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On Preserving Properties of Linear Maps on C^*-algebras

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Abstract. Let A and B be two unital C^*-algebras and $\varphi : A \to B$ be a linear map. In this paper, we investigate the structure of linear maps between two C^*-algebras that preserve a certain property or relation. In particular, we show that if φ is unital, B is commutative and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism. It is also shown that if $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

1. Introduction

In 1970, Kaplansky asked the following question:
Let $\varphi : A \longrightarrow B$ be a unital and invertibility preserving linear map between unital Banach algebras A and B. Is φ a Jordan homomorphism?[6]. The Kaplansky's question was originated by Gleason-Kahane-Zelazko Theorem which states that every invertibility preserving unital linear functional on a unital complex Banach algebra is multiplicative [18]. In this paper we explain and prove a Gleason-Kahane-Zelazko type Theorem and show that if A is a unital C^*-algebra and φ is a unital linear functional on A such that $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism.
Another kind of linear preserver problems is absolute value preserving linear maps and in this paper, we will characterize this kind of maps.

Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is said to be a contraction if there exist $0 \leq k < 1$ such that

$$d'(f(x_1), f(x_2)) \leq kd(x_1, x_2); \quad x_1, x_2 \in X.$$

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Let A be a complex unital normed algebra, and

$$D(A,1)=\left\{f\in A', f(1)=\| f \|=1\right\},$$

where A' is the dual space of A. The elements of $D(A,1)$ are called the normalized states on A. For $a\in A$ let,

$$V(a)=\{f(a) : f\in D(A,1)\}, \qquad v(a)=\sup\{|\lambda| : \lambda \in V(A)\}.$$

The sets $V(a)$ and $v(a)$ are called the numerical range and numerical radius of a respectively and the spectrum of a is denoted by $\sigma(a)$, namely

$$\sigma(a)=\{\lambda\in\mathbb{C} : \lambda-a\in \operatorname{sing}(A)\}.$$

Also the convex hull of $\sigma(x)$ is denoted by $co\sigma(x)$. If H is a Hilbert space, for every $T\in B(H)$, the numerical range of T is the set

$$W(T)=\{\langle T(x),x\rangle : x\in H, \|x\|=1\},$$

and the numerical radius of T is defined by $w(T)=sup\{|\lambda| : \lambda\in W(T)\}$. When $A=B(H)$ and $T\in A$, the set $V(T)$ becomes the closure of $W(T)$. Let A and B be complex unital normed algebras. A linear map $\varphi : A\to B$ is said to be numerical range compressing if $V(\varphi(x))\subseteq V(x)$, numerical range preserving if $V(\varphi(x))=V(x)$ and Jordan homomorphism if $\varphi(x^2)=\varphi(x)^2$ for all $x\in A$. Also φ is said to be unital if $\varphi(1_A)=1_B$.

Let A be a unital C^*-algebra. An element a of A is said to be positive if $V(a)\subseteq\mathbb{R}^+$ or $a^*=a$ and $\sigma(a)\subseteq\mathbb{R}^+$. We denote by A^+ the set of all positive elements of A. Also a is called normal and unitary, if $a^*a=aa^*$ and $a^*a=aa^*=1$, respectively. We recall that if a is a unitary element of A, then $\sigma(a)=\{\lambda\in\mathbb{C} : |\lambda|=1\}$ [17].

A linear map φ from a unital C^*-algebra A into a unital C^*-algebra B is said unitary preserving if $\varphi(u)$ is unitary whenever u is unitary in A. We say that φ preserves absolute values if $\varphi(|x|)=|\varphi(x)|$, where $|x|^2=x^*x$ and $*$-homomorphism, if $\varphi(xy)=\varphi(x)\varphi(y)$ and $\varphi(x^*)=\varphi(x)^*$ for every $x,y\in A$. For any positive integer n we define $\varphi_n : M_n(A)\to M_n(B)$ by $\varphi_n((a_{i,j})_{i,j})=\varphi((a_{i,j}))_{i,j}$, where $M_n(A)$ denotes the set of all $n\times n$ matrices with entries in A. The map φ is called positive if $\varphi(a)\geq 0$ for all $a\in A^+$ and $n-$positive if φ_n is positive. Also φ is called completely positive if φ is $n-$positive for all n. Every positive linear map is not necessary completely positive, for example if $\varphi : M_2(\mathbb{C})\to M_2(\mathbb{C})$ is defined by $\varphi(A)=A^t$, then φ is positive but is not necessary completely positive, [10, Example 4.2]. Every $*$-homomorphism on a $*$-algebras is completely positive [2, Example II.6.9.3] but the converse is false. For example if $\varphi : M_2(\mathbb{C})\to\mathbb{C}$ is defined by $\varphi\left((a_{i,j})_{i,j}\right)=\sum a_{i,i}$, then φ is completely positive by [10, Exercise 3.5]. But φ is not a homomorphism because

