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An Example of Data Dependence Result for The Class of Almost Contraction Mappings

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Abstract. In the present paper, we show that S^* iteration method can be used to approximate fixed point of almost contraction mappings. Furthermore, we prove that this iteration method is equivalent to CR iteration method and it produces a slow convergence rate compared to the CR iteration method for the class of almost contraction mappings. We also present table and graphic to support this result. Finally, we obtain a data dependence result for almost contraction mappings by using S^* iteration method and in order to show validity of this result we give an example.

1. Introduction and Preliminaries

The iterative approximation is one of the significant tools in the fixed point theory. Hence, for certain classes of operators, many iteration methods have been introduced and analyzed by a great number of researches in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1], [11], [18]). The following iteration methods are called Noor [14] and SP [16] iteration methods, respectively:

$$\left\{\begin{array}{l} x_{n+1}=(1-\alpha_n)\,x_n+\alpha_n T y_n \\ y_n=(1-\beta_n)\,x_n+\beta_n T z_n \\ z_n=(1-\gamma_n)\,x_n+\gamma_n T x_n, \quad n\in\mathbb{N}, \end{array}\right. \tag{1.1}$$

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where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

$$(1.2)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)\, y_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, z_n + \beta_n T z_n \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

The following iteration method is called CR iteration method [5],

$$(1.3)\qquad \begin{cases} u_{n+1} = (1-\alpha_n)\, v_n + \alpha_n T v_n \\ v_n = (1-\beta_n)\, T u_n + \beta_n T w_n \\ w_n = (1-\gamma_n)\, u_n + \gamma_n T u_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

Karahan and Özdemir [8] have introduced an S^* iteration method as follows:

$$(1.4)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)\, T x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, T x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

Sometimes there can be two or more iteration methods which are convergent to a fixed point of a particular mapping (see [4],[11]). In such a case, it is an important problem from theoretical and practical aspects to determine that the iteration method converges faster than others (see [2],[6],[12],[15]).

In this study, we prove that S^* iteration method (1.4) is strongly convergent to the fixed point of almost contraction mappings (1.6). Moreover, we show the equivalence of convergence between S^* and CR iteration methods. We also compare the rate of convergence of CR and S^* iteration methods for these mappings. In order to support this result we give a numerical example. Finally, using S^* iteration method, we give a data dependence result for almost contraction mappings. Now, we give some lemmas and definitions which will be useful in obtaininig our main results.

Lemma 1.1 ([20])**.** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + b_n,$$

where $\mu_n \in [0,1]$ for all $n \geq n_0$, $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 ([19])**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following condition* *holds:*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\eta_n,$$

where $\mu_n \in (0,1)$ *such that* $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ *and* $\eta_n \geq 0$. *Then the following inequality holds:*

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup \eta_n.$$

In 2003, Berinde [3] introduced almost contraction type operators on a normed space X satisfying

$$\|Tx - Ty\| \leq \delta \|x-y\| + L. \|y - Tx\|, \tag{1.5}$$

for any $x, y \in X$, $\delta \in (0,1)$ and $L \geq 0$.

Theorem 1.3 ([3]). *Let X be a Banach space and $T : X \to X$ be an operator satisfying (1.5) such that*

$$\|Tx - Ty\| \leq \delta \|x-y\| + L_1. \|x - Tx\|. \tag{1.6}$$

Then, T has a unique fixed point.

Definition 1.4 ([15]). Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two iteration methods converging to the same fixed point p_* of a mapping T. We say that $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ to p_* if

$$\lim_{n\to\infty} \frac{\|a_n - p_*\|}{\|b_n - p_*\|} = 0.$$

Definition 1.5 ([19]). Let T, $S : C \to C$ be two operators. We say that S is an approximate operator of T for all $x \in C$ and a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.

2. Main Resuts

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be an almost contraction mapping satisfying condition (1.6). Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by (1.4) with a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point p_* of T.*

Proof. It can be easily seen from (1.6) that, p_* is the unique fixed point of T. We shall show that $x_n \to p_*$ as $n \to \infty$. From (1.6) and (1.4), we have

$$\begin{aligned} \|z_n - p_*\| &= \|(1-\gamma_n)\, x_n + \gamma_n T x_n - p_*\| \\ &\leq (1-\gamma_n) \|x_n - p_*\| + \gamma_n \|Tx_n - Tp_*\| \\ &\leq \{1 - \gamma_n (1-\delta)\} \|x_n - p_*\|, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \|y_n - p_*\| &= \|(1-\beta_n)\, Tx_n + \beta_n T z_n - p_*\| \\ &\leq (1-\beta_n)\, \|Tx_n - Tp_*\| + \beta_n \|Tz_n - Tp_*\| \\ &\leq (1-\beta_n)\, \delta \|x_n - p_*\| + \beta_n \delta \|z_n - p_*\| . \end{aligned} \tag{2.2}$$

Substituting (2.1) in (2.2), we obtain

$$\|y_n - p_*\| \leq \{(1-\beta_n)\,\delta + \beta_n \delta [1 - \gamma_n (1-\delta)]\} \|x_n - p_*\| . \tag{2.3}$$

