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A New Iterative Algorithm for Multivalued Nonexpansive Mappping and Equlibruim Problems with Applications

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Abstract. In this paper, we introduce two iterative schemes by a modified Krasnoselskii-Mann algorithm for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of multivalued nonexpansive mappings in Hilbert space. We prove that the sequence generated by the proposed method converges strongly to a common element of the set of solutions of equilibruim problems and the set of fixed points of multivalued nonexpansive mappings which is also the minimum-norm element of the above two sets. Finally, some applications of our results to optimization problems with constraint and the split feasibility problem are given. No compactness assumption is made. The methods in the paper are novel and different from those in early and recent literature.

1. Introduction

Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

It is easy to see that single-valued nonexpansive mapping is a particular case of multivalued nonexpansive mapping.

For several years, the study of fixed point theory for single-valued and multivalued nonlinear mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Kakutani [13], Nash [17, 18], Geanakoplos [11], Nadla [16], Downing and Kirk [8], Sow et. al [23]).

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Interest in the study of fixed point theory for multivalued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Non-Smooth Differential Equations.

1.1. **Nonsmooth differential equations.** A large number of problems from mechanics and electrical engineering leads to differential inclusions and differential equations with discontinuous right-hand sides, for example, a dry friction force of some electronic devices. Below are two models.

$$\frac{du}{dt} = f(t,u), \quad \text{a.e.}, t \in I := [-a,a],\, u(0) = u_0, \tag{1.1}$$

$a, u_0 \in \mathbb{R}$. These types of differential equations do not have solutions in the classical sense. A generalized notion of solution is what is called a solution in the sense of Fillipov.

Consider the following *multi-valued* initial value problem.

$$\begin{cases} -\frac{d^2u}{dt^2} \in u - \frac{1}{4} - \frac{1}{4}\text{sign}(u-1) & \text{on } \Omega = (0,\pi); \\ u(0) = 0; \\ u(\pi) = 0. \end{cases} \tag{1.2}$$

Under some conditions, the solutions set of equations (1.1) and (1.2) coincides with the fixed point set of some multivalued mappings.

Let D be a nonempty subset of a normed space E. The set D is called proximinal (see, e.g., [19]) if for each $x \in E$, there exists $u \in D$ such that

$$\begin{aligned} d(x,u) &= \inf\{\|x-y\| : y \in D\} \\ &= d(x,D), \end{aligned}$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The Hausdorff metric on $CB(K)$ is defined by:

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

for all $A, B \in CB(K)$. A multivalued mapping $T : D(T) \subseteq E \to CB(E)$ is called L-Lipschitzian if there exists $L > 0$ such that

$$H(Tx,Ty) \leq L\|x-y\|, \quad \forall x, y \in D(T). \tag{1.3}$$

Existence theorems for fixed point of multi-valued contractions and nonexpansive mappings using the Hausdorff metric have been proved by several authors (see, e.g., Nadler [16], Markin [15], Lim [14]). Later, an interesting and rich fixed point theory for such maps and more general maps was developed which has applications in control theory, convex optimization, differential inclusion, and economics (see, Gorniewicz [12] and references cited therein).

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for f is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions is denoted by $EP(f)$. Equilibrium problems which were introduced by Fan [9] and Blum and Oettli [1] have had a great impact and influence on the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. It has been shown [22, 27] that equilibrium, problems include variational inequalities, fixed point, the Nash equilibrium, and game theory as special cases. A number of iterative algorithms have recently been studying for fixed point and equilibrium problems, see [1, 20, 21] and the references therein. However, there were few results established for fixed point of set-valued mappings and equilibrium problems.

It is our purpose in this paper to construct and study a new iterative algorithm and prove strong convergence theorems for approximating a common element of the set of solutions of equilibrium problems and the set of fixed points of multivalued nonexpansive mappings in the setting of a real Hilbert spaces. Then, we apply our main results to optimization problems with constraint and the split feasiblity problem. No compactness assumption is made, the iterative algorithms and results presented in this paper generalize, unify and improve the previously known results in this area. Finally, our method of proof is of independent interest.

2. Preliminaries

This section collects some lemmas and definitions which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Definition 2.1. Let E be real Banach space and $T : D(T) \subset E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.

Lemma 2.2 ([5, Demi-closeness Principle]). *Let E be a uniformly convex Banach space satisfying the Opial condition, K be a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2.3 ([6]). *Let H be a real Hilbert space. Then, for any $x, y \in H$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.4 ([26, Xu]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([7, Chidume et al.]). *Let X be a reflexive real Banach space and let $A, B \in CB(X)$.*

Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that

$$\|a - b\| \leq H(A, B).$$

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

$(A1)$ $f(x, x) = 0$ for all $x \in C$;

$(A2)$ f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\to 0} f(tz + (1 - t)x, y) \leq f(x, y)$$

$(A4)$ for each $x \in C$, $y \to f(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.6 ([1]). *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in [25].