$$\varphi\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right)^2 = 9 \qquad , \varphi\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}^2\right) = 5.$$

A subspace S of a unital C^*-algebra A is called operator system if it is self-adjoint ($S = S^*$) and contains the unit of A.

Numerical range of operators is very important and was studied by many authors, see, e.g., [1, 5, 7, 11, 13]. In this paper, we characterize a linear map φ from a unital C^*-algebra A into a unital commutative C^*-algebra B and show that if φ is unital and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ or $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism. Also if $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism. We show that every Jordan homomorphism from a complex Banach algebra into $\mathbb{C}$ is a numerical range compressing, but the converse is false.

Also in this paper, we discuss about absolute value preserving linear maps and show that if φ is a linear map from a unital C^*-algebra A into a unital commutative C^*-algebra B such that $|\varphi(a)\varphi(b)| = \varphi(|ab|)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

2. Preliminaries

Let A be a unital C^*-algebra. If a and b are positive elements of A such that $a^2 = b^2$, then $\sigma\left(a^2 - b^2\right) = -\sigma\left(b^2 - a^2\right) = \sigma(0) = \{0\}$. Since $a^2 - b^2$ and $b^2 - a^2$ are self-adjoint, $a^2 \geq b^2$ and $a^2 \leq b^2$, so by [9, Theorem 2.2.6], $a \geq b$ and $a \leq b$. This implies that $a = b$, [2, Proposition II.3.1.2]. Let X be a compact Hausdorff space. We denote by $C(X)$ the algebra of all continuous complex functionals on X.

Theorem 2.1. *Let A be a unital C^*-algebra and $\varphi : A \to C(X)$ be a positive linear map. If φ is a unitary preserving map, then φ is a unital $*$-homomorphism.*

Proof. Since φ is positive and preserves unitary elements, $\varphi(1_A)$ is positive and $\varphi(1_A)^2 = \varphi(1_A)^*\varphi(1_A) = 1_{C(X)}$, so $\varphi(1_A) = 1_{C(x)}$.

Since φ is unital and positive by [2, Proposition II.6.9.4], φ is a contraction. Also, since φ is a bounded linear map and A is an operator system, φ is completely positive [10, Proposition 3.9].

Now, let a and b be elements of A such that $\|a\| < 1$ and $\|b\| < 1$. Since -1_A is a unitary element of A by [2, Poroposition II.3.2.13], there exist unitary elements u_1,u_2,v_1,v_2 of A such that $a - 1_A = u_1 + u_2$ and $b - 1_A = v_1 + v_2$. So

$$\varphi(ab) = \varphi(1_A) + \varphi(v_1) + \varphi(v_2) + \varphi(u_1) + \varphi(u_2)$$

$$+\varphi(u_1v_1)+\varphi(u_1v_2)+\varphi(u_2v_1)+\varphi(u_2v_2).$$

Since v_1 is a unitary element of A and φ preserves unitary elements, we have

$$\begin{aligned}\varphi(v_1)^*\varphi(v_1) &= 1_{C(X)}\\ &= \varphi(1_A)\\ &= \varphi(v_1^*v_1),\end{aligned}$$

and therefore $\varphi(xv_1)=\varphi(x)\varphi(v_1)$ for all $x\in A$ by [2, Poroposition II.6.9.18]. Similarly, we can show that $\varphi(xv_2)=\varphi(x)\varphi(v_2)$ for all $x\in A$. Therefore

$$\begin{aligned}\varphi(ab) = 1_{C(X)}&+\varphi(v_1)+\varphi(v_2)+\varphi(u_1)+\varphi(u_2)+\varphi(u_1)\varphi(v_1)\\ &+\varphi(u_1)\varphi(v_2)+\varphi(u_2)\varphi(v_1)+\varphi(u_2)\varphi(v_2)\\ &= (1_{C(X)}+\varphi(u_1)+\varphi(u_2))(1+\varphi(v_1)+\varphi(v_2))\\ &= \varphi(a)\varphi(b).\end{aligned}$$

