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A Common Fixed Point Theorem Using an Iterative Method

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Abstract. Let H be a Hilbert space and C be a closed, convex and nonempty subset of H. Let $T : C \to H$ be a non-self and non-expansive mapping. V. Colao and G. Marino with particular choice of the sequence $\{\alpha_n\}$ in Krasonselskii-Mann algorithm, $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n)$, proved both weak and strong converging results. In this paper, we generalize their algorithm and result, imposing some conditions upon the set C and finite many mappings from C in to H, to obtain a converging sequence to a common fixed point for these non-self and non-expansive mappings.

1. Introduction and Preliminaries

In the last decades an iterative scheme defined as follows which has been studied in very much papers such as [5] and the references therein and it is often called 'segmenting Mann' [7, 9, 11] or 'Krasnoselskii-Mann' (e.g., [6, 10]) iteration as follows:

Let C be a closed, convex and nonempty subset of a Hilbert space H and let $T : C \to H$ be a non-expansive mapping with nonempty fixed point set. For a real sequence $\{\alpha_n\} \subseteq (0, 1)$ the following iteration is called Krasnoselskii-Mann itertion a

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n) \tag{1.1}$$

If T is selfmapping and $\Sigma_{i=1}^{\infty}\alpha_i(1-\alpha_i) = \infty$, a general result about weakly convergence of $\{x_n\}$ is proved in Reich [13]. When T is non-selfmapping, to guarantee the existence of a fixed point of T, often

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impose some kind of boundary conditions upon the set C and the mapping T. The inward condition and its generalization is some of these conditions, which were studied by many authors. (see [2, 12, 15, 16]).

Moreover, the common fixed point theorems are interesting and attractive subject to investigate. Wang [14] proved a common fixed point theorem for two asymptotically nonexpansive non-self mappings in uniformly convex Banach spaces. Later J. Ayaragarnchanakul presented a common fixed point iterative process with errors for quasi-nonexpansive non-selfmappings in arbitrary real Banach spaces and proved some strong convergence theorems for such iterative process [1].

Colao and Marino using Krasonselskii-Mann method with particular choice of the sequence $\{\alpha_n\}$ based on the values of the map T and geometry of the set C, proved both weak and strong convergence results [4]. They presented some open question as the conclusion of the paper, which the second one is about commoon fixed point for a countable family of mappings and were answered by Gua et al. [8]. In this paper, we want to generalize their result in some direction, which is completely differant from the mentioned open question and prove a common fixed point theorem for many finite mappings.

2. Main Result

We state some elementary definitions and lemmas which have essential roles in our main result.

Definition 2.1. A mapping $T : C \to H$ is said to be inward (or to satisfy the inward condition) if, for any $x \in C$, it holds

$$Tx \in I_{C(x)} := \{x + c(u - x) : c \geq 1 \text{ and } u \in C\}. \tag{2.1}$$

The properties of the inward mappings are explained in [4].

Definition 2.2. A sequence $\{y_n\} \in C$ is called Fejér-monotone with respect to a set $D \subseteq C$ if for every $y \in D$, $\| y_{n+1} - y \| \leq \| y_n - y \|$ for all $n \in \mathbb{N}$.

Lemma 2.3 ([3], Lemma 7)**.** *Let X be a strictly convex Banach space and C convex subset of X . If $T : C \to X$ is a nonexpansive mapping, then the fixed point set of T in C is convex.*

Lemma 2.4 ([13], Lemma 6)**.** *Let X be a uniformly convex Banach space, $\{x_n\}, \{y_n\} \subseteq X$ be two sequences; if there exists a constant $d \geq 0$ such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d,$$
$$\limsup_{n\to\infty} \|y_n\| \leq d,$$

$$\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$, *where* $t_n \in [a,b] \subseteq (0,1)$ *and* a, b *are two constants.*

Now, we are ready to define a function which will use it in sequel.

Definition 2.5. For a closed and convex set $C \in H$ and mappings $T_i : C \to H, i = 1, 2, \ldots, m$, we define a mapping $h : C \to \mathbb{R}$, as

(2.2)

$$\begin{aligned} h(x_0) := \inf \{\lambda \geq 0 :\ & y^1 = \lambda y^0 + (1-\lambda)T_m y^0 \in C, \\ & y^2 = \lambda y^1 + (1-\lambda)T_{m-1} y^1 \in C, \ldots, \\ & y^m = \lambda y^{m-1} + (1-\lambda)T_1 y^{m-1} \in C, y^0 = x_0\}, \end{aligned}$$

for every $x_0 \in C$.

