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Abstract. Norm and essential norm of differences of differentiation composition operators between Bloch spaces have been estimated in this paper. As a result, we find characterizations for boundedness and compactness of these operators.

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1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit ball in $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. The study of composition operators on subspaces of $H(\mathbb{D})$ is of more importance, since these operators play an important role in the study of isometries on some spaces. The main purpose of the study of composition operators is to find the relations between operator-theoretic properties of these operators like continuity and compactness and function-theoretic of the symbol induces of the operator. First of all, we should determine the space that the composition operator acting on. Such spaces containing analytic functions are Bergman, Bloch and so on. Another interesting subject is to study of differences of two or more composition operators between spaces of analytic functions which is related to the topological structure of space of all composition operator. Specific information on these operators and spaces can be found in [2, 13, 20]. The composition operators can be generalized to weighted composition operators and to generalized composition operators using integration or derivative. The definition of the last one D^n_φ which is

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called generalized composition operator or differentiation composition operator is

$$D_\varphi^n f = f^{(n)} \circ \varphi,$$

where $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function, n is a nonnegative integer and $f \in H(\mathbb{D})$. In case $n = 0$, we just have composition operator $D_\varphi^0 f = C_\varphi f = f \circ \varphi$. For $0 < \alpha < \infty$, the weighted Banach space of analytic functions is defined by

$$H_\alpha^\infty = \left\{ f \in H(\mathbb{D}) : \ \|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^\alpha |f(z)| \right\} < \infty,$$

and the weighted Bloch space $\mathcal{B}_\alpha$ is the space of all analytic functions f for which

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

$\mathcal{B}_\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right|.$$

Differences of (weighted) composition operators between Bloch spaces and Bloch spaces into H^∞ spaces are studied in [4, 5]. The boundedness and compactness of the differences of two (integration) generalized composition operators on the Bloch space are investigated in [6]. A comprehensive study of differences of composition operators has been done by Moorhouse [11]. Compact differences of composition operators on some spaces characterized in [14] and between Bloch and Lipschitz spaces one can see [12]. A characterization of differences of generalized weighted composition operators between growth spaces to be bounded or compact, were completely done in [15] and between Bloch and Bers-type spaces, the similar work has been done by Liu and Li [9]. For other references, see [1, 7, 8, 10, 16–19].

The aim of this paper is to estimate the norm and essential norm of differences of two generalized composition operators between weighted Bloch spaces. As a result, we can obtain conditions on which the differences of two generalized composition operators are bounded or compact.

Study of differences of these operators is mainly depended on the pseudo-hyperbolic distance $\rho(z, w)$ between z and w, $\rho(z, w) = |\sigma_w(z)|$, where σ_a is the Mobius transformation of $\mathbb{D}$ defined by $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, for $a, z \in \mathbb{D}$. All constants will be shown by C and the notation $A \approx B$ means that there exits positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. We need the following lemma which is Proposition 8 of [19].

Lemma 1.1. *Let $n \geq 1$ be an integer. Then $f \in \mathcal{B}_\alpha$ if and only if* $\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+n-1} \left|f^{(n)}(z)\right| < \infty$. *Moreover*

$$\|f\|_{\mathcal{B}_\alpha} = \sum_{k=0}^{n-1} \left|f^{(k)}(0)\right| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+n-1} \left|f^{(n)}(z)\right|.$$

If $f \in \mathcal{B}_\alpha$, then

$$\left|f^{(n)}(z)\right| \leq \frac{\|f\|_{\mathcal{B}_\alpha}}{(1-|z|^2)^{\alpha+n-1}}.$$

Also if $f \in \mathcal{B}_\alpha$, then $f^{(n)} \in H^\infty_{\alpha+n-1}$ and $\left\|f^{(n)}\right\|_{H^\infty_{\alpha+n-1}} \leq C\,\|f\|_{\mathcal{B}_\alpha}$, where C is a positive constant. So by Lemma 2.3 of [12],

$$\begin{aligned}(1.1)\qquad \left|\left(1-|z|^2\right)^{\alpha+n-1} f^{(n)}(z) - \left(1-|w|^2\right)^{\alpha+n-1} f^{(n)}(w)\right| \\ \leq C \left\|f^{(n)}\right\|_{H^\infty_{\alpha+n-1}} \rho(z,w) \\ \leq C\,\|f\|_{\mathcal{B}_\alpha}\, \rho\,(z,w)\,.\end{aligned}$$

Also from Remark 3.3 of [3] we have

$$\begin{aligned}(1.2)\qquad \left|\left(1-|z|^2\right)^{\alpha+n-1} f^{(n)}(z) - \left(1-|w|^2\right)^{\alpha+n-1} f^{(n)}(w)\right| \\ \leq C \sup_{z\in\mathbb{D}_r} \left(1-|z|^2\right)^{\alpha+n-1} \times \left|f^{(n)}(z)\right| \rho\,(z,w)\,,\end{aligned}$$

where $\mathbb{D}_r = \{z : |z| \leq r < 1\}$.