Let $\|a\|\geq 1$ and $\|b\|\geq 1$ and let $a^{'}=\frac{a}{1+\|a\|}$ and $b^{'}=\frac{b}{1+\|b\|}$. Then $\|a^{'}\|<1$ and $\|b^{'}\|<1$. Therefore $\varphi(a^{'}b^{'})=\varphi(a^{'})\varphi(b^{'})$ and hence $\varphi(ab)=\varphi(a)\varphi(b)$. Also if $a\in A$ is self-adjoint, then there exist positive elements $b,c\in A$ such that $a=b-c$. Thus $\varphi(a^*)=\varphi(b)-\varphi(c)=(\varphi(b)-\varphi(c))^*=\varphi(a)^*$. Let $a\in A$ be arbitrary. Then there exist self-adjoint elements $x,y\in A$ such that $a=x+iy$, so

$$\begin{aligned}\varphi(a^*) &= \varphi(x-iy)\\ &= \varphi(x)-i\varphi(y)\\ &= (\varphi(x)+i\varphi(y))^*\\ &= \varphi(a)^*.\end{aligned}$$

□

Remark 2.2. Since by [9, Theorem 2.1.10], every non-zero commutative C^*-algebra B is isomorphic to $C(\Omega(B))$, where $\Omega(B)$ is the set of all linear homomorphisms from B into $\mathbb{C}$, every unitary preserving positive linear map from a unital C^*-algebra A into a unital commutative C^*-algebra B is a unital $*$-homomorphism.

3. Numerical Range Preserving Maps

Let H and K be complex Hilbert spaces, $A,B\in B(H)$ and $\varphi: B(H)\to B(K)$ be a surjective map. Theorem 2.1 in [5] shows that $W(\varphi(A)\varphi(B))=W(AB)$ if and only if there exists unitary operator U in $B(H,K)$ such that φ is of the form $\varphi(A)=\epsilon UAU^*$ for all $A\in B(H)$, where $\epsilon=\pm 1$. Similarly Theorem 2.2 in [5] states that

$W(\varphi(A)\varphi(B)\varphi(A)) = W(ABA)$ if and only if there exist a scalar λ with $\lambda^3 = 1$ and a unitary operator $U : H \to K$ such that either $\varphi(A) = \lambda UAU$ or $\varphi(A) = \lambda UA^tU$, where A^t is the transpose of A with respect to an arbitrary fixed orthonormal basis of H. Also $W(\varphi(A)^*\varphi(B)) = W(A^*B)$ if and only if there exist unitary operators U and V in $B(H, K)$ such that φ is of the form $\varphi(A) = UAV^*$ [5, Corollary 4.3].

Let A and B be unital C^*-algebras and φ be a unital linear mapping from A onto B. Theorem 2.3 in [13] states that if $W(\varphi(a)) = W(a)$ for all $a \in A$, then φ is a Jordan $*-$isomorphism. Furthermore, if B is prime, then φ is a C^*-isomorphism or C^*-anti-isomorphism.

In this section, we characterize the numerical range of a map from a unital C^*-algebra into a unital commutative C^*-algebra.

Lemma 3.1. *Let A be a unital complex Banach algebra and $\varphi : A \to \mathbb{C}$ be a linear functional. If $V(\varphi(a)) = V(a)$ for all $a \in A$, then φ is a unital monomorphism.*

Proof. Since $V(\varphi(1_A)) = V(1_A) = \{1\}$, $v(\varphi(1) - 1) = 0$ and by [3, Theorem 1.4.1] $\varphi(1)-1 = 0$, therefore φ is unital. Let $a \in A$ be invertible but $\varphi(a)$ is singular, then $\varphi(a) = 0$. Thus $V(a) = V(\varphi(a)) = 0$, so $a = 0$ by [3, Theorem, 1.4.1]. But this is a contraction, so φ is an invertibility preserving functional, therefore by Gleason-Kahane-Zelazko Theorem, φ is multiplicative.