Since C is closed, the above quantity is a minimum. For $\lambda = 1$, we have $x_0 = y^0 = y^1 = y^2 = \cdots = y^m \in C$, so the above set is not empty and $h(x_0)$ is well-defined. The main properties of the mapping h are stated in the following Lemma. We give some notation for the rest. We set $S_1 = T_m,\ S_2 = T_{m-1}T_m,\ S_3 = T_{m-2}T_{m-1}T_m, \ldots,\ S_m = T_1T_2 \ldots T_m$. Also

$$y_i = h(x_0)y_{i-1} + (1 - h(x_0))T_{m-i+1}y_{i-1} \tag{2.3}$$

for every $1 \leq i \leq m$, whereas $y_0 = y^0 = x_0$. By definition of $h, y_i \in C$, for every $0 \leq i \leq m$.

Lemma 2.6. *Let C be a nonempty, closed and convex subset of H and $T_i : C \to H, i = 1, 2, \ldots, m$, be mappings and $h : C \to \mathbb{R}$ is defined as in (2.2). Then the following properties hold:*

P(1) *for any $x_0 \in C$, $h(x_0) \in [0,1]$ and $h(x_0) = 0$ if and only if $S_i(x_0) = 0, 1 \leq i \leq m$;*

P(2) *for any $x_0 \in C$ and $\alpha \in [h(x_0), 1]$, $z^i = \alpha y_{i-1} + (1-\alpha) T_{m-i+1} y_{i-1} \in C, 1 \leq i \leq m$;*

P(3) *if $T_i, 1 \leq i \leq m$, be inward mappings, then $h(x_0) < 1$, for any $x_0 \in C$;*

P(4) *whenever $S_i(x_0) \neq 0, 1 \leq i \leq m$, then $y_i \in \partial C$.*

Proof. P(1) is clear. P(2) holds, since $y_{i-1} = 1y_{i-1} + (1-1)T_{m-i+1}y_{i-1}$ and $y_i = h(x_0)y_{i-1} + (1-h(x_0))\,T_{m-i+1}y_{i-1}$ belong to C for every $1 \leq i \leq m$, by convexity of C. To prove P(3), suppose that T_is are inward and $x_0 \in C$ is given. Since T_m is inward, so

$$T_m(x_0) \in I_{C(x)} := \{x_0 + c(u - x_0) : c \geq 1 \text{ and } u \in C\},$$

hence there exist a real number $c_m \geq 1$, such that $\left(1 - \dfrac{1}{c_m}\right) x_0 + \frac{1}{c_m} T_m x_0 \in C$. We set $\alpha_m = 1 - \dfrac{1}{c_m} < 1$ and hence $x_1 = \alpha_m x_0 + (1 - \alpha_m) T_m x_0 \in C$. Repeating this process with x_1 and T_{m-1} and using P(2) yeilds an $\alpha_{m-1} \leq \alpha_m < 1$ and $x_2 = \alpha_{m-1} x_1 + (1 - \alpha_{m-1}) T_{m-1} x_1 \in C$ and so on. So we have $\alpha_1 \leq \cdots \leq \alpha_{m-1} \leq \alpha_m < 1$. It is clear that $h(x_0) \leq \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\} < 1$ by definition (2.2) and using P(2).

To prove P(4), suppose that $S_i(x_0) \neq 0, 1 \leq i \leq m$. It is clear that $h(x_0) < 1$ by P(1) and assumption. Let $\{t_n\} \subseteq (0, h(x_0))$ be a sequence of real numbers converging to $h(x_0)$ and note that, by the definition of h;

$$\begin{cases} y_1^n := t_n x_0 + (1 - t_n) T_m x_0 \notin C, \\ y_2^n := t_n y_1^n + (1 - t_n) T_{m-1} y_1^n \notin C, \\ \vdots \\ y_m^n := t_n y_{m-1}^n + (1 - t_n) T_1 y_{m-1}^n \notin C, \end{cases}$$

for any $n \in \mathbb{N}$. Since $t_n \to h(x_0)$ and

$$\|y_1^n - h(x_0)x_0 - (1 - h(x_0))T_m x_0\| = (t_n - h(x_0))\|x_0 - T_1 x_0\| \to 0 \text{ as } n \to \infty,$$

$y_1^n \to y_1$, also since $y_1 \in C$ and $y_1^n \notin C$, $y_1^n \in \partial C$. Similarly, we have

$$\|y_2^n - y_2\| = \|y_1^n + (1 - t_n)T_{m-1}y_1^n - h(x_0)y_1 - (1 - h(x_0))T_{m-1}y_1\|.$$

Since $t_n \to h(x_0)$ and $y_1^n \to y_1$, $y_2^n \to y_2$, moreover $y_2 \in C$ and $y_2^n \notin C$ hence, $y_2^n \in \partial C$. The rest of the proof is similar. □

Our main result is the following.