2. Operator Norm

In this section, estimation for the operator norm of differences of two operators has been obtained. We set

$$M_{u,\varphi}(z) = \frac{\left(1-|z|^2\right)^\alpha u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha+n}}, \qquad M_{u,\psi}(z) = \frac{\left(1-|z|^2\right)^\alpha u(z)}{\left(1-|\psi(z)|^2\right)^{\alpha+n}}.$$

Theorem 2.1. *Suppose that $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic functions. Then there exists a positive constant C such that*

$$\begin{aligned}\left\|D^n_\varphi - D^n_\psi\right\| \leq C \max\Big\{\sup_{z\in\mathbb{D}} M_{\varphi',\varphi}(z)\rho\left(\varphi(z),\psi(z)\right), \\ \sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right|, M_{1,\varphi}(0), M_{1,\psi}(0)\Big\}\end{aligned}$$

or

$$\left\|D^n_\varphi - D^n_\psi\right\| \leq C \max\Big\{\sup_{z\in\mathbb{D}} M_{\psi',\psi}(z)\rho\left(\varphi(z),\psi(z)\right),$$

$$\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|, M_{1,\varphi}(0), M_{1,\psi}(0)\Big\}.$$

Proof. For every $f\in\mathcal{B}_\alpha$ we have

$$\begin{aligned}
\left\|\left(D_\varphi^n-D_\psi^n\right)f\right\|_{\mathcal{B}_\alpha} &= \left|f^{(n)}\left(\varphi(0)\right)-f^{(n)}\left(\psi(0)\right)\right| \\
&\quad+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\Big|\varphi'(z)f^{(n+1)}\left(\varphi(z)\right) \\
&\quad-\psi'(z)f^{(n+1)}\left(\psi(z)\right)\Big| \\
&\le \frac{\|f\|_{\mathcal{B}_\alpha}}{\left(1-|\varphi(0)|^2\right)^{\alpha+n-1}}+\frac{\|f\|_{\mathcal{B}_\alpha}}{\left(1-|\psi(0)|^2\right)^{\alpha+n-1}} \\
&\quad+\sup_{z\in\mathbb{D}}\Big|M_{\varphi',\varphi}(z)\left(1-|\varphi(z)|^2\right)^{\alpha+n}f^{(n+1)}\left(\varphi(z)\right) \\
&\quad-M_{\psi',\psi}(z)\left(1-|\psi(z)|^2\right)^{\alpha+n}f^{(n+1)}\left(\psi(z)\right)\Big| \\
&\le \frac{\|f\|_{\mathcal{B}_\alpha}}{\left(1-|\varphi(0)|^2\right)^{\alpha+n-1}}+\frac{\|f|_{\mathcal{B}_\alpha}}{\left(1-|\psi(0)|^2\right)^{\alpha+n-1}} \\
&\quad+\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|\left(1-|\varphi(z)|^2\right)^{\alpha+n} \\
&\quad\times\left|f^{(n+1)}\left(\varphi(z)\right)\right| \\
&\quad+\sup_{z\in\mathbb{D}}\left|M_{\psi',\psi}(z)\right|\Big|\left(1-|\varphi(z)|^2\right)^{\alpha+n}f^{(n+1)}\left(\varphi(z)\right) \\
&\quad-\left(1-|\psi(z)|^2\right)^{\alpha+n}f^{(n+1)}\left(\psi(z)\right)\Big| \\
&\le M_{1,\varphi}(0)\|f\|_{\mathcal{B}_\alpha}+M_{1,\psi}(0)\|f\|_{\mathcal{B}_\alpha} \\
&\quad+C\|f\|_{\mathcal{B}_\alpha}\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right| \\
&\quad+C\|f\|_{\mathcal{B}_\alpha}\sup_{z\in\mathbb{D}}\left|M_{\psi',\psi}(z)\right|\rho\left(\varphi(z),\psi(z)\right).
\end{aligned}$$

By the definition of the operator norm

$$\begin{aligned}
\left\|D_\varphi^n-D_\psi^n\right\| \le C\max\Big\{&\sup_{z\in\mathbb{D}}M_{\varphi',\varphi}(z)\rho\left(\varphi(z),\psi(z)\right), \\
&\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|, M_{1,\varphi}(0), M_{1,\psi}(0)\Big\}
\end{aligned}$$

Changing the role of φ and ψ, one can obtain

$$\left\|D_{\varphi}^{n}-D_{\psi}^{n}\right\| \leq C \max \left\{\sup_{z \in \mathbb{D}} M_{\psi^{\prime}, \psi}(z) \rho(\varphi(z), \psi(z)),\right.$$
$$\left.\sup_{z \in \mathbb{D}}\left|M_{\varphi^{\prime}, \varphi}(z)-M_{\psi^{\prime}, \psi}(z)\right|, M_{1, \varphi}(0), M_{1, \psi}(0)\right\}.$$

