right)^{j}\phi\left(\frac{x}{(a+b)^{\frac{j}{3}}},\frac{x}{(a+b)^{\frac{j}{3}}}\right)<\infty, \tag{2.8}$$

and

$$\lim_{n\to\infty}(a+b)^{\beta n}\phi\left(\frac{x}{(a+b)^{\frac{n}{3}}},\frac{y}{(a+b)^{\frac{n}{3}}}\right)=0,$$

for all $x,y\in\mathbb{R}$, then there exists a unique mapping $F:\mathbb{R}\to X$ satisfying the functional equation (1.5) *and the following inequality*

$$\|f(x)-F(x)\|\le\frac{K}{(a+b)^{\beta}}\Phi(x), \tag{2.9}$$

for all $x\in\mathbb{R}$.

Proof. By the same argument as we used in the proof of Theorem 2.1, we have the inequality (2.5). Replacing x by $\frac{x}{(a+b)^{\frac{1}{3}}}$ in (2.5), we have

$$\left\|f(x)-(a+b)f\left(\frac{x}{(a+b)^{\frac{1}{3}}}\right)\right\|\le\phi\left(\frac{x}{(a+b)^{\frac{1}{3}}},\frac{x}{(a+b)^{\frac{1}{3}}}\right), \tag{2.10}$$

for all $x\in\mathbb{R}$. Replacing x by $\frac{x}{(a+b)^{\frac{m}{3}}}$ in (2.10), we have

$$\left\|(a+b)^{m}f\left(\frac{x}{(a+b)^{\frac{m}{3}}}\right)-(a+b)^{m+1}f\left(\frac{x}{(a+b)^{\frac{m+1}{3}}}\right)\right\|$$
$$\le(a+b)^{m\beta}\phi\left(\frac{x}{(a+b)^{\frac{m+1}{3}}},\frac{x}{(a+b)^{\frac{m+1}{3}}}\right),$$

for all $x\in\mathbb{R}$ and $m\in\mathbb{N}$. Then, by an iterative, we get

$$\left\|f(x)-(a+b)^{m}f\left(\frac{x}{(a+b)^{\frac{m}{3}}}\right)\right\| \le\frac{K}{(a+b)^{\beta}}\sum_{j=1}^{m}\left(K(a+b)^{\beta}\right)^{j}\phi\left(\frac{x}{(a+b)^{\frac{j}{3}}},\frac{x}{(a+b)^{\frac{j}{3}}}\right), \tag{2.11}$$

for all $x\in\mathbb{R}$. From (2.11) and for any $l,m\in\mathbb{N}$ with $m>l\ge 0$, we have

$$\left\|(a+b)^{l}f\left(\frac{x}{(a+b)^{\frac{l}{3}}}\right)-(a+b)^{m}f\left(\frac{x}{(a+b)^{\frac{l}{3}}}x\right)\right\|$$
$$=\frac{K}{(a+b)^{\beta}}\sum_{j=l}^{m}\left(K(a+b)^{\beta}\right)^{j}\phi\left(\frac{x}{(a+b)^{\frac{j}{3}}},\frac{x}{(a+b)^{\frac{j}{3}}}\right)$$

$$\leq \frac{K}{(a+b)^{\beta}} \sum_{j=l}^{\infty} \left(K(a+b)^{\beta}\right)^{j} \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right),$$

for all $x \in \mathbb{R}$. By (2.8) and taking the limit $l \to \infty$ in the above inequality, the sequence $\left\{(a+b)^n f\left(\frac{x}{(a+b)^{\frac{n}{3}}}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in quasi-β-Banach space X. So, it converges in X. We define a mapping $F : \mathbb{R} \to X$ by

$$F(x) = \lim_{n\to\infty} (a+b)^n f\left(\frac{x}{(a+b)^{\frac{n}{3}}}\right),$$

for all $x \in \mathbb{R}$. Next, we consider

$$\begin{aligned} &\left\|F(\sqrt[3]{ax^3+by^3}) - aF(x) - bF(y)\right\| \\ &\quad \leq \lim_{n\to\infty} (a+b)^{\beta n} \phi\left(\frac{x}{(a+b)^{\frac{n}{3}}}, \frac{y}{(a+b)^{\frac{n}{3}}}\right) = 0, \end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore $F(\sqrt[3]{ax^3+by^3}) = aF(x) + bF(y)$, that is F satisfies the functional equation (1.5) on $\mathbb{R}$.