Lemma 2.7 ([25]). *Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*

$$T_r(x) = \left\{ z \in C,\ f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\},$$

for all $x \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for any $x, y \in H$;*
3. *$F(T_r) = EP(f)$;*
4. *$EP(f)$ is closed and convex.*

Lemma 2.8. *Let H be a real Hilbert space, K a nonempty, closed and convex subset of H. Let $S : K \to CB(K)$ be a multivalued nonexpansive mapping such that $F := EP(f) \cap F(S) \neq \emptyset$. Suppose that $Sp = \{p\}$ for all $p \in F$. Then*

$$\langle x - v, x - p\rangle \geq 0, \quad \forall x \in K, p \in F, v \in ST_r x.$$

Proof. Using Schwartz inequality, properties of S and T_r, we obtain

$$\begin{aligned}
\langle x - v, x - p\rangle &= \langle x - v + p - p, x - p\rangle \\
&= \|x - p\|^2 - \langle v - p, x - p\rangle \\
&\geq \|x - p\|^2 - \|v - p\|\|x - p\| \\
&\geq \|x - p\|^2 - H(ST_r x, ST_r p)\|x - p\| \\
&\geq \|x - p\|^2 - \|T_r x - T_r p\|\|x - p\| \\
&\geq \|x - p\|^2 - \|x - p\|^2 \geq 0.
\end{aligned}$$

Hence, $\langle x - v, x - p\rangle \geq 0$. □

3. Main Results

Let K be a nonempty, closed convex cone of a real Hilbert space and $S : K \to CB(K)$ be a multivalued nonexpansive mapping. Let λ be a constant in $(0, 1)$. Let $\{T_{r_n}\}$ be a sequence of mappings defined as Lemma 2.7. Consider a multivalued mapping S_n on K defined by

$$S_n x = \alpha_n(\lambda x) + (1 - \alpha_n)ST_{r_n} x, \quad \forall x \in K,\ n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$. We show that S_n is a contraction. For this, let $x, y \in K$. We have:

$$H(S_n x, S_n y) = \max\left\{ \sup_{z_1 \in S_n x} d\big(z_1, S_n y\big),\ \sup_{z_2 \in S_n y} d\big(z_2, S_n x\big) \right\}.$$

For $z_1 \in S_n x$, there exists $z_3 \in ST_{r_n} x$ such that

$$z_1 = \alpha_n(\lambda x) + (1-\alpha_n) z_3.$$

Hence,

$$\begin{aligned} d(z_1, S_n y) &= d\big(\alpha_n(\lambda x) + (1-\alpha_n) z_3, S_n y\big) \\ &= \inf_{z_2 \in S_n y} \|\alpha_n(\lambda x) + (1-\alpha_n) z_3 - z_2\| \\ &\leq \|\alpha_n(\lambda x) + (1-\alpha_n) z_3 - z_2\| \quad \forall z_2 \in S_n y. \end{aligned} \tag{3.1}$$

For $z_2 \in S_n y$, there exists $z_4 \in ST_{r_n} y$ such that

$$z_2 = \alpha_n(\lambda x) + (1-\alpha_n) z_4.$$

So, from (3.1), it follows that,

$$\begin{aligned} d(z_1, S_n y) &\leq \|\alpha_n(\lambda x) + (1-\alpha_n) z_3 - \alpha_n(\lambda y) - (1-\alpha_n) z_4\| \\ &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n)\|z_3 - z_4\| \\ &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n) d(z_3, ST_{r_n} y), \quad \forall z_4 \in ST_{r_n} y. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{z_1 \in S_n x} d(z_1, S_n y) &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n) \sup_{z_3 \in ST_{r_n} x} d(z_3, ST_{r_n} y) \\ &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n) H(ST_{r_n} x, ST_{r_n} y) \\ &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n)\|T_{r_n} x - T_{r_n} y\| \\ &\leq \alpha_n \lambda \|x - y\| + (1-\alpha_n)\|x - y\| \\ &\leq [1 - (1-\lambda)\alpha_n]\|x - y\|. \end{aligned}$$

Hence,

$$\sup_{z_1 \in S_n x} d(z_1, S_n y) \leq [1 - (1-\lambda)\alpha_n]\|x - y\|. \tag{3.2}$$

Similary, we have

$$\sup_{z_2 \in S_n y} d(z_2, S_n x) \leq [1 - (1-\lambda)\alpha_n]\|x - y\|. \tag{3.3}$$

Using (3.2) and (3.3), it follows that

$$H(S_n x, S_n y) \leq [1 - (1-\lambda)\alpha_n]\|x - y\|,$$

That implies that S_n is a contraction. Therefore, from the contraction mapping principle,(see.eg, [3]), there exists $z_n \in K$ such that,

$$z_n \in \alpha_n(\lambda z_n) + (1-\alpha_n) ST_{r_n} z_n. \tag{3.4}$$

Using (3.4), there exists $y_n \in ST_{r_n} z_n$ such that

$$z_n = \alpha_n(\lambda z_n) + (1-\alpha_n) y_n.$$

We now prove the following results.