□

Theorem 2.2. *Suppose that $\varphi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ be analytic functions. If $D_{\varphi}^{n}-D_{\psi}^{n}: \mathcal{B}_{\alpha} \rightarrow \mathcal{B}_{\alpha}$ is bounded, then there exists a positive constant C such that*

$$\left\|D_{\varphi}^{n}-D_{\psi}^{n}\right\| \geq C \max \left\{\sup_{z \in \mathbb{D}} M_{\varphi^{\prime}, \varphi}(z) \rho(\varphi(z), \psi(z)),\right.$$
$$\left.\sup_{z \in \mathbb{D}}\left|M_{\varphi^{\prime}, \varphi}(z)-M_{\psi^{\prime}, \psi}(z)\right|\right\}.$$

or

$$\left\|D_{\varphi}^{n}-D_{\psi}^{n}\right\| \geq C \max \left\{\sup_{z \in \mathbb{D}} M_{\varphi^{\prime}, \varphi}(z) \rho(\varphi(z), \psi(z)),\right.$$
$$\left.\sup_{z \in \mathbb{D}}\left|M_{\psi^{\prime}, \psi}(z)-M_{\psi^{\prime}, \psi}(z)\right|\right\}.$$

Proof. Suppose that $D_{\varphi}^{n}-D_{\psi}^{n}: \mathcal{B}_{\alpha} \rightarrow \mathcal{B}_{\alpha}$ be bounded. Fix $w \in \mathbb{D}$ with $|\varphi(w)| \geq r$ where r is constant and define

$$g_{w}(z)=\frac{1}{\tau(\alpha+n+1) \overline{\varphi(w)}^{n+1}} \frac{\left(1-|\varphi(w)|^{2}\right)^{2}}{(1-\overline{\varphi(w)} z)^{\alpha+1}},$$
$$f_{w}(z)=\frac{g_{w}(z)}{\lambda_{n}+1}\left(\sigma_{\varphi(w)} z+\lambda_{n} / \overline{\varphi(w)}\right),$$

where $\tau(\alpha)=1$, $\tau(\alpha+n+1)=(\alpha+n+1) \tau(\alpha+n)$ and

$$\lambda_{0}=0, \qquad \lambda_{n}=\sum_{i=0}^{n} \frac{(n+1-i)!\, \tau(\alpha+i)}{\tau(\alpha+n+1)}.$$

Direct calculations shows that $g_{w}, f_{w} \in \mathcal{B}_{\alpha}$ and

$$g_{w}^{(n+1)}(z)=\frac{\left(1-|\varphi(w)|^{2}\right)^{2}}{(1-\overline{\varphi(w)} z)^{\alpha+n+2}},$$
$$f_{w}^{(n+1)}(z)=\frac{\left(1-|\varphi(w)|^{2}\right)^{2}}{(1-\overline{\varphi(w)} z)^{\alpha+n+2}} \sigma_{\varphi(w)}(z).$$

The boundedness of $D_\varphi^n - D_\psi^n$ shows that

$$\begin{aligned}
\infty > C\left\|D_\varphi^n - D_\psi^n\right\| &\geq \left\|\left(D_\varphi^n - D_\psi^n\right) f_w\right\|_{\mathcal{B}_\alpha} \\
&\geq \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\varphi'(z) f_w^{(n+1)}\left(\varphi(z)\right) - \psi'(z) f_w^{(n+1)}\left(\psi(z)\right)\right| \\
&\geq \left(1-|w|^2\right)^\alpha\left|\varphi'(w) f_w^{(n+1)}\left(\varphi(w)\right) - \psi'(w) f_w^{(n+1)}\left(\psi(w)\right)\right| \\
&= \left|\frac{\left(1-|w|^2\right)^\alpha \psi'(w)\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}}\rho\left(\varphi(w),\psi(w)\right)\right| \\
&= \left|M_{\psi',\psi}(z)\frac{\left(1-|\psi(w)|^2\right)^{\alpha+n}\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}}\rho\left(\varphi(w),\psi(w)\right)\right|.
\end{aligned}$$

Again for g_w we have

$$\begin{aligned}
\infty > C\left\|D_\varphi^n - D_\psi^n\right\| &\geq \left\|\left(D_\varphi^n - D_\psi^n\right) g_w\right\|_{\mathcal{B}_\alpha} \\
&\geq \left(1-|w|^2\right)^\alpha\left|\varphi'(w) f_w^{(n+1)}\left(\varphi(w)\right) - \psi'(w) f_w^{(n+1)}\left(\psi(w)\right)\right| \\
&= \left|\frac{\left(1-|w|^2\right)^\alpha \varphi'(w)}{\left(1-|\varphi(w)|^2\right)^{\alpha+n}} - \frac{\left(1-|w|^2\right)^\alpha \psi'(w)\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}}\right| \\
&= \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\frac{\left(1-|\psi(w)|^2\right)^{\alpha+n}\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}}\right|.
\end{aligned}$$

Multiplying both sides by $\rho\left(\varphi(w),\psi(w)\right)$ and noting that ρ is bounded, we get

$$\sup_{w\in\mathbb{D},|\varphi(w)|\geq r}\left|M_{\varphi',\varphi}(w)\rho\left(\varphi(w),\psi(w)\right)\right| < \infty. \tag{2.1}$$