Next, we assume that there exists another mapping $G : \mathbb{R} \to X$ which satisfies the functional equation (1.5) and (2.9). Since G satisfies (1.5), we have $G(x) = (a+b)^n G\left(\frac{x}{(a+b)^{\frac{n}{3}}}\right)$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Similarly, we have $F(x) = (a+b)^n F\left(\frac{x}{(a+b)^{\frac{n}{3}}}\right)$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Next, for any $n \in \mathbb{N}$, we consider

(2.12)

$$\|F(x) - G(x)\| \leq \frac{2K^2}{(a+b)^{\beta}} \sum_{j=n}^{\infty} \left(K(a+b)^{\beta}\right)^{j} \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right),$$

for all $x \in \mathbb{R}$. Taking the limit $n \to \infty$ in (2.12), we get that

$$\begin{aligned} &\lim_{n\to\infty} \|F(x) - G(x)\| \\ &\quad \leq \frac{2K^2}{(a+b)^{\beta}} \lim_{n\to\infty} \sum_{j=n}^{\infty} \left(K(a+b)^{\beta}\right)^{j} \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right) = 0, \end{aligned}$$

for all $x \in \mathbb{R}$. We obtain that $\|F(x) - G(x)\| = 0$, so $F(x) = G(x)$ for all $x \in \mathbb{R}$. Therefore, F is unique. □

Lemma 2.6. *Let $a, b \in \mathbb{R}_+$, $p, q \in \mathbb{R}_+ \cup \{0\}$, $0 < \beta \leq 1$ and $K \geq 1$. If $a + b \neq 1$ and $3(\beta + \log_{(a+b)} K) < p + q$, then $K(a+b)^{\frac{3\beta-(p+q)}{3}} < 1$.*

Proof. Since $3(\beta + \log_{(a+b)} K) < p + q$, we get

$$\begin{aligned} 0 &> \frac{3\beta - (p+q)}{3} + \log_{(a+b)} K \\ &= \frac{3\beta - (p+q)}{3} \log_{(a+b)}(a+b) + \log_{(a+b)} K \\ &= \log_{(a+b)}(a+b)^{\frac{3\beta-(p+q)}{3}} + \log_{(a+b)} K \\ &= \log_{(a+b)} \left(K(a+b)^{\frac{3\beta-(p+q)}{3}} \right). \end{aligned}$$

Hence we obtain that $K(a+b)^{\frac{3\beta-(p+q)}{3}} < 1$. □

Corollary 2.7. *Let X, p, q be as Corollary 2.3 and $\varepsilon \geq 0$. If $f : \mathbb{R} \to X$ be a function satisfying the following inequality*

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \leq \varepsilon |x|^p |y|^q,$$

for all $x, y \in \mathbb{R}$ where $3(\beta + \log_{(a+b)} K) < p + q$, then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) *and the following inequality*

$$\|f(x) - F(x)\| \leq \frac{\varepsilon K^2}{(a+b)^\beta} \cdot \frac{|x|^{p+q}}{(a+b)^{\frac{p+q}{3} - \beta} - K},$$

for all $x \in \mathbb{R}$.

Proof. The proof follows from Theorem 2.5 by taking $\phi(x, y) = \varepsilon |x|^p |y|^q$ for all $x, y \in \mathbb{R}$. We have

$$\begin{aligned} &\sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right) \\ &\quad = \sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \varepsilon \left|\frac{x}{(a+b)^{\frac{j}{3}}}\right|^{p+q} \\ &\quad = \varepsilon K(a+b)^{\frac{3\beta-(p+q)}{3}} |x|^{p+q} \left[1 + K(a+b)^{\frac{3\beta-(p+q)}{3}} + \left(K(a+b)^{\frac{3\beta-(p+q)}{3}}\right)^2 \right. \\ &\qquad \left. + \left(K(a+b)^{\frac{3\beta-(p+q)}{3}}\right)^3 + \dots \right], \end{aligned}$$

for all $x \in \mathbb{R}$. By Lemma 2.6, we have $K(a+b)^{\frac{3\beta-(p+q)}{3}} < 1$. Therefore