If $a \in$A and $\varphi(a) = 0$, then $V(a) = V(\varphi(a)) = 0$ and by [3, Theorem 1.4.1] $a = 0$, so φ is injective. □

Lemma 3.2. *Let A be a unital complex Banach algebra and $\varphi : A \to \mathbb{C}$ be a linear functional. If φ is a Jordan homomorphism, then φ is a unital numerical range compressing.*

Proof. Since φ is a Jordan homomorphism, then by [4, Proposition, II.16.6] φ is a homomorphism, so φ is continuous and $\|\varphi\| = \varphi(1_A) = 1$ [4, Proposition II.16.3]. Let $a \in A$ and $\lambda \in V(\varphi(a))$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\varphi(a))$. Let $g = f\varphi$. Then g is linear, continuous and for all $x \in A$

$$|g(x)| = |f(\varphi(x))| \leq \|f\|\|\varphi\|\|x\| = \|x\|,$$

so $\|g\| \leq 1$, also $g(1_A) = (f\varphi)(1_A) = 1$, thus $\|g\| = 1$ and it follows that $g \in D(A, 1_A)$, but $\lambda = g(a)$. Therefore $\lambda \in V(a)$. □

Remark 3.3. The converse of Lemma 3.2 is false. To see that, let $A = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ and define $\varphi : A \to \mathbb{C}$ by $\varphi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \frac{a+c}{2}$, then

$$V\left(\varphi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)\right) = V\left(\frac{a+c}{2}\right)$$

$$= \left\{ \frac{a+c}{2} \right\}.$$

Also $a, c \in \sigma\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$, so $a, c \in V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$ [3, Theorem I.2.6]. Also by [4, Proposition 1.10.4], $V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$ is convex, so $\frac{a+c}{2} \in V\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right)$. Therefore $V(\varphi(x)) \subseteq V(x)$ for all $x \in A$. But φ is not a Jordan homomorphism, because $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^2 = \frac{9}{4}$ and $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2\right) = \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right) = \frac{5}{2}$.

Theorem 3.4. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a unital linear map. If B is commutative and $V(\varphi(a)^*\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let $a \in A$ be positive. Then $V(\varphi(a)) = V(\varphi(1_A)^*\varphi(a)) \subseteq V(1_A a) \subseteq \mathbb{R}^+$, so φ is a positive map. Let $u \in A$ be unitary, then

$$\begin{aligned} V(\varphi(u)^*\varphi(u)) &\subseteq V(u^*u) \\ &= V(1) \\ &= \{1\}, \end{aligned}$$

thus $\varphi(u)^*\varphi(u) = 1_B$, so φ is a unitary preserving positive linear map. Therefore φ is a $*$-homomorphism by Remark 2.2. □

Corollary 3.5. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a surjective unital linear map. If B is commutative and $V(\varphi(a)^*\varphi(b)) = V(a^*b)$ for all $a, b \in A$, then φ is a $*$-isomorphism.*

Proof. If $a \in$A and $\varphi(a) = 0$, then

$$\begin{aligned} v(a) &= v(\varphi(1)^*\varphi(a)) \\ &= v(\varphi(a)) \\ &= v(0) \\ &= 0, \end{aligned}$$

and by [4, Theorem 1.10.14] $a = 0$, so φ is injective. Also φ is a $*$-homomorphism, by Theorem 3.4, so φ is a $*$-isomorphism. □

Theorem 3.6. *Let A be a unital C^*-algebra. If φ is a linear functional on A such that $V(\varphi(a)\varphi(b)) \subseteq V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-homomorphism .*

Proof. $V(\varphi(1_A)\varphi(1_A)) \subseteq V(1) = \{1\}$, so $\varphi(1_A)^2 = 1$, thus $\varphi(1_A) = 1$ or $\varphi(1_A) = -1$. Also if $a \in A$ is self-adjoint, then

$$V(\varphi(a)) = V(\varphi(a)\varphi(1))$$

$$\begin{aligned}&\subseteq V(a)\\&\subseteq \mathbb{R},\end{aligned}$$

or

$$\begin{aligned}V(\varphi(a)) &= V(\varphi(a)\varphi(-1))\\&\subseteq V(-a)\\&\subseteq \mathbb{R},\end{aligned}$$