If $|\varphi(w)| < r$, then we define another function

$$k_w(z) = \frac{\left(z-\psi(w)\right)^{n+2}}{(n+2)!}.$$

Then $k_w \in \mathcal{B}_\alpha$ and $k_w^{(n+1)}(z) = z - \psi(w)$. Using the boundedness of the operator, we have

$$\begin{aligned}
\infty > C\left\|D_\varphi^n - D_\psi^n\right\| &\geq \left\|\left(D_\varphi^n - D_\psi^n\right) k_w\right\|_{\mathcal{B}_\alpha} \\
&\geq \left(1-|w|^2\right)^\alpha\left|\varphi'(w) k_w^{(n+1)}\left(\varphi(w)\right) - \psi'(w) k_w^{(n+1)}\left(\psi(w)\right)\right| \\
&= \left(1-|w|^2\right)^\alpha\left|\varphi'(w)\left(\varphi(w)-\psi(w)\right)\right|.
\end{aligned}$$

So

$$\left|M_{\varphi',\varphi}(w)\right| \rho\left(\varphi(w),\psi(w)\right) = \frac{\left(1-|w|^2\right)^{\alpha} |\varphi'(w)| \; |\varphi(w)-\psi(w)|}{\left(1-|\varphi(w)|^2\right)^{\alpha+n} \left|1-\overline{\varphi(w)}\psi(w)\right|} \le C \left\|\left(D_\varphi^n - D_\psi^n\right) k_w\right\|_{\mathcal{B}_\alpha}. \tag{2.2}$$

From (2.1) and (2.2) we get

$$\sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z)\right| \rho\left(\varphi(z),\psi(z)\right) \le C \left\|D_\varphi^n - D_\psi^n\right\| < \infty. \tag{2.3}$$

In a similar way and changing the role of φ and ψ in the definition of test functions, the following condition can be obtained

$$\sup_{z\in\mathbb{D}} \left|M_{\psi',\psi}(z)\right| \rho\left(\varphi(z),\psi(z)\right) \le C \left\|D_\varphi^n - D_\psi^n\right\| < \infty. \tag{2.4}$$

Now we prove that

$$\sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| < \infty.$$

Using the test functions g_w again and noting the boundedness of the operator, we obtain

$$\begin{aligned}
\infty &> \left\|\left(D_\varphi^n - D_\psi^n\right) g_w\right\|_{\mathcal{B}_\alpha} \\
&\ge \left| M_{\varphi',\varphi}(w) - M_{\psi',\psi}(w) \frac{\left(1-|\psi(w)|^2\right)^{\alpha+n}\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}} \right| \\
&= \Bigg| M_{\varphi',\varphi}(w) - M_{\psi',\psi}(w) + M_{\psi',\psi}(w) \\
&\quad \times \left(1 - \frac{\left(1-|\psi(w)|^2\right)^{\alpha+n}\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}\psi(w)\right)^{\alpha+n+2}}\right) \Bigg| \\
&\ge \left|M_{\varphi',\varphi}(w) - M_{\psi',\psi}(w)\right| - \left|M_{\psi',\psi}(w)\right| \\
&\quad \times \left| g_w^{(n+1)}\left(\varphi(w)\right)\left(1-|\varphi(w)|^2\right)^{\alpha+n} - g_w^{(n+1)}\left(\psi(w)\right)\left(1-|\psi(w)|^2\right)^{\alpha+n} \right|.
\end{aligned}$$

Then

$$\begin{aligned}
\infty &> \left|M_{\psi',\psi}(w)\right| \|g_w\|_{\mathcal{B}_\alpha} \rho\left(\varphi(w),\psi(w)\right) \\
&\ge \left|M_{\psi',\psi}(w)\right| \Bigg| g_w^{(n+1)}\left(\varphi(w)\right)\left(1-|\varphi(w)|^2\right)^{\alpha+n} \\
&\quad - g_w^{(n+1)}\left(\psi(w)\right)\left(1-|\psi(w)|^2\right)^{\alpha+n} \Bigg|
\end{aligned}$$

where the inequalities come from Lemma 1.1 and relation (2.4). Thus

$$\sup_{w\in\mathbb{D},|\varphi(w)|\geq r}\left|M_{\varphi',\varphi}(w)-M_{\psi',\psi}(w)\right|<\infty.$$

If $|\varphi(w)|<r$ and $|\psi(w)|\geq\frac{1+r}{2}$ then $\rho\left(\varphi(w),\psi(w)\right)\geq\frac{1-r}{2(1+r)}$. So (2.3) and (2.4) show that $\sup_{|\varphi(w)|<r,|\psi(w)|\geq\frac{1+r}{2}}\left|M_{\varphi',\varphi}(w)-M_{\psi',\psi}(w)\right|<\infty$. In the last case $|\varphi(w)|<r$ and $|\psi(w)|<\frac{1+r}{2}$, the test function $h(z)=\frac{z^{n+1}}{(n+1)!}$ can be applied. Then $h\in\mathcal{B}_\alpha$ and $h^{(n+1)}(z)=1$. So

$$\begin{aligned}\infty &> \left\|\left(D^n_\varphi-D^n_\psi\right)h\right\|_{\mathcal{B}_\alpha}\\ &\geq\left(1-|w|^2\right)^\alpha\left|\varphi'(w)-\psi'(w)\right|\\ &\geq\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|\left(1-|\varphi(w)|^2\right)^{\alpha+n}\\ &\quad-\left|M_{\psi',\psi}(z)\right|\left|\left(1-|\varphi(w)|^2\right)^{\alpha+n}-\left(1-|\psi(w)|^2\right)^{\alpha+n}\right|.\end{aligned}$$

