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On Sum and Stability of Continuous G-Frames

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Abstract. In this paper, we give some conditions under which the finite sum of continuous g-frames is again a continuous g-frame. We give necessary and sufficient conditions for the continuous g-frames $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ and operators U and V on H such that $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is again a continuous g-frame. Moreover, we obtain some sufficient conditions under which the finite sum of continuous g-frames are stable under small perturbations.

1. Introduction

Frames were first introduced by Duffin and Schaeffer in the study of nonharmonic Fourier sereis [7] and reintroduced in 1986 by Daubechies, Grossmann and Meyer [6]. In [13], Sun introduced the concept of g-frames in a Hilbert space. The notion of continuous frames was introduced by Kaiser in [8] and independently by Ali, Antoine and Cazeau [2]. In 2008, continuous g-frames were introduced by Abdollahpour and Faroughi [1].

This paper is organized as follows. First, we summarize some facts about continuous g-frames from [1]. By generalizing some results of [5] and [9], in Section 2, we give some condtions that the finite sum of continous g-frames to be a continous g-frame and in Section 3, we study some new results in stability of finite sum of continuous g-frames.

Throughout this paper, H is a complex Hilbert space and (Ω, μ) is a measure space with positive measure μ and $\{K_w\}_{w\in\Omega}$ is a family of

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closed subspaces of Hilbert space K. We denote the space of all bounded linear operators from H into K by $B\left(H,K\right)$.

Definition 1.1. Let $F \in \prod_{w\in\Omega} K_w$. We say that F is strongly measurable if F as a mapping of Ω to K is measurable, where

$$\prod_{w\in\Omega} K_w = \left\{ f : \Omega \longrightarrow \bigcup_{w\in\Omega} K_w : f\left(w\right) \in K_w \right\}.$$

Definition 1.2. We say that $\Lambda = \{\Lambda_w \in B\left(H, K_w\right) : w \in \Omega\}$ is a continuous g-frame for H with respect to $\{K_w\}_{w\in\Omega}$ (or simply continuous g-frame) if

(i) for each $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable,
(ii) there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$A_\Lambda \left\|f\right\|^2 \leq \int_\Omega \left\|\Lambda_w f\right\|^2 d\mu\left(w\right) \leq B_\Lambda \left\|f\right\|^2, \quad f \in H. \tag{1.1}$$

We call A_Λ and B_Λ the lower and upper continuous g-frame bounds, respectively.
Λ is called a tight continuous g-frame if $A_\Lambda = B_\Lambda$ and a Parseval continuous g-frame if $A_\Lambda = B_\Lambda = 1$. If the right hand inequality of (1.1) holds for all $f \in H$, then we say that Λ is a continuous g-Bessel family for H with respect to $\{K_w\}_{w\in\Omega}$ (or simply continuous g-Bessel family). In this case, B_Λ is called the continuous g-Bessel constant. We denote by A_Λ and B_Λ the lower and upper bounds of continuous g-frame Λ, respectively.

Proposition 1.3 ([1])**.** *Let $\Lambda = \{\Lambda_w \in B\left(H, K_w\right) : w \in \Omega\}$ be a continuous g-frame for H with continuous g-frame bounds A_Λ, B_Λ. Then, there exists a unique positive and invertible operator $S_\Lambda : H \longrightarrow H$ such that for each $f, g \in H$,*

$$\langle S_\Lambda f, g\rangle = \int_\Omega \left\langle \Lambda_w^* \Lambda_w f, g\right\rangle d\mu\left(w\right),$$

and $A_\Lambda I_H \leq S_\Lambda \leq B_\Lambda I_H$.