$$\begin{aligned} \sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right) &= \frac{\varepsilon K(a+b)^{\frac{3\beta-(p+q)}{3}} |x|^{p+q}}{1 - K(a+b)^{\frac{3\beta-(p+q)}{3}}} \\ &= \frac{\varepsilon K |x|^{p+q}}{(a+b)^{\frac{p+q-3\beta}{3}} - K} \end{aligned}$$

$$= \frac{\varepsilon K|x|^{p+q}}{(a+b)^{\frac{p+q}{3}-\beta} - K} < \infty,$$

for all $x \in \mathbb{R}$. Next, we consider

$$\lim_{n\to\infty} (a+b)^{\beta n} \phi\left(\frac{x}{(a+b)^{\frac{n}{3}}}, \frac{y}{(a+b)^{\frac{n}{3}}}\right) = \varepsilon |x|^p |y|^q \lim_{n\to\infty} \left((a+b)^{\frac{3\beta-(p+q)}{3}}\right)^n,$$

for all $x, y \in \mathbb{R}$. Since $(a+b)^{\frac{3\beta-(p+q)}{3}} < \frac{1}{K} < 1$, we have

$$\lim_{n\to\infty} (a+b)^{\beta n} \phi\left(\frac{x}{(a+b)^{\frac{n}{3}}}, \frac{y}{(a+b)^{\frac{n}{3}}}\right) = 0,$$

for all $x, y \in \mathbb{R}$. By Theorem 2.5, there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the following inequality:

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{K}{(a+b)^\beta} \sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right) \\ &= \frac{\varepsilon K^2}{(a+b)^\beta} \cdot \frac{|x|^{p+q}}{(a+b)^{\frac{p+q}{3}-\beta} - K}, \end{aligned}$$

for all $x \in \mathbb{R}$. □

Corollary 2.8. *Let X, p, q be as Corollary 2.4 and $\varepsilon \geq 0$. If $f : \mathbb{R} \to X$ be a function satisfying the following inequality*

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) - af(x) - bf(y) \right\| \leq \varepsilon(|x|^p + |y|^q),$$

for all $x, y \in \mathbb{R}$ where $3(\beta + log_{(a+b)}K) < p + q$, then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) and the inequality

$$\|f(x) - F(x)\| \leq \frac{\varepsilon K^3}{(a+b)^\beta} \cdot \left(\frac{|x|^p}{(a+b)^{\frac{p}{3}-\beta} - K} + \frac{|x|^q}{(a+b)^{\frac{q}{3}-\beta} - K}\right),$$

for all $x \in \mathbb{R}$.

Proof. The result follows from Theorem 2.5 by taking $\phi(x, y) = \varepsilon(|x|^p + |y|^q)$ for all $x, y \in \mathbb{R}$. We have

$$\begin{aligned} &\sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}}, \frac{x}{(a+b)^{\frac{j}{3}}}\right) \\ &\quad = \sum_{j=1}^{\infty} \left(K(a+b)^\beta\right)^j \varepsilon \left(\left|\frac{x}{(a+b)^{\frac{j}{3}}}\right|^p + \left|\frac{x}{(a+b)^{\frac{j}{3}}}\right|^q\right) \\ &\quad = \varepsilon \left[K(a+b)^{\frac{3\beta-p}{3}} |x|^p \left(1 + K(a+b)^{\frac{3\beta-p}{3}} + \left(K(a+b)^{\frac{3\beta-p}{3}}\right)^2\right.\right. \end{aligned}$$

$$
\begin{aligned}
&+\left(K(a+b)^{\frac{3\beta-p}{3}}\right)^3+\cdots\Big) \\
&+K(a+b)^{\frac{3\beta-q}{3}}|x|^q\left(1+K(a+b)^{\frac{3\beta-q}{3}}+\left(K(a+b)^{\frac{3\beta-q}{3}}\right)^2\right. \\
&\left.+\left(K(a+b)^{\frac{3\beta-q}{3}}\right)^3+\cdots\right)\Bigg],
\end{aligned}
$$

for all $x \in \mathbb{R}$. By Lemma 2.6, we have $K(a+b)^{\frac{3\beta-p}{3}} < 1$ and $K(a+b)^{\frac{3\beta-q}{3}} < 1$. Therefore