so $\varphi(a)$ is self-adjoint. Now let $x \in A$, then there exist self-adjoint elements $a, b \in A$ such that $x = a + ib$, then

$$\begin{aligned}\varphi(x^*) &= \varphi(a - ib)\\&= \varphi(a) - i\varphi(b)\\&= (\varphi(a) + i\varphi(b))^*\\&= \varphi(x)^*,\end{aligned}$$

so $V(\varphi(a)^*\varphi(b)) = V(\varphi(a^*)\varphi(b)) \subseteq V(a^*b)$ for all $a, b \in A$. Then, by Theorem 3.4, φ is $*$-preserving and $\varphi(ab) = \pm\varphi(a)\varphi(b)$ for all $a, b \in A$. □

Corollary 3.7. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $V(\varphi(a)\varphi(b)) \subseteq V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B and $\psi = \tau\varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b))$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)) = f\tau(\varphi(a)\varphi(b))$. But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b))$, thus $\lambda \in V(ab)$. Therefore $V(\psi(a)\psi(b)) \subseteq V(ab)$ and by Theorem 3.9, for all $x, y \in A$, we have

$$\begin{aligned}\tau(\varphi(xy)) &= \psi(xy)\\&= \pm\psi(x)\psi(y)\\&= \pm\tau(\varphi(x))\tau(\varphi(y))\\&= \tau(\pm\varphi(x)\varphi(y),\end{aligned}$$

and

$$\begin{aligned}\tau(\varphi(x^*)) &= \psi(x^*)\\&= \psi(x)^*\\&= \tau(\varphi(x)^*).\end{aligned}$$

Since B is semi-simple, $\Omega(B)$ separates the points of B [4, Corollary II.17.7], so $\varphi(xy) = \pm\varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$. □

Corollary 3.8. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a surjective linear map. If B is commutative and $V(\varphi(a)\varphi(b)) = V(ab)$ for all $a, b \in A$, then φ is a scaler of a $*$-isomorphism.*

Proof. The map φ is injective. Also $\pm\varphi$ is a $*$-homomorphism, by Corollary 3.7, so φ is a scaler of a $*$-isomorphism. □

Theorem 3.9. *Let A be a unital C^*-algebra and φ be a unital linear functional on A. If $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let $a \in A$ be positive, then $V(\varphi(a)) \subseteq V(a) \subseteq \mathbb{R}^+$, so φ is positive. Therefore $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Since φ is unital and positive, then by [2, Proposition II.6.9.4] φ is a contraction. Also since φ is a bounded linear functional and A is an operator system, then φ is completely positive [10, Proposition 3.8], thus φ is 2-positive.

Let $a, b \in A$ and $ab = 0$, then $V(\varphi(a)\varphi(b)\varphi(a) \subseteq V(aba) = \{0\}$, so $\varphi(a) = 0$ or $\varphi(b) = 0$. Thus φ preserves zero product elements, therefore φ is a homomorphism [16, Theorem 2]. □

Theorem 3.10. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a unital linear map. If B is commutative and $V(\varphi(a)\varphi(b)\varphi(a)) \subseteq V(aba)$ for all $a, b \in A$, then φ is a $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B. Then $\tau(1) = \|\tau\| = 1$ by [4, Proposition I.16.3]. Let $\psi = \tau\varphi$, then $\psi(1) = 1$. Also if $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a))$, then there exists $f \in D(\mathbb{C}, 1)$ such that

$$\begin{aligned} \lambda &= f(\psi(a)\psi(b)\psi(a)) \\ &= f\tau(\varphi(a)\varphi(b)\varphi(a). \end{aligned}$$

But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a))$, thus $\lambda \in V(aba)$. Therefore $V(\psi(a)\psi(b)\psi(a)) \subseteq V(aba)$, so by Theorem 3.9, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \psi(x)\psi(y) \\ &= \tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\varphi(x)\varphi(y)), \end{aligned}$$

and

$$\begin{aligned} \tau(\varphi(x^*)) &= \psi(x^*) \\ &= \psi(x)^* \\ &= \tau(\varphi(x)^*). \end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$. □