Theorem 2.7. *Let C be a nonempty, closed and convex subset of H and $T_i : C \to H, i = 1, 2, \ldots, m$, be non-self mappings and $h : C \to \mathbb{R}$ is defined as in (2.2). Then the algorithm*

$$\begin{cases} x_0 \in C, \\ \alpha_0 = \max\{\frac{1}{2}, h(x_0)\}, \\ y_1^n := \alpha_n x_n + (1 - \alpha_n) T_m x_n, \\ y_2^n := \alpha_n y_1^n + (1 - \alpha_n) T_{m-1} y_1^n, \\ \vdots \\ y_m^n := \alpha_n y_{m-1}^n + (1 - \alpha_n) T_1 y_{m-1}^n, \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\} \\ x_{n+1} := y_m^n \end{cases}$$

is well defined.

If C is strictly convex, $T_i, i = 1, 2, \ldots, m$, are nonexpansive and $F = \cap_{i=1}^{m} Fix(T_i) \neq \emptyset$, then $\{x_n\}$ weakly converges to a point of F. Moreover, if each T_i satisfies the inward condition and $\Sigma_{i=1}^{\infty}(1-\alpha_i) < \infty$ then the convergence is strong.

Proof. We put for the rest $x_{n+1} = y_m^n := y_0^{n+1}$ and we have $y_i^n := \alpha_n y_{i-1}^n + (1-\alpha_n) T_{m-i+1} y_{i-1}^n$, for each $1 \leq i \leq m$ and $n \in \mathbb{N}$. By induction and recalling property P(2), $y_i^n \in C$, for each $n \in \mathbb{N}$ and $1 \leq i \leq m$ and so is $x_{n+1} = y_m^n$. We show that the sequences $\{y_i^n\}, 1 \leq i \leq m$, are Fejér-monotone with respect to F, especially $\{x_n\}$. Fix any $p \in F$. Since T_m is nonexpansive, we have:

$$\|y_1^n - p\| = \|\alpha_n x_n + (1-\alpha_n) T_m x_n - p\| \leq \|x_n - p\|;$$

similarly, since T_{m-1} is nonexpansive, we get

$$\|y_2^n - p\| = \|\alpha_n y_1^n + (1-\alpha_n) T_{m-1} y_1^n - p\| \leq \|y_1^n - p\|.$$

Repeating this process we have:

(2.4)
$$\|x_{n+1} - p\| = \|y_m^n - p\| \leq \|y_{m-1}^n - p\| \leq \cdots \leq |y_1^n - p\| \leq \|x_n - p\|.$$

So all of the sequences $\{y_i^n\}, 1 \leq i \leq m$, are-monotone with respect to F. Since the sequences $\|\{y_i^n - p\}\|$ are decreasing and bounded, they have same limit by (2.4). We put $\lim \|y_i^n - p\| = d, 1 \leq i \leq m$, and get

$$\limsup \|T_{m-i} y_i^n - p\| \leq \limsup \|y_i^n - p\| = d.$$

Moreover,

$$\begin{aligned}\|\alpha_n(y_i^n - p) + (1-\alpha_n)(T_{m-i} y_i^n - p)\| &= \|\alpha_n y_i^n + (1-\alpha_n)(T_{m-i} y_i^n - p)\| \\ &= \|y_{i+1}^n - p\| \to d, \text{ as } n \to \infty,\end{aligned}$$

for each $1 \leq i \leq m$. Using Lemma 2.4, we have

$$\|T_{m-i} y_i^n - y_i^n\| \to 0, \textit{as } n \to \infty, 1 \leq i \leq m, \tag{2.5}$$

especially we get $\|T_m x_n - x_n\| \to 0, \textit{as } n \to \infty$. Now we show that $\|y_i^n - y_j^n\| \to 0, \textit{as } n \to \infty$. Without loss of generality, let $i > j$. We have

$$\begin{aligned}\|y_i^n - y_j^n\| &\leq \|y_i^n - y_{i-1}^n\| + \cdots + \|y_{j+1}^n - y_j^n\| \\ &= \|\alpha_n y_{i-1}^n + (1-\alpha_n) T_{m-i+1} y_{i-1}^n - y_{i-1}^n\| + \cdots \\ &\quad + \|\alpha_n y_j^n + (1-\alpha_n) T_{m-j} y_j^n - y_j^n\| \\ &= (1-\alpha_n)(\|T_{m-i+1} y_{i-1}^n - y_{i-1}^n\| + \cdots + \|T_{m-j} y_j^n - y_j^n\|);\end{aligned}$$

so,

(2.6)
$$\|y_i^n - y_j^n\| \leq (1-\alpha_n)(\|T_{m-i+1} y_{i-1}^n - y_{i-1}^n\| + \cdots + \|T_{m-j} y_j^n - y_j^n\|)$$