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\left(K(a+b)^{\beta}\right)^j \phi\left(\frac{x}{(a+b)^{\frac{j}{3}}},\frac{x}{(a+b)^{\frac{j}{3}}}\right) \\
&\qquad=\varepsilon\left(\frac{K(a+b)^{\frac{3\beta-p}{3}}|x|^p}{1-K(a+b)^{\frac{3\beta-p}{3}}}+\frac{K(a+b)^{\frac{3\beta-q}{3}}|x|^q}{1-K(a+b)^{\frac{3\beta-q}{3}}}\right) \\
&\qquad=\varepsilon K\left(\frac{|x|^p}{(a+b)^{\frac{p}{3}-\beta}-K}+\frac{|x|^q}{(a+b)^{\frac{q}{3}-\beta}-K}\right) \\
&\qquad<\infty,
\end{aligned}
$$

for all $x \in \mathbb{R}$. Next, we have

$$
\begin{aligned}
&\lim_{n\to\infty}(a+b)^{\beta n}\phi\left(\frac{x}{(a+b)^{\frac{n}{3}}},\frac{y}{(a+b)^{\frac{n}{3}}}\right) \\
&\quad=\lim_{n\to\infty}(a+b)^{\beta n}\varepsilon\left(\left|\frac{x}{(a+b)^{\frac{n}{3}}}\right|^p+\left|\frac{y}{(a+b)^{\frac{n}{3}}}\right|^q\right) \\
&\quad=\varepsilon\left(|x|^p\lim_{n\to\infty}\left((a+b)^{\frac{3\beta-p}{3}}\right)^n+|y|^q\lim_{n\to\infty}\left((a+b)^{\frac{3\beta-q}{3}}\right)^n\right),
\end{aligned}
$$

for all $x, y \in \mathbb{R}$. Since $(a+b)^{\frac{3\beta-p}{3}} < \frac{1}{K} < 1$ and $(a+b)^{\frac{3\beta-q}{3}} < \frac{1}{K} < 1$, we have

$$
\lim_{n\to\infty}(a+b)^{\beta n}\phi\left(\frac{x}{(a+b)^{\frac{n}{3}}},\frac{y}{(a+b)^{\frac{n}{3}}}\right)=0,
$$

for all $x, y \in \mathbb{R}$. By Theorem 2.5, there exists a unique mapping $F:\mathbb{R}\to X$ satisfying the functional equation (1.5) and the following inequality

$$
\begin{aligned}
\|f(x)-F(x)\| &\le \frac{K}{(a+b)^{\beta}}\sum_{j=1}^{\infty}\left(K(a+b)^{\beta}\right)^j\phi\left(\frac{x}{(a+b)^{\frac{j}{3}}},\frac{x}{(a+b)^{\frac{j}{3}}}\right) \\
&=\frac{\varepsilon K^2}{(a+b)^{\beta}}\cdot\left(\frac{|x|^p}{(a+b)^{\frac{p}{3}-\beta}-K}+\frac{|x|^q}{(a+b)^{\frac{q}{3}-\beta}-K}\right),
\end{aligned}
$$

for all $x \in \mathbb{R}$. □

3. Stability of the functional equation (1.5) in (β, p)-Banach spaces

Now, we investigate the stability of the functional equation (1.5) in (β, p)-Banach spaces by using contractive subbadditive and expansively superadditive functions.

We recall that a subadditive function is a function $\varphi : A \to B$, having a domain A and a codomain $(B, \leq)$ that are both closed under addition, with the following property:

$$\phi(x+y) \leq \phi(x) + \phi(y),$$

for all $x, y \in A$. Now we say that a function $\phi : A \to B$ is contractively subadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \leq L\left(\phi(x) + \phi(y)\right),$$

for all $x, y \in A$. Then ϕ satisfies the property $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. It follows by the contractively subadditive condition of ϕ that $\phi(\lambda x) \leq \lambda L \phi(x)$ and so $\phi\left(\lambda^i x\right) \leq (\lambda L)^i \phi(x)$ for all $i \in \mathbb{N}$, for all $x \in \mathbb{A}$ and for all positive integers $\lambda \geq 2$.