Also,

$$\begin{aligned} \|x_{n+1} - p_*\| &= \|(1-\alpha_n)\, Tx_n + \alpha_n T y_n - p_*\| \\ &\leq (1-\alpha_n)\, \|Tx_n - Tp_*\| + \alpha_n \|Ty_n - Tp_*\| \\ &\leq (1-\alpha_n)\, \delta \|x_n - p_*\| + \alpha_n \delta \|y_n - p_*\| . \end{aligned} \tag{2.4}$$

Substituting (2.3) in (2.4), we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &\leq (1-\alpha_n)\, \delta \|x_n - p_*\| \\ &\quad + \alpha_n \delta \Big\{ (1-\beta_n)\, \delta + \beta_n \delta \,[1 - \gamma_n (1-\delta)] \Big\} \|x_n - p_*\| . \end{aligned}$$

Since $\delta \in (0{,}1)$ and $\alpha_n, \beta_n, \gamma_n \in [0{,}1]$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+1} - p_*\| &\leq \delta [1 - \alpha_n (1-\delta)] \|x_n - p_*\| \\ &\leq [1 - \alpha_n (1-\delta)] \|x_n - p_*\| . \end{aligned} \tag{2.5}$$

By induction, inequality (2.5) yields

$$\|x_{n+1} - p_*\| \leq \|x_0 - p_*\| \prod_{k=0}^{n} [1 - \alpha_k (1-\delta)].$$

It is well-known from classical analysis that $1 - x \leq e^{-x}$ for all $x \in [0{,}1]$. By considering this fact, we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &\leq \|x_0 - p_*\| \prod_{k=0}^{n} e^{-(1-\delta)\alpha_k} \\ &= \|x_0 - p_*\| \, e^{-(1-\delta) \sum_{k=0}^{n} \alpha_k} . \end{aligned} \tag{2.6}$$

Taking the limit of both sides of inequality (2.6), $x_n \to p_*$ as $n \to \infty$. $\square$

Theorem 2.2. *Let C, X and T with a fixed point p_* be as in Theorem 2.1. Let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be two iterative sequences defined by (1.3) for $u_0 \in C$ and (1.4) for $x_0 \in C$ with the same real sequences $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$,$\{\gamma_n\}_{n=0}^{\infty} \in [0{,}1]$. Then the following assertions are equivalent:*

(i) *The S^* iteration method (1.4) converges to the fixed point p_* of T.*

(ii) *The CR iteration method (1.3) converges to the fixed point* p_* *of* T.

Proof. We will show that $(i) \Rightarrow (ii)$, that is if the iteration method (1.4) converges, then the iteration method (1.3) does too. Now, by using (1.4), (1.3) and (1.6), we have

$$\begin{aligned} \|z_n - w_n\| &= \|(1-\gamma_n)\,x_n + \gamma_n T x_n - (1-\gamma_n)\,u_n - \gamma_n T u_n\| \\ &\le (1-\gamma_n)\,\|x_n - u_n\| + \gamma_n\,\|Tx_n - Tu_n\| \\ &\le [1-\gamma_n(1-\delta)]\,\|x_n - u_n\| + \gamma_n L\,\|x_n - Tx_n\|, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \|y_n - v_n\| &= \|(1-\beta_n)\,Tx_n + \beta_n T z_n - (1-\beta_n)\,Tu_n - \beta_n T w_n\| \\ &\le (1-\beta_n)\delta\,\|x_n - u_n\| + (1-\beta_n)L\,\|x_n - Tx_n\| \\ &\quad + \beta_n\delta\,\|z_n - w_n\| + \beta_n L\,\|z_n - Tz_n\|. \end{aligned} \tag{2.8}$$

Substituting (2.7) in (2.8), we obtain

$$\begin{aligned} \|y_n - v_n\| &\le \delta[1-\beta_n\gamma_n(1-\delta)]\,\|x_n - u_n\| \\ &\quad + \{(1-\beta_n)L + \beta_n\gamma_n\delta L\}\,\|x_n - Tx_n\| \\ &\quad + \beta_n L\,\|z_n - Tz_n\|. \end{aligned} \tag{2.9}$$

Also,

$$\begin{aligned} \|x_n - w_n\| &= \|x_n - (1-\gamma_n)\,u_n - \gamma_n T u_n\| \\ &\le (1-\gamma_n)\,\|x_n - u_n\| + \gamma_n\,\|x_n - Tx_n\| + \gamma_n\,\|Tx_n - Tx_n\| \\ &\le [1-\gamma_n(1-\delta)]\,\|x_n - u_n\| + (1+L)\gamma_n\,\|x_n - Tx_n\|, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \|x_n - v_n\| &= \|x_n - (1-\beta_n)\,Tu_n - \beta_n T w_n\| \\ &\le (1-\beta_n)\,\|x_n - Tu_n\| + \beta_n\,\|x_n - Tw_n\| \\ &\le (1-\beta_n)\,\|x_n - Tx_n\| + (1-\beta_n)\,\|Tx_n - Tu_n\| \\ &\quad + \beta_n\,\|x_n - Tx_n\| + \beta_n\,\|Tx_n - Tw_n\| \\ &\le (1-\beta_n)\,\delta\,\|x_n - u_n\| + \beta_n\delta\,\|x_n - w_n\| \\ &\quad + (1+L)\,\|x_n - Tx_n\|. \end{aligned} \tag{2.11}$$