Using (2.5), implies that

$$\|y_i^n - y_j^n\| \to 0, \quad as\ n \to \infty. \tag{2.7}$$

This shows that $\{y_i^n\}$s have the same limit point, if one of them converges to a point. Now we show that

$$\|T_{m-i}y_j^n - y_j^n\| \to 0, \quad \text{as } n \to \infty, 1 \le i, j \le m. \tag{2.8}$$

Since

$$\|T_{m-i}y_j^n - y_j^n\| \le \|T_{m-i}y_j^n - T_{m-i}y_i^n\| + \|T_{m-i}y_i^n - y_i^n\| + \|y_i^n - y_j^n\|,$$

using (2.5) and (2.7) and nonexpansivity of T_{m-i} yeild that the right hand converges to 0 as $n \to \infty$, and so is the left hand. This fact, together with the Fejér-monotonicity of $\{y_j^n\}$s and using ([2], Prop 2.1.) prove that these sequences are weakly converging in F to same point, by (2.7).

Now suppose that

$$\Sigma_{i=1}^{\infty} (1 - \alpha_i) < \infty. \tag{2.9}$$

By (2.6) and the defined algorithm we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \|y_m^n - y_1^n\| + \|y_1^n - x_n\|, \\ &\le (1 - \alpha_n) (\|T_1 y_{m-1}^n - y_{m-1}^n\| + \cdots + \|T_{m-1}y_1^n - y_1^n\|, \\ &\quad + \|T_m x_n - x_n\|), \end{aligned}$$

and by the boundedness of factors in the right hand, it is obtained that $\Sigma_{i=1}^{\infty}\|x_{n+1} - x_n\| < \infty$. i.e., $\{x_n\}$ is a strongly Cauchy sequence and hence $x_n \to x^* \in C$. If there exists a natural number N_0 such that $n > N_0$ implies $x_n = x^*$, the conclusion is right. In the other case, since T_is satisfy the inward condition, by applying properties P(2) and P(3) from Lemma 2.6, we obtain that $h(x^*) < 1$ and that for any $\mu \in (0, h(x^*))$ it holds

$$\mu x^* + (1 - \mu)T_i x^* \in C, \quad 1 \le i \le m. \tag{2.10}$$

Because of (2.9), $\lim \alpha_n = 1$ and since $\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}$, we can choose a subsequence $\{x_{n_k}\}$ with the property that $h(x_{n_k})$ is non-decreasing and $h(x_{n_k}) \to 1$. In particular, for any $\mu < 1$,

$$\mu x_{n_k} + (1 - \mu)T_i x_{n_k} \notin C, \quad 1 \le i \le m, \tag{2.11}$$

eventually holds.

Choose $\mu_1, \mu_2 \in (h(x^*), 1)$, with $\mu_1 \neq \mu_2$ and set $\nu_{ij} := \mu_j x^* + (1 - \mu_j)T_i x^*$ which $j = 1, 2$ and $1 \le i \le m$. Then, whenever $\mu \in (\mu_1, \mu_2)$ by (2.10), we have $\nu_i := \mu x^* + (1 - \mu)T_i x^* \in C$ and since $x_n \to x^*$ we have $\mu x_{n_k} + (1 - \mu)\, T_i x_{n_k} \to \nu_i$ as $k \to \infty$, so by using (2.11), $\nu_i \in \partial C$ and hence $[\nu_{i1}, \nu_{i2}] \subseteq \partial C$, Since μ is arbitrary. By the strict convexity of C, we derive that $\mu_1 x^* + (1 - \mu_1)T_i x^* = \mu_2 x^* + (1 - \mu_2)T_i x^*, 1 \le i \le m$

and $T_i x^* = x^*$ must necessarily hold, i.e., $\{x_n\}$ strongly converges to a common fixed point of T_is. □

Remark 2.8. It is clear that, if $m = 1$ then the previouse Lemma and Theorem reduce to Lemma1 and theorem1 of [4].

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