According to Lemma 1.1 and (2.4),

$$\begin{aligned}&\left|M_{\psi',\psi}(z)\right|\left|\left(1-|\varphi(w)|^2\right)^{\alpha+n}-\left(1-|\psi(w)|^2\right)^{\alpha+n}\right|\\ &\qquad\leq\left|M_{\psi',\psi}(z)\right|\|h\|_{\mathcal{B}_\alpha}\,\rho\left(\varphi(w),\psi(w)\right)<\infty.\end{aligned}$$

Therefore $\sup_{|\varphi(w)|<r,|\psi(w)|<\frac{1+r}{2}}\left|M_{\varphi',\varphi}(w)-M_{\psi',\psi}(w)\right|<\infty$. From all cases we conclude that

$$\sup_{w\in\mathbb{D}}\left|M_{\varphi',\varphi}(w)-M_{\psi',\psi}(w)\right|<\infty.$$

By a few calculation we can see that

$$\sup_{w\in\mathbb{D}}\left|M_{\varphi',\varphi}(w)-M_{\psi',\psi}(w)\right|\leq C\left\|D^n_\varphi-D^n_\psi\right\|<\infty. \tag{2.5}$$

The relations (2.3) and (2.5) imply that

$$\begin{aligned}\left\|D^n_\varphi-D^n_\psi\right\|\geq C\max\Big\{&\sup_{z\in\mathbb{D}}M_{\varphi',\varphi}(z)\rho\left(\varphi(z),\psi(z)\right),\\ &\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|\Big\}.\end{aligned}$$

□

Corollary 2.3. *Suppose that $\varphi,\psi:\mathbb{D}\to\mathbb{D}$ be analytic functions. Then the continuity of $D^n_\varphi-D^n_\psi:\mathcal{B}_\alpha\to\mathcal{B}_\alpha$ is equivalent to each of the following conditions*

(i) $\sup_{z\in\mathbb{D}}M_{\varphi',\varphi}(z)\rho\left(\varphi(z),\psi(z)\right)<\infty$,
$\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|<\infty$

(ii) $\sup_{z\in\mathbb{D}} M_{\psi',\psi}(z)\rho\left(\varphi(z),\psi(z)\right) < \infty$,
$\sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| < \infty$.

3. Essential Norm

In the next theorem, we directly obtain an estimate for the essential norm $\left\|D_\varphi^n - D_\psi^n\right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha}$. Recall that for an operator T between Banach spaces X and Y, the essential norm $\|T\|_{e,X\to Y}$ is the distance of T from the space of all compact operators. Moreover the essential norm is zero if and only if the operator is compact.

Theorem 3.1. *Let $\alpha > 0$ and φ, ψ be analytic self maps of $\mathbb{D}$ and $D_\varphi^n - D_\psi^n : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is bounded. Then*

$$\left\|D_\varphi^n - D_\psi^n\right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha} \approx \max\left\{\limsup_{|\varphi(z)|\to 1}\left|M_{\varphi',\varphi}(z)\right|\rho\left(\varphi(z),\psi(z)\right),\right.$$
$$\limsup_{|\psi(z)|\to 1}\left|M_{\psi',\psi}(z)\right|\rho\left(\varphi(z),\psi(z)\right),$$
$$\left.\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}\left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right|\right\}.$$

Proof. At first, we prove the upper estimate. For $0 \le r < 1$, the operator $K_r : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$, $\left(K_r f\right)(z) = f_r(z) = f(rz)$ is bounded by 1 and also compact. Moreover $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\}$ be a sequence in $(0,1)$ such that $r_j \to 1$ as $\to \infty$. So $\left(D_\varphi^n - D_\psi^n\right)K_{r_j} : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is compact. Then

$$\begin{aligned}
\left\|D_\varphi^n - D_\psi^n\right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha} &\le \left\|\left(D_\varphi^n - D_\psi^n\right) - \left(D_\varphi^n - D_\psi^n\right)K_{r_j}\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\\
&= \sup_{\|f\|_{\mathcal{B}_\alpha}\le 1}\left|\left(D_\varphi^n - D_\psi^n\right)f(0) - \left(D_\varphi^n - D_\psi^n\right)f_{r_j}(0)\right|\\
&\quad + \sup_{\|f\|_{\mathcal{B}_\alpha}\le 1}\left|\left(\left(D_\varphi^n - D_\psi^n\right)f\right)'(z)\right.\\
&\quad \left. - \left(\left(D_\varphi^n - D_\psi^n\right)f_{r_j}\right)'(z)\right|\\
&= \sup_{\|f\|_{\mathcal{B}_\alpha}\le 1}\left|\left(f - f_{r_j}\right)^{(n)}(\varphi(0)) - \left(f - f_{r_j}\right)^{(n)}(\psi(0))\right|\\
&\quad + \sup_{\|f\|_{\mathcal{B}_\alpha}\le 1}\sup_{z\in\mathbb{D}}\left(1 - |z|^2\right)^\alpha\left|\varphi'(z)\left(f - f_{r_j}\right)^{(n+1)}(\varphi(z))\right.\\
&\quad \left. - \psi'(z)\left(f - f_{r_j}\right)^{(n)}(\psi(z))\right|
\end{aligned}$$