Using the last two inequalities, we get

$$\begin{aligned} \|x_n - v_n\| &\le \delta[1-\beta_n\gamma_n(1-\delta)]\,\|x_n - u_n\| \\ &\quad + \{(1+L)(1+\beta_n\gamma_n\delta)\,\|x_n - Tx_n\|. \end{aligned} \tag{2.12}$$

Then,

(2.13)

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|(1-\alpha_n)\,Tx_n+\alpha_nTy_n-(1-\alpha_n)\,v_n-\alpha_nTv_n\| \\ &\le (1-\alpha_n)\,\|x_n-Tx_n\|+(1-\alpha_n)\,\|x_n-v_n\| \\ &\quad +\alpha_n\delta\,\|y_n-v_n\|+\alpha_nL\,\|y_n-Ty_n\|\,.\end{aligned}$$

Substituting (2.9) and (2.12) in (2.13), we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \le{}& (1-\alpha_n)\,\|x_n-Tx_n\| \\ &+(1-\alpha_n)\,\delta[1-\beta_n\gamma_n(1-\delta)]\,\|x_n-u_n\| \\ &+(1-\alpha_n)\,\{(1+L)(1+\beta_n\gamma_n\delta)\,\|x_n-Tx_n\| \\ &+\alpha_n\delta^2[1-\beta_n\gamma_n(1-\delta)]\,\|x_n-u_n\| \\ &+\alpha_n\delta\{(1-\beta_n)L+\beta_n\gamma_n\delta L\}\,\|x_n-Tx_n\| \\ &+\alpha_nL\,\|y_n-Ty_n\|+\alpha_n\delta\beta_nL\,\|z_n-Tz_n\|\,.\end{aligned}$$

Hence we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \le{}& \delta[1-\alpha_n(1-\delta)][1-\beta_n\gamma_n(1-\delta)]\,\|x_n-u_n\| \\ &+\Bigg\{(1-\alpha_n)[1+(1+L)(1+\beta_n\gamma_n\delta)] \\ &+\alpha_n\delta[(1-\beta_n)L+\beta_n\gamma_n\delta L]\Bigg\}\,\|x_n-Tx_n\| \\ &+\alpha_nL\,\|y_n-Ty_n\|+\alpha_n\delta\beta_nL\,\|z_n-Tz_n\|\,.\end{aligned}$$

Since $\delta\in(0,1)$ and $[1-\beta_n\gamma_n(1-\delta)]\le 1$, we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \le{}& [1-\alpha_n(1-\delta)]\,\|x_n-u_n\| \\ &+\Bigg\{(1-\alpha_n)[1+(1+L)(1+\beta_n\gamma_n\delta)] \\ &+\alpha_n\delta[(1-\beta_n)L+\beta_n\gamma_n\delta L]\Bigg\}\,\|x_n-Tx_n\| \\ &+\alpha_nL\,\|y_n-Ty_n\|+\alpha_n\delta\beta_nL\,\|z_n-Tz_n\|\,.\end{aligned}$$

Furthermore, using $Tp_*=p_*$ and $\|x_n-p_*\|\to 0$, we have

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-p_*\|+\delta\,\|x_n-p_*\|+L\,\|p_*-Tp_*\| \\ &= (1+\delta)\,\|x_n-p_*\|\,,\end{aligned}$$

so, $\|x_n-Tx_n\|\to 0$. Similarly,

$$\begin{aligned}\|y_n-Ty_n\| &\le \|y_n-p_*\|+\|Tp_*-Ty_n\| \\ &\le (1+\delta)\,\|y_n-p_*\| \\ &\le (1+\delta)\,(1-\beta_n)\,\|Tx_n-Tp_*\|+(1+\delta)\beta_n\,\|Tz_n-Tp_*\|\end{aligned}$$

$$\leq (1+\delta)\,(1-\beta_n)\left\{\delta\left\|x_n-p_*\right\|+L\left\|x_n-Tx_n\right\|\right\}$$
$$+\,(1+\delta)\beta_n\left\{\delta\left\|z_n-p_*\right\|+L\left\|z_n-Tz_n\right\|\right\}.$$

Moreover,

$$\begin{aligned}\left\|z_n-Tz_n\right\| &\leq \left\|z_n-p_*\right\|+\left\|Tp_*-Tz_n\right\| \\ &\leq \left\|z_n-p_*\right\|+\delta\left\|z_n-p_*\right\|+L\left\|p_*-Tp_*\right\| \\ &= (1+\delta)\left\|z_n-p_*\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|z_n-p_*\right\| &= \left\|(1-\gamma_n)\,x_n+\gamma_n Tx_n-p_*\right\| \\ &\leq (1-\gamma_n)\left\|x_n-p_*\right\|+\gamma_n\left\|Tx_n-Tp_*\right\| \\ &\leq (1-\gamma_n)\left\|x_n-p_*\right\|+\gamma_n\delta\left\|x_n-p_*\right\|+\gamma_n L\left\|p_*-Tp_*\right\| \\ &= [1-\gamma_n(1-\delta)]\left\|x_n-p_*\right\|,\end{aligned}$$