Similarly, w say that a function $\phi : A \to B$ is expansively superadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}\left(\phi(x) + \phi(y)\right),$$

for all $x, y \in \mathbb{A}$. Then ϕ satisfies the property $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi\left(\frac{x}{2^n}\right) \leq \left(\frac{L}{2}\right)^n \phi(x)$. We observe that an expansively superadditive mapping ϕ satisfies the following properties $\phi(\lambda x) \geq \frac{\lambda}{L}\phi(x)$ and so $\phi\left(\frac{x}{\lambda^i}\right) \leq \left(\frac{L}{\lambda}\right)^i \phi(x)$, for all $i \in \mathbb{N}$, for all $x \in A$ and for all positive integers $\lambda \geq 2$.

Theorem 3.1. *Let X be a (β, p)-Banach space and $f : \mathbb{R} \to X$ be a ϕ-approximately generalized radical cubic function. Assume that the following conditions are valid:*

(*i*) *the function ϕ is contractive subadditive with constant L satisfying $(a+b)^{1-3\beta}L < 1$;*
(*ii*) *$a+b$ is a positive integer with $a+b \geq 2$.*

Then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) *and the following inequality*

$$\|f(x) - F(x)\| \leq \frac{(a+b)^{3\beta}}{\sqrt[p]{(a+b)^{3\beta p} - ((a+b)L)^p}}\Phi(x), \tag{3.1}$$

for all $x \in \mathbb{R}$, where

$$\Phi(x) = \frac{K}{(a+b)^\beta}\phi(x,x) + \left(\frac{K}{(a+b)^\beta}\right)^2 \phi\left((a+b)^{\frac{1}{3}}x, (a+b)^{\frac{1}{3}}x\right)$$

$$+\left(\frac{K}{(a+b)^{\beta}}\right)^{3}\phi\left((a+b)^{\frac{2}{3}}x,(a+b)^{\frac{2}{3}}x\right).$$

Proof. By the same argument as we used in the proof of Theorem 2.1, for any $m\in\mathbb{N}$, we can show that

$$\left\|f(x)-\frac{1}{(a+b)^{m}}f\left((a+b)^{\frac{m}{3}}x\right)\right\| \leq \frac{K}{(a+b)^{\beta}}\sum_{j=0}^{m-1}\left(\frac{K}{(a+b)^{\beta}}\right)^{j}\phi\left((a+b)^{\frac{j}{3}}x,(a+b)^{\frac{j}{3}}x\right),$$

for all $x\in\mathbb{R}$. For $m=3$, we have

$$\left\|f(x)-\frac{1}{(a+b)^{3}}f\left((a+b)x\right)\right\| \leq \frac{K}{(a+b)^{\beta}}\phi(x,x)+\left(\frac{K}{(a+b)^{\beta}}\right)^{2}\phi\left((a+b)^{\frac{1}{3}}x,(a+b)^{\frac{1}{3}}x\right) +\left(\frac{K}{(a+b)^{\beta}}\right)^{3}\phi\left((a+b)^{\frac{2}{3}}x,(a+b)^{\frac{2}{3}}x\right), \tag{3.2}$$

for all $x\in\mathbb{R}$. Then (3.2) takes the following form

$$\left\|f(x)-\frac{1}{(a+b)^{3}}f\left((a+b)x\right)\right\|\leq\Phi(x), \tag{3.3}$$

for all $x\in\mathbb{R}$. By an iterative process, we have

$$\left\|\frac{1}{(a+b)^{3m}}f\left((a+b)^{m}x\right)-\frac{1}{(a+b)^{3(m+1)}}f\left((a+b)^{m+1}x\right)\right\| \leq\frac{1}{(a+b)^{3m\beta}}\Phi\left((a+b)^{m}x\right),$$

for all $x\in\mathbb{R}$ and $m\in\mathbb{N}$. For any $m,l\in\mathbb{N}$, $0\leq l<m$, we have

$$\begin{aligned}
&\left\|\frac{1}{(a+b)^{3l}}f\left((a+b)^{l}x\right)-\frac{1}{(a+b)^{3m}}f\left((a+b)^{m}x\right)\right\|^{p}\\
&\leq\sum_{j=l}^{m-1}\left\|\frac{1}{(a+b)^{3j}}f\left((a+b)^{j}x\right)-\frac{1}{(a+b)^{3(j+1)}}f\left((a+b)^{j+1}x\right)\right\|^{p}\\
&\leq\sum_{j=l}^{m-1}\frac{1}{(a+b)^{3j\beta p}}\Phi\left((a+b)^{j}x\right)^{p}\\
&\leq\Phi(x)^{p}\sum_{j=l}^{m-1}\left((a+b)^{1-3\beta}L\right)^{jp},
\end{aligned} \tag{3.4}$$