The operator S_Λ in Proposition 1.3 is called the continuous g-frame operator of Λ. Also, we have

$$\begin{aligned} \langle f, g\rangle &= \int_\Omega \left\langle S_\Lambda^{-1} f \Lambda_w^* \Lambda_w g\right\rangle d\mu\left(w\right) \\ &= \int_\Omega \left\langle f, \Lambda_w^* \Lambda_w S_\Lambda^{-1} g\right\rangle d\mu\left(w\right), \quad f, g \in H. \end{aligned} \tag{1.2}$$

Let

$$\widehat{K} = \left\{ F \in \prod_{w\in\Omega} K_w : F \text{ is storngly measurable, } \int_\Omega \left\|F\left(w\right)\right\|^2 d\mu\left(w\right) < \infty \right\}.$$

It is clear that, $\widehat{K}$ is a Hilbert space with pointwise operations and the inner product given by

$$\langle F,G\rangle = \int_{\Omega} \big\langle F(w), G(w)\big\rangle d\mu(w), \quad F,G \in \widehat{K}.$$

Proposition 1.4 ([1])**.** *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family. Then the mapping $T_\Lambda : \widehat{K} \longrightarrow H$ defined by*

$$\langle T_\Lambda F, g\rangle = \int_{\Omega} \big\langle \Lambda_w^* F(w), g\big\rangle d\mu(w), \quad F \in \widehat{K}, \quad g \in H,$$

is linear and bounded with $\|T_\Lambda\| \leq \sqrt{B}_\Lambda$. Also, for each $g \in H$ we have

$$T_\Lambda^*(g)(w) = \Lambda_w g, \quad w \in \Omega.$$

The operators T_Λ and T_Λ^* in Proposition 1.4 are called the synthesis and analysis operators of Λ, respectively.

2. The Sum of Continuous G-Frames

The authours in [12] have given some conditions under which the finite sum of frames can be also frames. In [10], Madadian and Rahmani have discussed that the finite sum of continuous g-frames can be a continuous g-frame under some conditions.

In this section, we study the sum of continuous g-frames and generalize some results of [5] and [9] to continuous g-frames.

The following example shows that the sum of two continuous g-frames is not necessarily a continuous g-frame.

Example 2.1. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame. Let $\Gamma_w = -\Lambda_w$ for all $w \in \Omega$, then $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame and $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is not a continuous g-frame.

Here we give some conditions under which $\Lambda + \Gamma$ is a continuous g-frame for H.

Theorem 2.2. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family. For non-zero constants a, b, if*

$$A_\Lambda |a|^2 - 2B_\Gamma |b|^2 > 0,$$

then $a\Lambda + b\Gamma = \{a\Lambda_w + b\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame.

Proof. By the Cauchy-Schwarz inequality, for any $f \in H$, we have

$$\int_{\Omega} \|(a\Lambda_w + b\Gamma_w) f\|^2 \, d\mu(w)$$

$$
\begin{aligned}
&= \int_\Omega \Big\langle a\Lambda_w f + b\Gamma_w f, a\Lambda_w f + b\Gamma_w f\Big\rangle d\mu\,(w) \\
&= \int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) + \int_\Omega 2Re\big\langle a\Lambda_w f, b\Gamma_w f\big\rangle d\mu\,(w) \\
&\quad + \int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w) \\
&\le \int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) + 2\int_\Omega \big|\big\langle a\Lambda_w f, b\Gamma_w f\big\rangle\big| d\mu\,(w) \\
&\quad + \int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w) \\
&\le \int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) + 2\int_\Omega \|a\Lambda_w f\|\,\|b\Gamma_w f\| \,d\mu\,(w) \\
&\quad + \int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w) \\
&\le \int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) + 2\left(\int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w)\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w)\right)^{\frac{1}{2}} + \int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w) \\
&= \left[\left(\int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w)\right)^{\frac{1}{2}} + \left(\int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w)\right)^{\frac{1}{2}}\right]^2 \\
&\le 2\int_\Omega |a|^2 \,\|\Lambda_w f\|^2 \,d\mu\,(w) + 2\int_\Omega |b|^2 \,\|\Gamma_w f\|^2 \,d\mu\,(w) \\
&= 2|a|^2 \int_\Omega \|\Lambda_w f\|^2 \,d\mu\,(w) + 2|b|^2 \int_\Omega \|\Gamma_w f\|^2 \,d\mu\,(w) \\
&\le 2\left(|a|^2 B_\Lambda + |b|^2 B_\Gamma\right) \|f\|^2 .
\end{aligned}
$$