$$\leq \sup_{\|f\|_{\mathcal{B}_\alpha}\leq 1}\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))-\left(f-f_{r_j}\right)^{(n)}(\psi(0))\right| + \sup_{\|f\|_{\mathcal{B}_\alpha}\leq 1}(I+J),$$

where

$$I=\sup_{z\in\mathbb{D}}\left|M_{\varphi',\varphi}(z)-M_{\psi',\psi}(z)\right|\left(1-|\varphi(z)|^2\right)^{\alpha+n}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|$$

and

$$J=\sup_{z\in\mathbb{D}}\left|M_{\psi',\psi}(z)\right|\Bigg|\left(1-|\varphi(z)|^2\right)^{\alpha+n}\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z)) - \left(1-|\psi(z)|^2\right)^{\alpha+n}\left(f-f_{r_j}\right)^{(n+1)}(\psi(z))\Bigg|.$$

It is easy to see that $\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))-\left(f-f_{r_j}\right)^{(n)}(\psi(0))\right| \to 0$ as $j\to\infty$, since $\{\varphi(0)\}$ and $\{\psi(0)\}$ are compact subsets of $\mathbb{D}$ and $f_{r_j}^{(n)}\to f^{(n)}$ uniformly on compact subsets of $\mathbb{D}$. Suppose that $f\in\mathcal{B}_\alpha$ is arbitrary and $\|f\|_{\mathcal{B}_\alpha}\leq 1$. We divide into four cases:

Case 1: $|\varphi(z)|\leq r_N$ and $|\psi(z)|\leq r_N$, where N is a positive integer such that for all $j\geq N$, $r_j\geq 1/2$. Since $f_{r_j}\to f$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, $r_j^{n+1}f_{r_j}^{(n+1)}\to f^{(n+1)}$ uniformly on compact subsets of $\mathbb{D}$. This fact and the boundedness of operator imply that

$$I<C\sup_{|\varphi)(z)|\leq r_N}\left(1-|\varphi(z)|^2\right)^{\alpha+n}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\to 0$$

as $j\to\infty$. Again using Theorem 2.1 and (1.2), we have

$$\begin{aligned}J&\leq C\sup_{|\varphi(z)|\leq r_N,|\psi(z)|\leq r_N}\left|M_{\psi',\psi}(z)\right|\rho\left(\varphi(z),\psi(z)\right)\\&\quad\times\sup_{\varphi(z)\in\mathbb{D}_r}\left(1-|\varphi(z)|^2\right)^{\alpha+n}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\&\leq C\sup_{\varphi(z)\in\mathbb{D}_r}\left(1-|\varphi(z)|^2\right)^{\alpha+n}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\end{aligned}$$

which tends to zero as $j\to\infty$. So $J\to 0$.

Case 2: $|\varphi(z)|\leq r_N$ and $|\psi(z)|>r_N$. A similar argument shows that $I\to 0$ as $j\to\infty$. Using (1.1), we obtain

$$\begin{aligned}J&\leq C\sup_{|\varphi(z)|>r_N}\left|M_{\psi',\psi}(z)\right|\left\|f-f_{r_j}\right\|_{\mathcal{B}_\alpha}\rho\left(\varphi(z),\psi(z)\right)\\&\leq C\sup_{|\varphi(z)|>r_N}\left|M_{\psi',\psi}(z)\right|\rho\left(\varphi(z),\psi(z)\right).\end{aligned}$$

By taking the limit over $j \to \infty$,

$$\begin{aligned} J &\le C \lim_{j\to\infty} \sup_{|\varphi(z)|>r_N} \left|M_{\psi',\psi}(z)\right| \rho\left(\varphi(z),\psi(z)\right) \\ &= \limsup_{|\varphi(z)|\to 1} \left|M_{\psi',\psi}(z)\right| \rho\left(\varphi(z),\psi(z)\right). \end{aligned}$$

Case 3: $|\varphi(z)| > r_N$ and $|\psi(z)| > r_N$. For J we have the same argument as previous case. About I, (1.1) implies that

$$\begin{aligned} I &\le C \sup_{|\varphi(z)|>r_N, |\psi(z)|>r_N} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| \left\|f - f_{r_j}\right\|_{\mathcal{B}_\alpha} \\ &< \sup_{|\varphi(z)|>r_N, |\psi(z)|>r_N} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right|. \end{aligned}$$

Thus

$$\begin{aligned} I &\le C \lim_{j\to\infty} \sup_{|\varphi(z)|>r_N, |\psi(z)|>r_N} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| \\ &= \limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right|. \end{aligned}$$

Case 4: $|\varphi(z)| > r_N$ and $|\psi(z)| \le r_N$. Rewrite the statements and define

$$I = \sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| \left| \left(1 - |\psi(z)|^2\right)^{\alpha+n} \left(f - f_{r_j}\right)^{(n+1)} (\psi(z)) \right|$$

and

$$\begin{aligned} J = \sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z)\right| \Bigg| &\left(1 - |\varphi(z)|^2\right)^{\alpha+n} \left(f - f_{r_j}\right)^{(n+1)} (\varphi(z)) \\ &- \left(1 - |\psi(z)|^2\right)^{\alpha+n} \left(f - f_{r_j}\right)^{(n+1)} (\psi(z)) \Bigg|. \end{aligned}$$