for all $x \in \mathbb{R}$. Indeed, for any $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{(a+b)^{3j\beta p}} \Phi\left((a+b)^j x\right)^p \\
&= \frac{1}{(a+b)^{3j\beta p}} \left[\frac{K}{(a+b)^\beta} \phi((a+b)^j x, (a+b)^j x) \right. \\
&\quad + \left(\frac{K}{(a+b)^\beta}\right)^2 \phi\left((a+b)^{\frac{1}{3}}(a+b)^j x, (a+b)^{\frac{1}{3}}(a+b)^j x\right) \\
&\quad \left. + \left(\frac{K}{(a+b)^\beta}\right)^3 \phi\left((a+b)^{\frac{2}{3}}(a+b)^j x, (a+b)^{\frac{2}{3}}(a+b)^j x\right)\right]^p \\
&\le \frac{1}{(a+b)^{3j\beta p}} \left[\frac{K}{(a+b)^\beta} ((a+b)L)^j \phi(x,x) \right. \\
&\quad + \left(\frac{K}{(a+b)^\beta}\right)^2 ((a+b)L)^j \phi((a+b)^{\frac{1}{3}} x, (a+b)^{\frac{1}{3}} x) \\
&\quad \left. + \left(\frac{K}{(a+b)^\beta}\right)^3 ((a+b)L)^j \phi((a+b)^{\frac{2}{3}} x, (a+b)^{\frac{2}{3}} x)\right]^p \\
&= \frac{1}{(a+b)^{3j\beta p}} \left[((a+b)L)^j \left(\frac{K}{(a+b)^\beta} \phi(x,x) \right.\right. \\
&\quad + \left(\frac{K}{(a+b)^\beta}\right)^2 \phi((a+b)^{\frac{1}{3}} x, (a+b)^{\frac{1}{3}} x) \\
&\quad \left.\left. + \left(\frac{K}{(a+b)^\beta}\right)^3 \phi((a+b)^{\frac{2}{3}} x, (a+b)^{\frac{2}{3}} x)\right)\right]^p \\
&= \frac{1}{(a+b)^{3j\beta p}} \left(((a+b)L)^j \Phi(x)\right)^p \\
&\le \Phi(x)^p \left((a+b)^{1-3\beta} L\right)^{jp},
\end{aligned}$$

for all $x \in \mathbb{R}$. Since $(a+b)^{1-3\beta}L < 1$, we get $\left((a+b)^{1-3\beta}L\right)^{jp} < 1$ for all $j \in \mathbb{N}$. Then, the geometric series converges

$$\sum_{j=0}^{\infty} \left((a+b)^{1-3\beta} L\right)^{jp} = \frac{1}{1-((a+b)^{1-3\beta}L)^p} < \infty.$$

From the inequality (3.4), we have

$$\left\|\frac{1}{(a+b)^{3l}} f\left((a+b)^l x\right) - \frac{1}{(a+b)^{3m}} f\left((a+b)^m x\right)\right\|^p \tag{3.5}$$

$$\leq \Phi(x)^p \sum_{j=l}^{m-1} \left((a+b)^{1-3\beta}L\right)^{jp}$$
$$\leq \Phi(x)^p \sum_{j=l}^{\infty} \left((a+b)^{1-3\beta}L\right)^{jp},$$

for all $x \in \mathbb{R}$. Taking the limit $l \to \infty$ in (3.5), we obtain that

$$\lim_{l\to\infty} \left\| \frac{1}{(a+b)^{3l}} f\left((a+b)^l x\right) - \frac{1}{(a+b)^{3m}} f\left((a+b)^m x\right) \right\|^p$$
$$\leq \Phi(x)^p \lim_{l\to\infty} \sum_{j=l}^{\infty} \left((a+b)^{1-3\beta}L\right)^{jp} = 0.$$