On the other hand, for each $f \in H$,

$$
\begin{aligned}
&\int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) \\
&\quad = \int_\Omega \|(a\Lambda_w + b\Gamma_w)\, f - b\Gamma_w f\|^2 \,d\mu\,(w) \\
&\quad \le 2\int_\Omega \|a\Lambda_w f + b\Gamma_w f\|^2 \,d\mu\,(w) + 2\int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w) ,
\end{aligned}
$$

so

$$
\begin{aligned}
&2\int_\Omega \|a\Lambda_w f + b\Gamma_w f\|^2 \,d\mu\,(w) \\
&\qquad \ge \int_\Omega \|a\Lambda_w f\|^2 \,d\mu\,(w) - 2\int_\Omega \|b\Gamma_w f\|^2 \,d\mu\,(w)
\end{aligned}
$$

$$= \int_\Omega |a|^2 \|\Lambda_w f\|^2 d\mu(w) - 2\int_\Omega |b|^2 \|\Gamma_w f\|^2 d\mu(w)$$
$$= |a|^2 \int_\Omega \|\Lambda_w f\|^2 d\mu(w) - 2|b|^2 \int_\Omega \|\Gamma_w f\|^2 d\mu(w)$$
$$\geq \left(|a|^2 A_\Lambda - 2|b|^2 B_\Gamma\right) \|f\|^2, \quad f \in H.$$

□

Corollary 2.3. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family. If $B_\Gamma < \frac{A_\Lambda}{2}$, then $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame.*

Proof. It is sufficient to put $a = b = 1$, in Theorem 2.2. □

Theorem 2.4. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g-frames. Let $U, V \in B(H)$. If $T_\Lambda T_\Gamma^* = 0$ and U or V is a self-adjoint surjective operator, then $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame.*

Proof. Since $T_\Lambda T_\Gamma^* = 0$, for any $f \in H$, we have

$$\int_\Omega \|(\Lambda_w U + \Gamma_w V) f\|^2 d\mu(w)$$
$$= \int_\Omega \|\Lambda_w U f\|^2 d\mu(w) + \int_\Omega \left\langle \Gamma_w^* \Lambda_w U f, V f\right\rangle d\mu(w)$$
$$+ \int_\Omega \left\langle \Lambda_w^* \Gamma_w V f, U f\right\rangle d\mu(w) + \int_\Omega \|\Gamma_w V f\|^2 d\mu(w)$$
$$= \int_\Omega \|\Lambda_w U f\|^2 d\mu(w) + \left\langle T_\Gamma T_\Lambda^* U f, V f\right\rangle$$
$$+ \left\langle T_\Lambda T_\Gamma^* V f, U f\right\rangle + \int_\Omega \|\Gamma_w V f\|^2 d\mu(w)$$
$$\leq B_\Lambda \|U f\|^2 + B_\Gamma \|V f\|^2$$
$$\leq \left(B_\Lambda \|U\|^2 + B_\Gamma \|V\|^2\right) \|f\|^2, \quad f \in H.$$

Now, suppose that U is a self-adjoint surjective operator. By Lemma 2.4.1 of [3], there exists a constant $C > 0$ such that

$$\|U f\|^2 \geq C \|f\|^2, \quad f \in H.$$

Then

$$\int_\Omega \|(\Lambda_w U + \Gamma_w V) f\|^2 d\mu(w)$$
$$= \int_\Omega \|\Lambda_w U f\|^2 d\mu(w) + \int_\Omega \|\Gamma_w V f\|^2 d\mu(w)$$

$$\begin{aligned}
&\geq \int_{\Omega} \|\Lambda_w U f\|^2 \, d\mu(w) \\
&\geq A_{\Lambda} \|Uf\|^2 \\
&\geq A_{\Lambda} C \|f\|^2, \quad f \in H.
\end{aligned}$$