then $\|z_n-p_*\|\to 0$ as $n\to\infty$. Thus, $\|z_n-Tz_n\|\to 0$ as $n\to\infty$. Then, we obtain $\|y_n-Ty_n\|\to 0$ as $n\to\infty$. Denote

$$\begin{aligned}\mu_n &= \alpha_n(1-\delta)\in(0,1) \\ a_n &= \left\|x_n-u_n\right\| \\ b_n &= \{(1-\alpha_n)[1+(1+L)(1+\beta_n\gamma_n\delta)] \\ &\quad +\alpha_n\delta[(1-\beta_n)L+\beta_n\gamma_n\delta L]\}\left\|x_n-Tx_n\right\| \\ &\quad +\alpha_n L\left\|y_n-Ty_n\right\|+\alpha_n\delta\beta_n L\left\|z_n-Tz_n\right\|.\end{aligned}$$

Thus, from Lemma 1.1, $a_n=\|x_n-u_n\|\to 0$ as $n\to\infty$. Consequently,

$$\left\|x_{n+1}-u_{n+1}\right\|\to 0 \text{ as } n\to\infty.$$

Now, we show that $(ii)\Rightarrow(i)$:

$$\begin{aligned}\left\|u_n-z_n\right\| &= \left\|u_n-(1-\gamma_n)x_n-\gamma_n Tx_n\right\| \\ &\leq (1-\gamma_n)\left\|u_n-x_n\right\|+\gamma_n\left\|u_n-Tu_n\right\|+\gamma_n\left\|Tu_n-Tx_n\right\| \\ &\leq [1-\gamma_n(1-\delta)]\left\|u_n-x_n\right\|+\gamma_n(1+L)\left\|u_n-Tu_n\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|w_n-z_n\right\| &\leq (1-\gamma_n)\left\|u_n-x_n\right\|+\gamma_n\left\|Tu_n-Tx_n\right\| \\ &\leq [1-\gamma_n(1-\delta)]\left\|u_n-x_n\right\|+\gamma_n L\left\|u_n-Tu_n\right\|.\end{aligned}\tag{2.14}$$

Also,

$$\begin{aligned}\left\|v_n-y_n\right\| &\leq (1-\beta_n)\,\delta\left\|u_n-x_n\right\|+(1-\beta_n)\,L\left\|u_n-Tu_n\right\| \\ &\quad +\beta_n\delta\left\|w_n-z_n\right\|+\beta_n L\left\|w_n-Tw_n\right\|.\end{aligned}\tag{2.15}$$

Substituting (2.14) in (2.15), we obtain

(2.16)

$$\left\|v_n-y_n\right\|\leq\delta[1-\beta_n\gamma_n(1-\delta)]\left\|u_n-x_n\right\|$$

$$+\left\{(1-\beta_n)L+\beta_n\gamma_n\delta L\right\}\|u_n-Tu_n\|+\beta_n L\|w_n-Tw_n\|.$$

Moreover,

$$\begin{aligned}\|w_n-x_n\|&=\|(1-\gamma_n)u_n+\gamma_n Tu_n-x_n\|\\&\le(1-\gamma_n)\|u_n-x_n\|+\gamma_n\|Tu_n-x_n\|\\&\le\|u_n-x_n\|+\gamma_n\|u_n-Tu_n\|,\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}\|v_n-x_n\|&=\|(1-\beta_n)Tu_n+\beta_n Tw_n-x_n\|\\&\le(1-\beta_n)\|u_n-x_n\|+(1-\beta_n)\|u_n-Tu_n\|\\&\quad+\beta_n\|w_n-Tw_n\|+\beta_n\|w_n-x_n\|.\end{aligned}\tag{2.18}$$

Substituting (2.17) in (2.18), we obtain

$$\begin{aligned}\|v_n-x_n\|\le\|u_n-x_n\|&+[1-\beta_n(1-\gamma_n)]\|u_n-Tu_n\|\\&+\beta_n\|w_n-Tw_n\|.\end{aligned}\tag{2.19}$$

Then,

$$\begin{aligned}\|u_{n+1}-x_{n+1}\|\le(1-\alpha_n)\delta\|v_n-x_n\|&+\alpha_n\delta\|v_n-y_n\|\\&+(1-\alpha_n+L)\|v_n-Tv_n\|.\end{aligned}\tag{2.20}$$

Substituting (2.16) and (2.19) in (2.20), we obtain

$$\begin{aligned}\|u_{n+1}-x_{n+1}\|\le&\left\{(1-\alpha_n)\delta+\alpha_n\delta^2[1-\beta_n\gamma_n(1-\delta)]\right\}\|u_n-x_n\|\\&+\left\{(1-\alpha_n)\delta[1-\beta_n(1-\gamma_n)]\right.\\&\left.+\alpha_n\delta[(1-\beta_n)L+\beta_n\gamma_n\delta L\right\}\|u_n-Tu_n\|\\&+(1-\alpha_n+L)\|v_n-Tv_n\|\\&+\left\{(1-\alpha_n)\delta\beta_n+\alpha_n\delta\beta_n L\right\}\|w_n-Tw_n\|.\end{aligned}$$