Then the sequence $\left\{\frac{1}{(a+b)^{3n}} f\left((a+b)^n x\right)\right\}$ is a Cauchy sequence in the (β, p)-Banach space X. So, it converges in X. We define a function $F : \mathbb{R} \to X$ by

$$F(x) = \lim_{n\to\infty} \frac{1}{(a+b)^{3n}} f\left((a+b)^n x\right),$$

for all $x \in \mathbb{R}$. Then we get that

$$\left\| F(\sqrt[3]{ax^3+by^3}) - aF(x) - bF(y) \right\| \leq \phi(x,y)^p \lim_{n\to\infty} \left((a+b)^{1-3\beta}L\right)^{np} = 0,$$

for all $x, y \in \mathbb{R}$. Then $F(\sqrt[3]{ax^3+by^3}) = aF(x) + bF(y)$, i.e. F satisfies the functional equation (1.5) on $\mathbb{R}$. It follows form (3.4) with $l = 0$ and taking the limit $m \to \infty$, that we have

$$\|f(x) - F(x)\|^p \leq \Phi(x)^p \frac{1}{1 - ((a+b)^{1-3\beta}L)^p},$$

so,

$$\|f(x) - F(x)\| \leq \frac{(a+b)^{3\beta}}{\sqrt[p]{(a+b)^{3\beta p} - ((a+b)L)^p}} \Phi(x),$$

for all $x \in \mathbb{R}$.

Next, we assume that there exists another mapping $G : \mathbb{R} \to X$ which satisfies the functional equation (1.5) and (3.1). Since G satisfies (1.5), we have

$$G(x) = \frac{1}{(a+b)^n} G\left((a+b)^{\frac{n}{3}} x\right), \tag{3.6}$$

for all $x, y \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Form (3.6), for $n = 3$, we have

$$G(x) = \frac{1}{(a+b)^3} G\left((a+b)x\right), \tag{3.7}$$

for all $x \in \mathbb{R}$. Replacing x in (3.7) by $(a+b)x$, we have

$$G\left((a+b)x\right) = \frac{1}{(a+b)^3} G\left((a+b)^2 x\right),$$

so,

$$\frac{1}{(a+b)^3} G\left((a+b)x\right) = \frac{1}{(a+b)^6} G\left((a+b)^2 x\right),$$

for all $x \in \mathbb{R}$. Continuing this process, we have

$$G\left(x\right) = \frac{1}{(a+b)^{3n}} G\left((a+b)^n x\right),$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Next, for any $n \in \mathbb{N}$, we have

$$\|F(x) - G(x)\|^p \leq \frac{2\Phi(x)^p}{1 - ((a+b)^{1-3\beta}L)^p} \left(\left((a+b)^{1-3\beta}L\right)^p\right)^n \tag{3.8}$$

for all $x \in \mathbb{R}$. Taking the limit $n \to \infty$ in the inequality (3.8), we obtain that $F(x) = G(x)$ for all $x \in \mathbb{R}$, so F is unique. This completes the proof. □

Theorem 3.2. *Let X be a (β, p)-Banach space and $f : \mathbb{R} \to X$ be a ϕ-approximately generalized radical cubic functional equation. Assume that the following conditions are valid:*

(*i*) *the function ϕ is expansively superadditive with constant L satisfying $(a+b)^{3\beta-1}L < 1$;*
(*ii*) *$a + b$ is a positive integer with $a + b \geq 2$.*

Then there exists a unique mapping $F : \mathbb{R} \to X$ satisfying the functional equation (1.5) *and the following inequality*

$$\|f(x) - F(x)\| \leq \frac{(a+b)^{3\beta}}{\sqrt[p]{((a+b)L^{-1})^p - (a+b)^{3\beta p}}} \Phi(x),$$

for all $x \in \mathbb{R}$, where

$$\Phi(x) = \frac{K}{(a+b)^\beta}\phi(x,x) + \left(\frac{K}{(a+b)^\beta}\right)^2 \phi\left((a+b)^{\frac{1}{3}}x, (a+b)^{\frac{1}{3}}x\right) + \left(\frac{K}{(a+b)^\beta}\right)^3 \phi\left((a+b)^{\frac{2}{3}}x, (a+b)^{\frac{2}{3}}x\right).$$