Therefore, $\Lambda U + \Gamma V = \{\Lambda_w U + \Gamma_w V \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame. □

Corollary 2.5. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g-frames. If $T_{\Lambda} T_{\Gamma}^* = 0$, then $\Lambda + \Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame.*

Proof. It is sufficient to put $U = V = I_H$, in Theorem 2.4. □

Corollary 2.6. *Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ and $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ be two continuous g-frames. If $T_{\Lambda} T_{\Gamma}^* = 0$ and $U \in B(H)$, then $\Lambda U + \Gamma = \{\Lambda_w U + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-frame.*

Proof. It is sufficient to put $V = I_H$, in Theorem 2.4. □

Theorem 2.7. *Let $\Lambda^i = \left\{\Lambda_w^i \in B(H, K_w) : w \in \Omega\right\}$ be a continuous g-frame for $i \in I = \{1, 2, \ldots, M\}$. Let $\{\alpha_i\}_{i \in I}$ be a sequence of scalars. Then*

$$\sum_{i \in I} \alpha_i \Lambda^i = \left\{ \sum_{i \in I} \alpha_i \Lambda_w^i \in B(H, K_w) : w \in \Omega \right\},$$

is a continuous g-frame if and only if there exist $\beta > 0$ and some $j \in I$ such that

$$\beta \int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) \leq \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w), \quad f \in H.$$

Proof. First, suppose that $\sum_{i \in I} \alpha_i \Lambda^i$ is a continuous g-frame for H. Then

(2.1)

$$A_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2 \leq \int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) \leq B_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2, \quad f \in H.$$

Since for $j \in I$, Λ^j is a continuous g-frame for H, we have

$$A_{\Lambda^j} \|f\|^2 \leq \int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) \leq B_{\Lambda^j} \|f\|^2, \quad f \in H, \ j \in I. \tag{2.2}$$

By the inequalities (2.1) and (2.2), for any $f \in H$, we have

$$\int_{\Omega} \left\| \sum_{i \in I} \alpha_i \Lambda_w^i f \right\|^2 d\mu(w) \geq A_{\sum_{i \in I} \alpha_i \Lambda^i} \|f\|^2$$

$$\geq \frac{A_{\sum_{i\in I}\alpha_i\Lambda^i}}{B_{\Lambda^j}}\int_{\Omega}\left\|\Lambda_w^j f\right\|^2 d\mu\left(w\right),$$

thus, it is sufficient to put $\beta = \frac{A_{\sum_{i\in I}\alpha_i\Lambda^i}}{B_{\Lambda^j}}$.

Conversely, we suppose that there exists $\beta > 0$ such that

$$\beta\int_{\Omega}\left\|\Lambda_w^j f\right\|^2 d\mu\left(w\right) \leq \int_{\Omega}\left\|\sum_{i\in I}\alpha_i\Lambda_w^i f\right\|^2 d\mu\left(w\right), \quad f\in H,$$

for some $j \in I$. Thus

$$\begin{aligned}\int_{\Omega}\left\|\sum_{i\in I}\alpha_i\Lambda_w^i f\right\|^2 d\mu\left(w\right) &\geq \beta\int_{\Omega}\left\|\Lambda_w^j f\right\|^2 d\mu\left(w\right)\\ &\geq \beta A_{\Lambda^j}\left\|f\right\|^2, \quad f\in H.\end{aligned}$$