Since $\delta\in(0,1)$ and $[1-\beta_n\gamma_n(1-\delta)]\le 1$, we get

$$\begin{aligned}\|u_{n+1}-x_{n+1}\|\le&[1-\alpha_n(1-\delta)]\|u_n-x_n\|\\&+\left\{(1-\alpha_n)\delta[1-\beta_n(1-\gamma_n)]\right.\\&\left.+\alpha_n\delta[(1-\beta_n)L+\beta_n\gamma_n\delta L\right\}\|u_n-Tu_n\|\\&+(1-\alpha_n+L)\|v_n-Tv_n\|\\&+\left\{(1-\alpha_n)\delta\beta_n+\alpha_n\delta\beta_n L\right\}\|w_n-Tw_n\|.\end{aligned}$$

Furthermore, using $Tp_*=p_*$ and $\|u_n-p_*\|\to 0$, we have

$$\begin{aligned}\|u_n-Tu_n\|&\le\|u_n-p_*\|+\delta\|u_n-p_*\|+L\|p_*-Tp_*\|\\&=(1+\delta)\|u_n-p_*\|,\end{aligned}$$

so, $\|u_n - Tu_n\| \to 0$. Similarly, we have $\|v_n - Tv_n\| \to 0$ and $\|w_n - Tw_n\| \to 0$ as $n \to \infty$. Denote

$$\begin{aligned} \mu_n &= \alpha_n(1-\delta) \in (0,1) \\ a_n &= \|x_n - u_n\| \\ b_n &= \{(1-\alpha_n)\delta[1-\beta_n(1-\gamma_n)] + \alpha_n\delta[(1-\beta_n)L + \beta_n\gamma_n\delta L\} \|u_n - Tu_n\| \\ &\quad + (1-\alpha_n + L)\|v_n - Tv_n\| + \{(1-\alpha_n)\delta\beta_n + \alpha_n\delta\beta_n L\}\|w_n - Tw_n\|. \end{aligned}$$

Thus, from Lemma 1.1, $a_n = \|u_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\|u_{n+1} - x_{n+1}\| \to 0 \text{ as } n \to \infty.$$

□

As a consequence of Theorem 2.2, we can give the following corollary:

Corollary 2.3. *Let X be a Banach space, C be a nonempty, closed and convex subset of X and $T : C \to C$ be an almost contraction mapping satisfying condition (1.6) with fixed point p_*. If the initial point is the same for all iterations, then the following assertions are equivalent:*

(i) *the Picard iteration [17] converges to p_*,*
(ii) *the Mann iteration [13] converges to p_*,*
(iii) *the Ishikawa iteration [7] converges to p_*,*
(iv) *the Noor iteration (1.1) converges to p_*,*
(v) *the SP iteration (1.2) converges to p_*,*
(vi) *the CR iteration (1.3) converges to p_*,*
(vii) *the S^* iteration (1.4) converges to p_*.*

Theorem 2.4. *Let C, X and T with a fixed point p_* be as in Theorem 2.1. For given $u_0 = x_0 \in C$, consider the iterative sequences $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ defined by (1.3) and (1.4), respectively. Then $\{u_n\}_{n=0}^{\infty}$ converges to p_* faster than $\{x_n\}_{n=0}^{\infty}$ does.*

Proof. From Theorem 2.1, we have

$$\|x_{n+1} - p_*\| \le \|x_0 - p_*\| \prod_{k=0}^{n} [1 - \alpha_k(1-\delta)].$$

Then, we obtain

$$\|x_{n+1} - p_*\| \le \|x_0 - p_*\| [1 - \alpha_1 (1-\delta)]^{n+1}. \tag{2.21}$$

From CR iteration method (1.3), we obtain

$$\begin{aligned} \|w_n - p_*\| &= \|(1-\gamma_n)u_n + \gamma_n Tu_n - p_*\| \\ &\le (1-\gamma_n)\|u_n - p_*\| + \gamma_n\delta\|u_n - p_*\| + \gamma_n L\|p_* - Tp_*\| \\ &= [1 - \gamma_n(1-\delta)]\|u_n - p_*\|, \end{aligned}$$

thus, we have

(2.22)
$$\begin{aligned}\|v_n - p_*\| &= \|(1-\beta_n)Tu_n + \beta_n Tw_n - p_*\| \\ &\le (1-\beta_n)\delta \|u_n - p_*\| + \beta_n\delta \|w_n - p_*\| \\ &\le (1-\beta_n)\delta \|u_n - p_*\| + \beta_n\delta [1-\gamma_n(1-\delta)] \|u_n - p_*\| \\ &= \{(1-\beta_n)\delta + \beta_n\delta [1-\gamma_n(1-\delta)]\} \|u_n - p_*\| .\end{aligned}$$