Theorem 3.11. *Let A be a unital C^*-algebra and φ be a linear functional on A. If $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Since $V(\varphi(1_A)\varphi(1_A)\varphi(1_A)^*) \subseteq V(1) = \{1\}$, $\varphi(1_A)\varphi(1_A)\varphi(1_A)^* = 1$, thus $\varphi(1_A) = 1$. Let $a \in A$ be positive. Then

$$\begin{aligned} V(\varphi(a)) &= V\left(\varphi(1_A)\varphi(a)\varphi(1_A)^*\right) \\ &\subseteq V(a) \\ &\subseteq \mathbb{R}^+, \end{aligned}$$

so $\varphi(a)$ is positive. Let $u \in A$ be unitary. Then

$$\begin{aligned} V(\varphi(u)\varphi(u)^*) &= V\left(\varphi(u)\varphi(1)\varphi(u)^*\right) \\ &\subseteq V(uu^*) \\ &= \{1\}, \end{aligned}$$

so $\varphi(u)\varphi(u)^* = 1$. Therefore, φ is a unital $*$-homomorphism by Remark 2.2. □

Lemma 3.12. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $V(\varphi(a)\varphi(b)\varphi(a)^*) \subseteq V(aba^*)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B. Then $\tau(1) = \|\tau\| = 1$ by [4, Proposition I.16.3].

Let $\psi = \tau\varphi$. If $a, b \in A$ and $\lambda \in V(\psi(a)\psi(b)\psi(a)^*)$, then there exists $f \in D(\mathbb{C}, 1)$ such that $\lambda = f(\psi(a)\psi(b)\psi(a)^*)$. Since B is a C^*-algebra, then by [9, Theorem 2.1.9.], $\tau(x^*) = \tau(x)^*$ for all $x \in B$, so $\lambda = f\tau(\varphi(a)\varphi(b)\varphi(a)^*)$. But $f\tau(1) = \|f\tau\| = 1$, so $\lambda \in V(\varphi(a)\varphi(b)\varphi(a)^*)$, thus $\lambda \in V(aba^*)$. Therefore $V(\psi(a))\psi(b)\psi(a)^*) \subseteq V(aba^*)$, so ψ is a unital $*$-homomorphism by Theorem 3.11. Therefore, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \psi(x)\psi(y) \\ &= \tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\varphi(x)\varphi(y)). \end{aligned}$$

Similarly

$$\begin{aligned} \tau(\varphi(x^*)) &= \psi(x^*) \\ &= \psi(x)^* \\ &= \tau(\varphi(x)^*), \end{aligned}$$

and

$$\begin{aligned}\tau(\varphi(1)) &= \psi(1)\\ &= 1\\ &= \tau(1).\end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ and $\varphi(1) = 1$. □

4. Absolute Value Preserving Maps

Let H and K be Hilbert spaces and $\varphi : B(H) \to B(K)$ be an additive map. Theorem 2 in [12] states that if $\varphi(|A|) = |\varphi(A)|$ for every $A \in B(H)$, $\varphi(iI)K \subset \varphi(I)K$ and $\varphi(I)$ is a projection, then φ is the sum of two $*$-homomorphisms which one is $\mathbb{C}$-linear and the other is $\mathbb{C}$-antilinear.

Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a map satisfying $\varphi(|a|) = |\varphi(a)|$ for every $a \in A$. Theorem 2 in [16] states that, if φ is linear, then φ is positive and $\varphi(a_1a_2) = \varphi(1_A)\varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$. Also Theorem 2.2 in [14] says that, if φ is additive and surjective and $\varphi(1)$ is a projection, then φ is unital and the restriction of φ to both A_s and A_{sk} is a Jordan $*$-homomorphism onto the corresponding set in B where A_s is the set of all self-adjoint elements of A and A_{sk} is the set of all skew-self-adjoint elements of A. Furthermore, if B is a C^*-algebra of real-rank zero, then φ is a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear $*$-homomorphism on A [14, Theorem 2.5].

Theorem 2.9 in [15] states that, if $\varphi : A \to B$ is a additive map which satisfies $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for every $a, b \in A$ and $\varphi(c) = 1$ for some $c \in A$, then φ is unital and the restriction of φ to A_s is a Jordan homomorphism. Moreover, if φ is surjective and B is a real rank zero, then φ is a $\mathbb{C}-$linear or $\mathbb{C}-$antilinear $*$-homomorphism.