On the other hand, by the Cauchy- Schwarz inequality, we have

$$\begin{aligned}\int_{\Omega}\left\|\sum_{i\in I}\alpha_i\Lambda_w^i f\right\|^2 d\mu\left(w\right) &\leq \int_{\Omega}\left(\sum_{i\in I}\left\|\alpha_i\Lambda_w^i f\right\|\right)^2 d\mu\left(w\right)\\ &\leq \int_{\Omega} M\sum_{i\in I}\left|\alpha_i\right|^2\left\|\Lambda_w^i f\right\|^2 d\mu\left(w\right)\\ &\leq \int_{\Omega} M^2\left(\max_{i\in I}\left|\alpha_i\right|^2\right)\sum_{i\in I}\left\|\Lambda_w^i f\right\|^2 d\mu\left(w\right)\\ &= M^2\left(\max_{i\in I}\left|\alpha_i\right|^2\right)\sum_{i\in I}\int_{\Omega}\left\|\Lambda_w^i f\right\|^2 d\mu\left(w\right)\\ &\leq M^2\left(\max_{i\in I}\left|\alpha_i\right|^2\right)\sum_{i\in I}B_{\Lambda^i}\left\|f\right\|^2\\ &\leq M^3\left(\max_{i\in I}\left|\alpha_i\right|^2\right)\left(\max_{i\in I}B_{\Lambda^i}\right)\left\|f\right\|^2, \quad f\in H.\end{aligned}$$

So, $\sum_{i\in I}\alpha_i\Lambda^i$ is a continuous g-frame for H. □

3. The Stability of Continuous G-Frames

In [4], Christensen has discussed the stability of frames in the Hilbert spaces under perturbations. Also, Sun has proved that g-frames are stable under small perturbations [14]. The perturbation result was generalized to continous g-frames in [1]. In this section, we study the stability of continuous g-frames.

Theorem 3.1. *Suppose that* $\Gamma = \{\Gamma_w \in B\left(H, K_w\right) : w\in\Omega\}$ *is a family of operators such that for each* $f\in H$, $\{\Gamma_w f\}_{w\in\Omega}$ *is strongly measurable.*

Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and $\Lambda+\Gamma = \{\Lambda_w + \Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a continuous g-Bessel family. If $A_\Lambda - 2B_{\Lambda+\Gamma} > 0$, then Γ is a continuous g-frame for H.

Proof. We have

$$\begin{aligned}\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) &= \int_\Omega \|(\Gamma_w + \Lambda_w) f - \Lambda_w f\|^2 \, d\mu(w) \\ &\leq 2\left(\int_\Omega \|(\Gamma_w + \Lambda_w) f\|^2 \, d\mu(w) + \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w)\right) \\ &\leq 2(B_{\Lambda+\Gamma} + B_\Lambda) \|f\|^2, \quad f \in H.\end{aligned}$$

Also,

$$\begin{aligned}2\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) &\geq \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) - 2\int_\Omega \|(\Lambda_w + \Gamma_w) f\|^2 \, d\mu(w) \\ &\geq A_\Lambda \|f\|^2 - 2B_{\Lambda+\Gamma} \|f\|^2 \\ &= (A_\Lambda - 2B_{\Lambda+\Gamma}) \|f\|^2, \quad f \in H.\end{aligned}$$

So, Γ is a continuous g-frame for H. □

Theorem 3.2. *Suppose that $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Gamma_w f\}_{w\in\Omega}$ is strongly measurable. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame. Then Γ is a continuous g-Bessel family for H if and only if there exists a constant $\lambda > 0$ such that*

$$\int_\Omega \|(\Lambda_w - \Gamma_w) f\|^2 \, d\mu(w) \leq \lambda \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w), \quad f \in H.$$

Proof. First, suppose that Γ is a continuous g-Bessel family for H. Since Λ is a continuous g-frame for H, we have

$$\|f\|^2 \leq \frac{1}{A_\Lambda} \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w), \quad f \in H,$$

thus

$$\begin{aligned}\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) &\leq B_\Gamma \|f\|^2 \\ &\leq \frac{B_\Gamma}{A_\Lambda} \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w), \quad f \in H.\end{aligned}$$

So

$$\begin{aligned}\int_\Omega \|(\Lambda_w - \Gamma_w) f\|^2 \, d\mu(w) &\leq 2\int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) + 2\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) \\ &\leq 2\left(1 + \frac{B_\Gamma}{A_\Lambda}\right) \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w), \quad f \in H.\end{aligned}$$