Now by using (2.22), we have

$$\begin{aligned}\|u_{n+1} - p_*\| &= \|(1-\alpha_n)v_n + \alpha_n Tv_n - p_*\| \\ &\le (1-\alpha_n) \|v_n - p_*\| + \alpha_n\delta \|v_n - p_*\| \\ &= [1-\alpha_n(1-\delta)] \|v_n - p_*\| \\ &\le [1-\alpha_n(1-\delta)] \{(1-\beta_n)\delta + \beta_n\delta [1-\gamma_n(1-\delta)]\} \|u_n - p_*\| \\ &= \delta [1-\alpha_n(1-\delta)] [1-\beta_n\gamma_n(1-\delta)] \|u_n - p_*\| .\end{aligned}$$

Since $\delta \in (0,1)$ and $[1-\beta_n\gamma_n(1-\delta)] < 1$, we obtain

$$\|u_{n+1} - p_*\| \le \delta [1-\alpha_n(1-\delta)] \|u_n - p_*\| . \tag{2.23}$$

Then from (2.23), we have

$$\|u_{n+1} - p_*\| \le \|u_0 - p_*\| \delta^{n+1} \prod_{k=0}^{n} [1-\alpha_k(1-\delta)].$$

Then, we have

$$\|u_{n+1} - p_*\| \le \|u_0 - p_*\| \delta^{n+1} [1-\alpha_1(1-\delta)]^{n+1} . \tag{2.24}$$

From (2.21) and (2.24), we can choose $\{a_n\}$ and $\{b_n\}$,

$$\begin{aligned}a_n &= \|u_0 - p_*\| \delta^{n+1} [1-\alpha_1(1-\delta)]^{n+1} \\ b_n &= \|x_0 - p_*\| [1-\alpha_1(1-\delta)]^{n+1} ,\end{aligned}$$

respectively. Define

$$\begin{aligned}\psi_n &= \frac{a_n}{b_n} \\ &= \frac{\|u_0 - p_*\| \delta^{n+1} [1-\alpha_1(1-\delta)]^{n+1}}{\|u_0 - p_*\| [1-\alpha_1(1-\delta)]^{n+1}} \\ &= \delta^{n+1}.\end{aligned}$$

Since $\delta \in (0,1)$ we obtain $\lim_{n\to\infty} \psi_n = 0$ which implies that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{x_n\}_{n=0}^{\infty}$. □

In order to show validity of Theorem 2.4, we give a numerical example.

Example 2.5. Let $X = \mathbb{R}$ and $C = [0, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = x - 1 + \frac{1}{e^x}$ for all $x \in C$. It is easy to show that T satisfies condition (1.6) with fixed point $p_* = 0$. Choose $\alpha_n = \frac{n+4}{n+6}$, $\beta_n = \frac{n+3}{n+5}$, $\gamma_n = \frac{n+2}{n+4}$ and an initial value $x_1 = 1$. The following table and figure show that the CR iteration method (1.3) converges faster than all S^* (1.4), SP (1.2) and Noor (1.1) iteration methods.

TABLE 1. Comparison rate of convergence among various iteration methods

Iter. No	SP	Noor	CR	S*
1	1	1	1	1
2	0,12076334335371	0,34544918777689	0,08243789076071	0,12236144117368
⋮	⋮	⋮	⋮	⋮
5	0,00000061322593	0,00406927930341	0,00000000000000	0,00000000000001
6	0,00000000495537	0,00074014007758	0,00000000000000	0,00000000000000
⋮	⋮	⋮	⋮	⋮
9	0,00000000000000	0,00000271133087	0,00000000000000	0,00000000000000
⋮	⋮	⋮	⋮	⋮
18	0,00000000000000	0,00000000000000	0,00000000000000	0,00000000000000
⋮	⋮	⋮	⋮	⋮

Table 1 shows that CR iteration reaches the fixed point at the 5^{th} step while S* iteration method reaches at the 6^{th} step.

The following figure is graphical presentation of the above result:

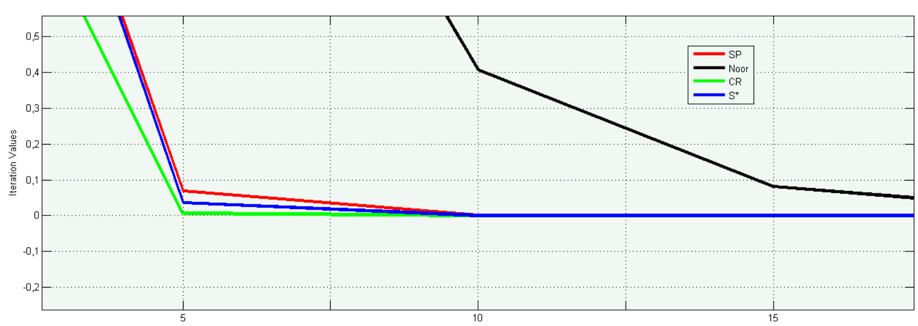

FIGURE 1. Graph of SP, Noor, CR and S* iterations

In recent years, the data dependence of fixed point in a normed space has been studied extensively by researchers (see [9], [10]).