The boundednes of the operator implies that

$$\sup_{z\in\mathbb{D}} \left|M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z)\right| < \infty.$$

So

$$I \le C \left(1 - |\psi(z)|^2\right)^{\alpha+n} \left| \left(f - f_{r_j}\right)^{(n+1)} (\psi(z)) \right|$$

which tends to zero as $j \to \infty$. Using (1.1), we obtain

$$\begin{aligned} J &\le C \sup_{|\varphi(z)|>r_N} \left|M_{\varphi',\varphi}(z)\right| \left\|f - f_{r_j}\right\|_{\mathcal{B}_\alpha} \rho\left(\varphi(z),\psi(z)\right) \\ &\le C \sup_{|\varphi(z)|>r_N} \left|M_{\varphi',\varphi}(z)\right| \rho\left(\varphi(z),\psi(z)\right). \end{aligned}$$

By taking the limit over $j \to \infty$, we can write

$$J \le C \lim_{j\to\infty} \sup_{|\varphi(z)|>r_N} \left|M_{\varphi',\varphi}(z)\right| \rho\left(\varphi(z),\psi(z)\right)$$

$$= C \limsup_{|\varphi(z)|\to 1} \left|M_{\varphi',\varphi}(z)\right| \rho\left(\varphi(z),\psi(z)\right).$$

From all cases we can get the upper bound.

For the proof of lower estimate, let $\{r_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ and $|\psi(z_j)| \to 1$ as $j \to \infty$. Consider the functions like we used in the proof of Theorem 2.2 as follows

$$g_j(z) = \frac{1}{\tau\left(\alpha+n+1\right)\overline{\varphi(z_j)}^{n+1}} \frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}z\right)^{\alpha+1}},$$

$$f_j(z) = \frac{g_j(z)}{\lambda_n+1}\left(\sigma_{\varphi(z_j)}(z) + \lambda_n/\overline{\varphi(z_j)}\right),$$

where $\tau(\alpha)=1$, $\tau\left(\alpha+n+1\right) = \left(\alpha+n+1\right)\tau\left(\alpha+n\right)$ and

$$\lambda_0 = 0, \qquad \lambda_n = \sum_{i=0}^{n} \frac{(n+1-i)!\,\tau\left(\alpha+i\right)}{\tau\left(\alpha+n+1\right)}.$$

Then $\{g_j\}$ and $\{f_j\}$ are the bounded sequences in $\mathcal{B}_\alpha$ which converge to zero uniformly on compact subsets of $\mathbb{D}$. Do $f_j, g_j \to 0$ weakly in $\mathcal{B}_\alpha$. Therefore, for any compact operator $K : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$,

$$\lim_{j\to\infty} \|Kf_j\|_{\mathcal{B}_\alpha} = 0, \qquad \lim_{j\to\infty} \|Kg_j\|_{\mathcal{B}_\alpha} = 0.$$

Hence

(3.1)

$$\begin{aligned}
\left\|\left(D^n_\varphi - D^n_\psi\right) - K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha} &\geq C \limsup_{j\to\infty} \left\|\left(D^n_\varphi - D^n_\psi\right) f_j - Kf_j\right\|_{\mathcal{B}_\alpha} \\
&\geq C \limsup_{j\to\infty} \left\|\left(D^n_\varphi - D^n_\psi\right) f_j\right\|_{\mathcal{B}_\alpha} - \limsup_{j\to\infty} \|Kf_j\|_{\mathcal{B}_\alpha} \\
&\geq C \limsup_{j\to\infty} \Bigg|M_{\psi',\psi}(z_j)\rho\left(\varphi(z_j),\psi(z_j)\right) \\
&\qquad \times \frac{\left(1-|\psi(z_j)|^2\right)^{\alpha+n}\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{\alpha+n+2}}\Bigg|,
\end{aligned}$$

(3.2)

$$\begin{aligned}
\left\|\left(D^n_\varphi - D^n_\psi\right) - K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha} &\geq C \limsup_{j\to\infty} \left\|\left(D^n_\varphi - D^n_\psi\right) g_j - Kg_j\right\|_{\mathcal{B}_\alpha} \\
&\geq C \limsup_{j\to\infty} \left\|\left(D^n_\varphi - D^n_\psi\right) g_j\right\|_{\mathcal{B}_\alpha} - \limsup_{j\to\infty} \|Kg_j\|_{\mathcal{B}_\alpha}
\end{aligned}$$

$$\begin{aligned}
&\geq C\limsup_{j\to\infty}\Bigg|M_{\varphi',\varphi}(z_j)-M_{\psi',\psi}(z_j)\\
&\quad\times\frac{\left(1-|\psi(z_j)|^2\right)^{\alpha+n}\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{\alpha+n+2}}\Bigg|\\
&\geq C\limsup_{j\to\infty}\Bigg|M_{\varphi',\varphi}(z_j)\rho\left(\varphi(z_j),\psi(z_j)\right)\\
&\quad-M_{\psi',\psi}(z_j)\rho\left(\varphi(z_j),\psi(z_j)\right)\\
&\quad\times\frac{\left(1-|\psi(z_j)|^2\right)^{\alpha+n}\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{\alpha+n+2}}\Bigg|.
\end{aligned}$$