Conversely, for any $f \in H$, we have

$$\begin{aligned}\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) &= \int_\Omega \|(\Gamma_w - \Lambda_w) f + \Lambda_w f\|^2 \, d\mu(w) \\ &\leq 2\int_\Omega \|(\Gamma_w - \Lambda_w) f\|^2 \, d\mu(w) + 2\int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) \\ &\leq 2\left(\lambda \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) + \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w)\right) \\ &\leq 2(\lambda+1) B_\Lambda \|f\|^2,\end{aligned}$$

therefore Γ is a continuous g-Bessel family for H. □

Theorem 3.3. *Suppose that $\Gamma = \{\Gamma_w \in B(H, K_w) : w \in \Omega\}$ is a family of operators such that for each $f \in H$, $\{\Gamma_w f\}_{w\in\Omega}$ is strongly measurable. Let $\Lambda = \{\Lambda_w \in B(H, K_w) : w \in \Omega\}$ be a continuous g-frame and a and b be non-zero constants. Suppose that there exist constants $0 \leq \lambda, \mu < \frac{1}{2}$ such that for any $f \in H$,*

$$\begin{aligned}&\int_\Omega \|(a\Lambda_w - b\Gamma_w) f\|^2 \, d\mu(w) \\ &\quad \leq \lambda \int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) + \mu \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w).\end{aligned}$$

Then Γ is a continuous g-frame for H.

Proof. For any $f \in H$,

$$\begin{aligned}&\int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w) \\ &= \int_\Omega \|(b\Gamma_w - a\Lambda_w) f + a\Lambda_w f\|^2 \, d\mu(w) \\ &\leq 2\int_\Omega \|(b\Gamma_w - a\Lambda_w) f\|^2 \, d\mu(w) + 2\int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) \\ &\leq 2\lambda \int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) + 2\mu \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w) \\ &\quad + 2\int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) \\ &= 2(\lambda+1) \int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) + 2\mu \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w).\end{aligned}$$

So

$$(1-2\mu) \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w) \leq 2(\lambda+1) \int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w), \quad f \in H,$$

therefore, for any $f \in H$,

$$(1-2\mu)|b|^2 \int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) \leq 2(\lambda+1)|a|^2 \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w).$$

Thus

$$\begin{aligned}\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) &\leq \frac{2(\lambda+1)|a|^2}{(1-2\mu)|b|^2} \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) \\ &\leq \frac{2(\lambda+1)|a|^2}{(1-2\mu)|b|^2} B_\Lambda \|f\|^2, \quad f \in H.\end{aligned}$$

On the other hand, for any $f \in H$, we have

$$\begin{aligned}\int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) &= \int_\Omega \|(a\Lambda_w - b\Gamma_w) f + b\Gamma_w f\|^2 \, d\mu(w) \\ &\leq 2\lambda \int_\Omega \|a\Lambda_w f\|^2 \, d\mu(w) + 2\mu \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w) \\ &\quad + 2 \int_\Omega \|b\Gamma_w f\|^2 \, d\mu(w).\end{aligned}$$

Also, for each $f \in H$, we have

$$(1-2\lambda)|a|^2 \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) \leq 2(1+\mu)|b|^2 \int_\Omega \|\Gamma_w f\|^2 \, d\mu(w).$$

Thus

$$\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) \geq \frac{(1-2\lambda)|a|^2}{2(1+\mu)|b|^2} A_\Lambda \|f\|^2, \quad f \in H,$$

therefore, Γ is a continuous g-frame for H. □

Theorem 3.4. *For $i \in I = \{1, 2, \ldots, M\}$, let $\Lambda^i = \{\Lambda_w^i \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family. Let $\Gamma^i = \{\Gamma_w^i \in B(H, K_w) : w \in \Omega\}$ be a continuous g-Bessel family such that*