Theorem 2.6. *Let S be an approximate operator of T. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1.4) for T and define an iterative sequence $\{u_n\}_{n=0}^{\infty}$ as follows:*

$$\begin{cases} u_0 \in C \\ u_{n+1} = (1-\alpha_n)Su_n + \alpha_n Sv_n \\ v_n = (1-\beta_n)Su_n + \beta_n Sw_n \\ w_n = (1-\gamma_n)u_n + \gamma_n Su_n, \quad n \in \mathbb{N} \end{cases} \tag{2.25}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1] satisfying $(*)$ $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$. If $Tp_* = p_*$ and $Sx_* = x_*$ such that $u_n \to x_*$ as $n \to \infty$, then we have

$$\|p_* - x_*\| \leq \frac{4\varepsilon}{1-\delta},$$

where $\varepsilon > 0$ is a fixed number.

Proof. Let us consider the iteration method (2.25) according to (1.4), using (1.6),(1.4) and (2.25) we have

(2.26)

$$\begin{aligned} \|z_n - w_n\| &\leq (1-\gamma_n)\|x_n - u_n\| + \gamma_n \|Tx_n - Su_n\| \\ &\leq (1-\gamma_n)\|x_n - u_n\| + \gamma_n \|Tx_n - Tu_n\| + \gamma_n \|Tu_n - Su_n\| \\ &\leq [1-\gamma_n(1-\delta)]\|x_n - u_n\| + \gamma_n L \|x_n - Tx_n\| + \gamma_n \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \|y_n - v_n\| &\leq (1-\beta_n)\|Tx_n - Su_n\| + \beta_n \|Tz_n - Sw_n\| \\ &\leq (1-\beta_n)\{\|Tx_n - Tu_n\| + \|Tu_n - Su_n\|\} \\ &\quad + \beta_n \{\|Tz_n - Tw_n\| + \|Tw_n - Sw_n\|\} \\ &\leq (1-\beta_n)\{\delta \|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad + \beta_n \{\delta \|z_n - w_n\| + L\|z_n - Tz_n\| + \varepsilon\}. \end{aligned} \tag{2.27}$$

Substituting (2.26) in (2.27), we obtain

$$\begin{aligned} \|y_n - v_n\| &\leq (1-\beta_n)\{\delta \|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad + \beta_n \{\delta [1-\gamma_n(1-\delta)]\|x_n - u_n\| + \delta\gamma_n L \|x_n - Tx_n\| \\ &\quad + \delta\gamma_n\varepsilon + L\|z_n - Tz_n\| + \varepsilon\} \\ &\leq \delta [1-\beta_n\gamma_n(1-\delta)]\|x_n - u_n\| + L[1-\beta_n(1-\delta\gamma_n)]\|x_n - Tx_n\| \\ &\quad + \beta_n L \|z_n - Tz_n\| + (1-\beta_n)\varepsilon + \beta_n\gamma_n\delta\varepsilon + \beta_n\varepsilon. \end{aligned}$$

Since $\delta \in (0,1)$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \in [0,1]$ for all $n \in \mathbb{N}$, we have

$$[1-\beta_n\gamma_n(1-\delta)] < 1,$$

$$[1-\beta_n(1-\delta\gamma_n)] < 1.$$

Using these inequalities, we obtain

$$\|y_n - v_n\| \le \delta \|x_n - u_n\| + L\|x_n - Tx_n\| + L\|z_n - Tz_n\| + 2\varepsilon.$$

Moreover,

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|Tx_n - Su_n\| + \alpha_n \|Ty_n - Sv_n\| \\
&\le (1-\alpha_n)\{\|Tx_n - Tu_n\| + \|Tu_n - Su_n\|\} \\
&\quad + \alpha_n\{\|Ty_n - Tv_n\| + \|Tv_n - Sv_n\|\} \\
&\le (1-\alpha_n)\{\delta\|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\
&\quad + \alpha_n\{\delta\|y_n - v_n\| + L\|y_n - Ty_n\| + \varepsilon\} \\
&\le (1-\alpha_n)\{\delta\|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\
&\quad + \alpha_n\delta^2\|x_n - u_n\| + \alpha_n\delta L\|x_n - Tx_n\| \\
&\quad + \alpha_n\delta L\|z_n - Tz_n\| + 2\alpha_n\delta\varepsilon + \alpha_n L\|y_n - Ty_n\| + \alpha_n\varepsilon.
\end{aligned}$$

Since $\delta \in (0,1)$, we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le [1-\alpha_n(1-\delta)]\|x_n - u_n\| + L\|x_n - Tx_n\| \\
&\quad + \alpha_n L\|y_n - Ty_n\| + \alpha_n L\|z_n - Tz_n\| \\
&\quad + 2\alpha_n\varepsilon + \varepsilon.
\end{aligned} \tag{2.28}$$

Using assumption $(*)$, we obtain

$$1-\alpha_n \le \alpha_n.$$

Hence from (2.28), we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le [1-\alpha_n(1-\delta)]\|x_n - u_n\| + 2\alpha_n L\|x_n - Tx_n\| \\
&\quad + \alpha_n L\|y_n - Ty_n\| + \alpha_n L\|z_n - Tz_n\| + 4\alpha_n\varepsilon.
\end{aligned}$$

Denote that

$$\begin{aligned}
a_n &= \|x_n - u_n\|, \\
\mu_n &= \alpha_n(1-\delta) \in (0,1) \\
\eta_n &= \frac{\{2L\|x_n - Tx_n\| + L\|y_n - Ty_n\| + L\|z_n - Tz_n\| + 4\varepsilon\}}{(1-\delta)}.
\end{aligned}$$