Proof. It follows from (3.3) of the proof of Theorem 3.1 that

$$\left\| f\left(\frac{x}{a+b}\right) - \frac{1}{(a+b)^3} f(x) \right\| \leq \Phi\left(\frac{x}{a+b}\right),$$

and so

$$\left\|(a+b)^3 f\left(\frac{x}{a+b}\right) - f(x)\right\| \leq (a+b)^{3\beta}\Phi\left(\frac{x}{a+b}\right), \tag{3.9}$$

for all $x \in \mathbb{R}$. For any $m \in \mathbb{N}$, replacing x by $\frac{x}{(a+b)^m}$ in (3.9), we have

$$\left\|(a+b)^3 f\left(\frac{x}{(a+b)^{m+1}}\right) - f\left(\frac{x}{(a+b)^m}\right)\right\| \leq (a+b)^{3\beta}\Phi\left(\frac{x}{(a+b)^{m+1}}\right),$$

and so

$$\left\|(a+b)^{3(m+1)} f\left(\frac{x}{(a+b)^{m+1}}\right) - (a+b)^{3m} f\left(\frac{x}{(a+b)^m}\right)\right\| \\ \leq (a+b)^{3(m+1)\beta}\Phi\left(\frac{x}{(a+b)^{m+1}}\right),$$

for all $x \in \mathbb{R}$. Using an iterative process, we have

(3.10)

$$\left\|(a+b)^{3l} f\left(\frac{x}{(a+b)^l}\right) - (a+b)^{3m} f\left(\frac{x}{(a+b)^m}\right)\right\|^p \\ \leq \sum_{j=l}^{m-1} \left\|(a+b)^{3j} f\left(\frac{x}{(a+b)^j}\right) - (a+b)^{3(j+1)} f\left(\frac{x}{(a+b)^{j+1}}\right)\right\|^p \\ \leq \sum_{j=l}^{m-1} (a+b)^{3(j+1)\beta p}\Phi\left(\frac{x}{(a+b)^{j+1}}\right)^p \\ \leq \Phi(x)^p \sum_{j=l}^{m-1} \left((a+b)^{3\beta-1}L\right)^{(j+1)p},$$

for all $x \in \mathbb{R}$. Since $(a+b)^{3\beta-1}L < 1$, we get $\left((a+b)^{3\beta-1}L\right)^{(j+1)p} < 1$ for all $j \in \mathbb{N}$. Then we have

$$\sum_{j=0}^{\infty} \left((a+b)^{3\beta-1}L\right)^{(j+1)p} = \frac{(a+b)^{3\beta p}}{((a+b)L^{-1})^p - ((a+b)^{3\beta})^p} < \infty.$$

From inequality (3.10), we have

$$\left\|(a+b)^{3l} f\left(\frac{x}{(a+b)^l}\right) - (a+b)^{3m} f\left(\frac{x}{(a+b)^m}\right)\right\|^p \\ \leq \Phi(x)^p \sum_{j=l}^{m-1} \left((a+b)^{3\beta-1}L\right)^{(j+1)p} \\ \leq \Phi(x)^p \sum_{j=l}^{\infty} \left((a+b)^{3\beta-1}L\right)^{(j+1)p}, \tag{3.11}$$

for all $x \in \mathbb{R}$. Taking limit $l \to \infty$ in (3.11), we obtain that

$$\lim_{l\to\infty} \left\|(a+b)^{3l} f\left(\frac{x}{(a+b)^l}\right) - (a+b)^{3m} f\left(\frac{x}{(a+b)^m}\right)\right\|^p$$

$$\leq \Phi(x)^p \sum_{j=l}^{\infty} \left((a+b)^{3\beta-1}L\right)^{(j+1)p}$$
$$= 0,$$

for all $x \in \mathbb{R}$. Then the sequence $\left\{(a+b)^{3n} f\left(\frac{x}{(a+b)^n}\right)\right\}$ is a Cauchy sequence in the (β, p)-Banach space X. So, it converges in X. We define a function $F : \mathbb{R} \to X$ by

$$F(x) = \lim_{n\to\infty} (a+b)^{3n} f\left(\frac{x}{(a+b)^n}\right)$$

for all $x \in \mathbb{R}$. The remaining follows from the proof of Theorem 3.1. This completes the proof. □

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