$$\int_\Omega \left\| \left(\Lambda_w^i - \Gamma_w^i\right) f \right\|^2 d\mu(w) \leq \lambda \int_\Omega \left\| \Lambda_w^i f \right\|^2 d\mu(w), \quad f \in H,$$

for $i \in I$ and $\lambda \geq 0$. If for some $j \in I$, there exists $A_{\Lambda^j} > 0$ such that

$$\int_\Omega \left\| \Lambda_w^j f \right\|^2 d\mu(w) \geq A_{\Lambda^j} \|f\|^2, \quad f \in H,$$

and

$$2(M-1) \sum_{i \neq j} \|T_{\Lambda^i}\|^2 + 4M\lambda \sum_{i \in I} \|T_{\Lambda^i}\|^2 < A_{\Lambda^j},$$

then,

$$\sum_{i \in I} \Gamma^i = \left\{ \sum_{i \in I} \Gamma_w^i \in B(H, K_w) : w \in \Omega \right\}$$

is a continuous g-frame.

Proof. For each $f \in H$,

$$\int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) = \int_{\Omega} \left\|\sum_{i\in I} \Lambda_w^i f - \sum_{i\neq j} \Lambda_w^i f\right\|^2 d\mu(w)$$
$$\leq 2\int_{\Omega} \left\|\sum_{i\in I} \Lambda_w^i f\right\|^2 d\mu(w) + 2\int_{\Omega} \left\|\sum_{i\neq j} \Lambda_w^i f\right\|^2 d\mu(w),$$

thus for any $f \in H$,

(3.1)
$$\int_{\Omega} \left\|\sum_{i\in I} \Lambda_w^i f\right\|^2 d\mu(w) \geq \frac{1}{2}\int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) - \int_{\Omega} \left\|\sum_{i\neq j} \Lambda_w^i f\right\|^2 d\mu(w).$$

Also,

$$\int_{\Omega} \left\|\Lambda_w^i f\right\|^2 d\mu(w) = \left\|(T_{\Lambda^i})^* f\right\|^2 \tag{3.2}$$
$$\leq \left\|(T_{\Lambda^i})^*\right\|^2 \|f\|^2$$
$$= \|T_{\Lambda^i}\|^2 \|f\|^2, \quad f \in H.$$

Then, by the inequalities (3.1) and (3.2), for any $f \in H$, we have

$$\int_{\Omega} \left\|\sum_{i\in I} \Gamma_w^i f\right\|^2 d\mu(w)$$
$$\geq \frac{1}{2}\left(\int_{\Omega} \left\|\sum_{i\in I} \Lambda_w^i f\right\|^2 d\mu(w) - 2\int_{\Omega} \left\|\sum_{i\in I} \left(\Gamma_w^i - \Lambda_w^i\right) f\right\|^2 d\mu(w)\right)$$
$$\geq \frac{1}{2}\left(\frac{1}{2}\int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) - \int_{\Omega} \left\|\sum_{i\neq j} \Lambda_w^i f\right\|^2 d\mu(w)\right.$$
$$\left. -2\int_{\Omega} \left\|\sum_{i\in I} \left(\Gamma_w^i - \Lambda_w^i\right) f\right\|^2 d\mu(w)\right)$$
$$\geq \frac{1}{2}\left(\frac{1}{2}\int_{\Omega} \left\|\Lambda_w^j f\right\|^2 d\mu(w) - (M-1)\sum_{i\neq j}\int_{\Omega} \left\|\Lambda_w^i f\right\|^2 d\mu(w)\right.$$
$$\left. -2M\sum_{i\in I}\int_{\Omega} \left\|\left(\Gamma_w^i - \Lambda_w^i\right) f\right\|^2 d\mu(w)\right)$$

$$\geq \frac{1}{2}\left(\frac{1}{2}A_{\Lambda^j}\|f\|^2-(M-1)\sum_{i\neq j}\|T_{\Lambda^i}\|^2\|f\|^2-2M\lambda\sum_{i\in I}\|T_{\Lambda^i}\|^2\|f\|^2\right)$$
$$=\frac{1}{4}\left(A_{\Lambda^j}-2(M-1)\sum_{i\neq j}\|T_{\Lambda^i}\|^2-4M\lambda\sum_{i\in I}\|T_{\Lambda^i}\|^2\right)\|f\|^2.$$