It follows from Lemma 1.2 that

$$\begin{aligned}
0 &\le \lim_{n\to\infty}\sup\|x_n - u_n\| \\
&\le \lim_{n\to\infty}\sup\left\{\frac{\{2L\|x_n - Tx_n\| + L\|y_n - Ty_n\| + L\|z_n - Tz_n\| + 4\varepsilon\}}{(1-\delta)}\right\} \\
&= \frac{4\varepsilon}{(1-\delta)}.
\end{aligned}$$

We know from Theorem 2.1 that $x_n \to p_*$ and using hypotesis, we obtain

$$\|p_* - x_*\| \leq \frac{4\varepsilon}{1-\delta}.$$

□

Example 2.7. Let $C = [-1, 1]$ be endowed with usual metric. Define operator $T : C \to C$ by

$$T(x) = \begin{cases} \frac{1}{2}\sin\frac{x}{2}; & -1 \leq x < 0 \\ -\frac{1}{2}\sin\frac{x}{2}; & 0 \leq x \leq 1. \end{cases}$$

It is easy to check that T satisfies condition (1.6) with $\delta \in [\frac{1}{4}, 1)$ and hence it has a unique fixed point $p_* = 0$. Define operator $S : C \to C$ by

(2.29)

$$Sx = \begin{cases} \frac{(x-0.07)}{3.95} + \frac{(x+0.1)^3}{95.04} - \frac{(x-0.3)^5}{7581.27} - \frac{(x+0.2)^7}{130160.02}; & -1 \leq x < 0 \\ -\frac{x}{4.88} - \frac{(x-0.2)^3}{109.85} - \frac{(x+0.1)^5}{7614.18} + \frac{(x-0.5)^7}{129970.84}; & 0 \leq x \leq 1. \end{cases}$$

By utilizing Wolfram Mathematica 9 software package, we get

$$\max_{x \in C} |T - S| = 0.0217145.$$

Hence, for all $x \in C$ and for a fixed $\varepsilon = 0.0217145 > 0$, we have

$$|Tx - Sx| \leq 0.0217145.$$

Thus, S is an approximate operator of T in the sense of Definition 1.5. Moreover, from (1.5), $x_* = 0.0000603$ is a fixed point for the operator S in $C = [-1, 1]$. Hence the distance between two fixed points p_* and x_* is $|p_* - x_*| = 0.0000603$.

If $Su = -\frac{x}{4.88} - \frac{(x-0.2)^3}{109.85} - \frac{(x+0.1)^5}{7614.18} + \frac{(x-0.5)^7}{129970.84}$ and we put $\alpha_n = \frac{n+2}{n+3}$, $\beta_n = \frac{n+3}{n+4}$ and $\gamma_n = \frac{n+4}{n+5}$ for all $n \in \mathbb{N}$ in (1.4), then we obtain

(2.30)

$$\begin{cases} u_0 \in C, \\ u_{n+1} = \left(1 - \frac{n+2}{n+3}\right)\left(-\frac{u_n}{4.88} - \frac{(u_n-0.2)^3}{109.85} - \frac{(u_n+0.1)^5}{7614.18} + \frac{(u_n-0.5)^7}{129970.84}\right) \\ \qquad + \left(\frac{n+2}{n+3}\right)\left(-\frac{v_n}{4.88} - \frac{(v_n-0.2)^3}{109.85} - \frac{(v_n+0.1)^5}{7614.18} + \frac{(v_n-0.5)^7}{129970.84}\right) \\ v_n = \left(1 - \frac{n+3}{n+4}\right)\left(-\frac{u_n}{4.88} - \frac{(u_n-0.2)^3}{109.85} - \frac{(u_n+0.1)^5}{7614.18} + \frac{(u_n-0.5)^7}{129970.84}\right) \\ \qquad + \left(\frac{n+3}{n+4}\right)\left(-\frac{w_n}{4.88} - \frac{(w_n-0.2)^3}{109.85} - \frac{(w_n+0.1)^5}{7614.18} + \frac{(w_n-0.5)^7}{129970.84}\right) \\ w_n = \left(1 - \frac{n+4}{n+5}\right)u_n + \left(\frac{n+4}{n+5}\right)\left(-\frac{u_n}{4.88} - \frac{(u_n-0.2)^3}{109.85} - \frac{(u_n+0.1)^5}{7614.18} + \frac{(u_n-0.5)^7}{129970.84}\right). \end{cases}$$

The following table shows that the sequence $\{u_n\}_{n=0}^{\infty}$ generated by (2.30) converges to the fixed point $x_* = 0,0000603$.

TABLE 2. Convergence test for the iteration method (2.30)

Iter.No	*Iter.Method* (2.30)
1	0.5
2	$-0,0225150$
3	$0,0008848$
4	$0,0000348$
5	$0,0000610$
6	$0,0000603$

Then, we can find the following estimate using Theorem 2.6,

$$|p_* - x_*| \leq \frac{4 \times (0.0217145)}{1 - \frac{1}{4}} = 0.1158107.$$

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