Molnar in [8, Theorem 3] was proved if A and B are von Neumann algebras, $A \neq \mathbb{C}I$ is a factor and $\varphi : A \to B$ is a bijective map which satisfies $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for every $a, b \in A$. Then φ is of the form $\varphi(a) = \tau(a)\psi(a)$ for all $a \in A$, where $\psi : A \to B$ is either a linear or a conjugate-linear $*$-algebra isomorphism and $\tau : A \to \mathbb{C}$ is a scalar function of modulus 1.

In this section, we show that if φ is a linear map from a unital C^*-algebra into a unital commutative C^*-algebra B such that $|\varphi(a)\varphi(b)| = \varphi(|ab|)$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.

Remark 4.1. Let A and B be unital C^*-algebras. If B is commutative and $\varphi : A \to B$ is a unital linear map such that $\varphi(|a|) = |\varphi(a)|$ for all $a \in A$, then by using Remark 2.2 we can show that φ is a unital $*$-homomorphism which compares with Theorem 2 in [16].

Theorem 4.2. *Let A be a unital C^*-algebra and φ be a linear functional on A. If $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. If $a \in A^+$, then $|a|^2 = a^2$ and since a and $|a|$ are positive, $|a| = a$, thus $\varphi(a) = \varphi(|a|) = |\varphi(a)| \geq 0$, so φ is a positive. Since $\varphi(1_A)$ is positive, $\varphi(1_A) = \varphi(|1_A|) = |\varphi(1_A)^2| = \varphi(1_A)^2$. So $\varphi(1_A) = 0$ or $\varphi(1_A) = 1$. If $\varphi(1_A) = 0$, then $\varphi(a) = \varphi(|a|) = |\varphi(a)\varphi(1_A)| = 0$ for all $a \in A^+$ and by [10, Remark 2.2.2] $\varphi(a) = 0$ for all $a \in A$, so $\varphi = 0$ and it is a contraction, therefore $\varphi(1) = 1$. If $u \in A$ is unitary, then $|u| = 1$, so $|\varphi(u)| = |\varphi(u)\varphi(1)| = \varphi(|u|) = \varphi(1) = 1$. Thus $\varphi(u)^*\varphi(u) = |\varphi(u)|^2 = 1$, so φ is unitary preserving. Therefore φ is a unital $*$-homomorphism by Remark 2.2. □

Corollary 4.3. *Let A and B be unital C^*-algebras and $\varphi : A \to B$ be a linear map. If B is commutative and $\varphi(|ab|) = |\varphi(a)\varphi(b)|$ for all $a, b \in A$, then φ is a unital $*$-homomorphism.*

Proof. Let τ be a multiplicative functional on B and $\psi = \tau\varphi$. Since τ is $*$-homomorphism, then for all $a \in A$, we have:

$$\begin{aligned} |\tau(a)|^2 &= \tau(a)^*\tau(a) \\ &= \tau(a^*a) \\ &= \tau(|a|^2) \\ &= (\tau(|a|))^2. \end{aligned}$$

Since $|\tau(a)|$ and $\tau(|a|)$ are positive, $|\tau(a)| = \tau(|a|)$. Now let $a, b \in A$. Then

$$\begin{aligned} \psi(|ab|) &= \tau\varphi(|ab|) \\ &= \tau(\varphi(|ab|)) \\ &= \tau(|\varphi(a)\varphi(b)|) \\ &= |\tau(\varphi(a)\varphi(b))| \\ &= |\tau(\varphi(a))\tau(\varphi(b))| \\ &= |\psi(a)\psi(b)|. \end{aligned}$$

So ψ is a unital $*$-homomorphism by Theorem 4.2. Therefore, for all $x, y \in A$, we have

$$\begin{aligned} \tau(\varphi(xy)) &= \psi(xy) \\ &= \psi(x)\psi(y) \\ &= \tau(\varphi(x))\tau(\varphi(y)) \\ &= \tau(\varphi(x)\varphi(y)). \end{aligned}$$

Similarly

$$\begin{aligned}\tau(\varphi(x^*)) &= \psi(x^*)\\ &= \psi(x)^*\\ &= \tau(\varphi(x)^*),\end{aligned}$$

and

$$\begin{aligned}\tau(\varphi(1)) &= \psi(1)\\ &= 1\\ &= \tau(1).\end{aligned}$$

Since B is semi-simple, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ and $\varphi(1) = 1$. □

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