Then (3.1) and (3.2) results in

$$\left\|\left(D_\varphi^n-D_\psi^n\right)-K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\geq C\limsup_{j\to\infty}\left|M_{\varphi',\varphi}(z_j)\right|\rho\left(\varphi(z_j),\psi(z_j)\right).$$

We get from the definition of the essential norm

$$\begin{aligned}
\left\|D_\varphi^n-D_\psi^n\right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha}&=\inf_{K\ compact}\left\|\left(D_\varphi^n-D_\psi^n\right)-K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\\
&\geq C\limsup_{|\varphi(z)|\to1}\left|M_{\varphi',\varphi}(z)\right|\rho\left(\varphi(z),\psi(z)\right).
\end{aligned}$$

Changing the role of φ and ψ in the definition of the functions f_j and g_j, we can obtain

$$\begin{aligned}
\left\|D_\varphi^n-D_\psi^n\right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha}&=\inf_{K\ compact}\left\|\left(D_\varphi^n-D_\psi^n\right)-K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\\
&\geq C\limsup_{j\to\infty}\left|M_{\psi',\psi}(z_j)\right|\rho\left(\varphi(z_j),\psi(z_j)\right)\\
&=C\limsup_{|\psi(z)|\to1}\left|M_{\psi',\psi}(z)\right|\rho\left(\varphi(z),\psi(z)\right).
\end{aligned}$$

Again using the functions g_j, we have

$$\begin{aligned}
\left\|\left(D_\varphi^n-D_\psi^n\right)-K\right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}&\geq C\limsup_{j\to\infty}\left\|\left(D_\varphi^n-D_\psi^n\right)g_j-Kg_j\right\|_{\mathcal{B}_\alpha}\\
&\geq\limsup_{j\to\infty}\Bigg|M_{\varphi',\varphi}(z_j)-M_{\psi',\psi}(z_j)\\
&\quad\times\frac{\left(1-|\psi(z_j)|^2\right)^{\alpha+n}\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{\alpha+n+2}}\Bigg|\\
&=\limsup_{j\to\infty}\Big|M_{\varphi',\varphi}(z_j)-M_{\psi',\psi}(z_j)+M_{\psi',\psi}(z_j)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(1 - \frac{\left(1-|\psi(z_j)|^2\right)^{\alpha+n}\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{\alpha+n+2}} \right) \Bigg| \\
&\geq \limsup_{j\to\infty} \left| M_{\varphi',\varphi}(z_j) - M_{\psi',\psi}(z_j) \right| \\
&\quad - \limsup_{j\to\infty} \left| M_{\psi',\psi}(z_j) \right| \\
&\quad \times \Bigg| g_j^{(n+1)}\left(\varphi(z_j)\right)\left(1-|\varphi(z_j)|^2\right)^{\alpha+n} \\
&\quad - g_j^{(n+1)}\left(\psi(z_j)\right)\left(1-|\psi(z_j)|^2\right)^{\alpha+n} \Bigg| .
\end{aligned}
$$

It can be easily proved by Theorem 2.1 and (1.2) that

$$
\begin{aligned}
- \limsup_{j\to\infty} \left| M_{\psi',\psi}(z_j) \right| &\Bigg| g_j^{(n+1)}\left(\varphi(z_j)\right)\left(1-|\varphi(z_j)|^2\right)^{\alpha+n} \\
&- g_j^{(n+1)}\left(\psi(z_j)\right)\left(1-|\psi(z_j)|^2\right)^{\alpha+n} \Bigg|
\end{aligned}
$$

tends to zero. So

$$
\begin{aligned}
\left\| \left(D_\varphi^n - D_\psi^n\right) - K \right\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha} &\geq C \limsup_{j\to\infty} \left| M_{\varphi',\varphi}(z_j) - M_{\psi',\psi}(z_j) \right| \\
&= C \limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left| M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z) \right| .
\end{aligned}
$$

Finally, we have

$$
\left\| D_\varphi^n - D_\psi^n \right\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\alpha} \geq C \limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left| M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z) \right| .
$$

$\square$

Corollary 3.2. *Let $\alpha > 0$ and φ, ψ be analytic self maps of $\mathbb{D}$ and $D_\varphi^n - D_\psi^n : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is bounded. Then $D_\varphi^n - D_\psi^n$ is compact if and only if*

$$
\begin{aligned}
\limsup_{|\varphi(z)|\to 1} \left| M_{\varphi',\varphi}(z) \right| \rho\left(\varphi(z), \psi(z)\right) &= 0, \\
\limsup_{|\psi(z)|\to 1} \left| M_{\psi',\psi}(z) \right| \rho\left(\varphi(z), \psi(z)\right) &= 0, \\
\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left| M_{\varphi',\varphi}(z) - M_{\psi',\psi}(z) \right| &= 0.
\end{aligned}
$$

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