For any $i\in I$ and $f\in H$, we have

$$\begin{aligned}
&\int_\Omega\left\|\sum_{i\in I}\Gamma_w^i f\right\|^2 d\mu(w)\\
&=\int_\Omega\left\|\sum_{i\in I}\left(\Gamma_w^i-\Lambda_w^i\right)f+\sum_{i\in I}\Lambda_w^i f\right\|^2 d\mu(w)\\
&\leq 2\left(\int_\Omega\left\|\sum_{i\in I}\left(\Gamma_w^i-\Lambda_w^i\right)f\right\|^2 d\mu(w)+\int_\Omega\left\|\sum_{i\in I}\Lambda_w^i f\right\|^2 d\mu(w)\right)\\
&\leq 2M\left(\int_\Omega\sum_{i\in I}\left\|\left(\Gamma_w^i-\Lambda_w^i\right)f\right\|^2 d\mu(w)+\int_\Omega\sum_{i\in I}\left\|\Lambda_w^i f\right\|^2 d\mu(w)\right)\\
&\leq 2M\left(\lambda\int_\Omega\sum_{i\in I}\left\|\Lambda_w^i f\right\|^2 d\mu(w)+\int_\Omega\sum_{i\in I}\left\|\Lambda_w^i f\right\|^2 d\mu(w)\right)\\
&=2M(1+\lambda)\sum_{i\in I}\int_\Omega\left\|\Lambda_w^i f\right\|^2 d\mu(w)\\
&\leq 2M(1+\lambda)\sum_{i\in I}B_{\Lambda^i}\|f\|^2\\
&\leq 2M^2(1+\lambda)\max_{i\in I}B_{\Lambda^i}\|f\|^2.
\end{aligned}$$

Therefore,

$$\sum_{i\in I}\Gamma^i=\left\{\sum_{i\in I}\Gamma_w^i\in B(H,K_w):w\in\Omega\right\},$$

is a continuous g-frame. □

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References

1. M.R. Abdollahpour and M.H. Faroughi, *Continuous g-frames in Hilbert spaces*, Southeast Asian Bull. Math., 32 (2008), pp. 1-19.

2. S.T. Ali, J.P. Antoine and J.P. Cazeau, *Continuous frames in Hilbert spaces*, Ann. Physics, 222 (1993), pp. 1-37.
3. O. Christensen, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, 2016.
4. O. Christensen, *Frame perturbations*, Proc. Amer. Math. Soc., 123 (1995), pp. 1217-1220.
5. R. Chugh and S. Goel, *On finite sum of g-frames and near exact g-frames*, Electron. J. Math. Anal. Appl., 2 (2014), pp. 73-80.
6. I. Daubechies, A. Grossmann and Y. Mayer, *Painless nonorthogonal expansions*, J. Math. Phys., 27 (1986), pp. 1271-128.
7. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341-366.
8. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser Boston, MA, 1995.
9. D. Li, J. Leng, T. Huang and G. Sun, *On sum and stability of g-frames in Hilbert spaces*, Linear Multilinear Algebra, 66 (2018), pp. 1578-1592.
10. M. Madadian and M. Rahmani, *G-frame sequence oprtators, cg-Riesz bases and sum of cg-frames*, Int. Math. Forum, 68 (2011), 3357-3369.
11. G.J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, San Diego, California, 1990.
12. S. Obeidat, S. Samarah, P.G. Casazza and J.C. Tremain, *Sums of Hilbert space frames*, J. Math. Anal. Appl., 351 (2009), pp. 579-585.
13. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl., 322 (2006), pp. 437-452.
14. W. Sun, *Stability of g-frames*, J. Math. Anal. Appl., 326 (2007), pp. 858-868.

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