Theorem 3.1. *Let K be a nonempty, closed convex cone of a real Hilbert space H. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4), let $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F := EP(f) \cap F(S) \neq \emptyset$ and $Sp=\{p\}$, for all $p \in F$. Let λ be a constant in $(0,1)$ and $\{z_n\}$ and $\{u_n\}$ be sequences defined by:*

$$(3.5) \quad \begin{cases} z_n = \alpha_n(\lambda z_n) + (1-\alpha_n)y_n, y_n \in Su_n, & n \geq 0, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, & \forall y \in K, \end{cases}$$

where $u_n = T_{r_n} z_n$, $\{r_n\} \subset]0,\infty[$ and $\{\alpha_n\} \subset (0,1)$, satisfy:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \inf r_n > 0.$$

Then, $\{z_n\}$ and $\{u_n\}$ defined by (3.5) converge strongly to $x^* \in F$, where x^* is the minimum-norm element of F.

Proof. We split the proof into four steps.
Step 1. We prove that $\{z_n\}$ is bounded. Let $p \in F$. Then from $u_n = T_{r_n} z_n$, we have

$$\|u_n - p\| = \|T_{r_n} z_n - T_{r_n} p\| \leq \|z_n - p\|, \quad \forall n \geq 0.$$

Using (3.5), the fact that $Sp = \{p\}$ and S is nonexpansive, we have

$$\begin{aligned} \|z_n - p\| &= \|\alpha_n(\lambda z_n) + (1-\alpha_n)y_n - p\| \\ &\leq \lambda\alpha_n\|z_n - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n(1-\lambda)\|p\| \\ &\leq \lambda\alpha_n\|x_n - p\| + (1-\alpha_n)H(Su_n, Sp) + \alpha_n(1-\lambda)\|p\| \\ &\leq [1-(1-\lambda)\alpha_n]\|z_n - p\| + \alpha_n(1-\lambda)\|p\|, \end{aligned}$$

which implies that

$$\|z_n - p\| \leq \|p\|.$$

Hence, $\{z_n\}$ is bounded and so $\{y_n\}$.
Step 2. We show that $\{z_n\}$ is relatively norm compact as $n \to \infty$. Using (3.5) and the boundeness of $\{z_n\}$, we have

$$(3.6) \qquad \|z_n - y_n\| = \alpha_n\|\lambda z_n - y_n\| \to 0, \text{ as } n \to \infty.$$

For $p \in F$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n} z_n - T_{r_n} p\|^2 \\ &\leq \langle T_{r_n} z_n - T_{r_n} p, z_n - p\rangle \\ &\leq \langle u_n - p, z_n - p\rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|z_n - p\|^2 - \|z_n - u_n\|^2\right), \end{aligned}$$

and hence

$$(3.7) \qquad \|u_n - p\|^2 \leq \|z_n - p\|^2 - \|z_n - u_n\|^2.$$

Therefore, from (3.5) and (3.7), we get that

$$\begin{aligned}
\|z_n - p\|^2 &= \|\alpha_n(\lambda z_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \|\alpha_n((\lambda z_n) - p) + (1-\alpha_n)(y_n - p)\|^2 \\
&\leq (1-\alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\langle(\lambda z_n) - p, z_n - p\rangle \\
&\leq (1-\alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\lambda\langle z_n - p, z_n - p\rangle \\
&\quad + 2(1-\lambda)\alpha_n\langle p, p - z_n\rangle \\
&\leq (1-\alpha_n)^2(\|z_n - p\|^2 - \|z_n - u_n\|^2) + 2\alpha_n\lambda\|z_n - p\|^2 \\
&\quad + 2\alpha_n(1-\lambda)\|p\|\|z_n - p\| \\
&= (1 - 2\alpha_n + \alpha_n^2)\|z_n - p\|^2 - (1-\alpha_n)^2\|z_n - u_n\|^2 \\
&\quad + 2\alpha_n\lambda\|z_n - p\|^2 + 2(1-\lambda)\alpha_n\|p\|\|z_n - p\| \\
&\leq \|z_n - p\|^2 + \alpha_n\|z_n - p\|^2 - (1-\alpha_n)^2\|z_n - u_n\|^2 \\
&\quad + 2\alpha_n\lambda\|z_n - p\|^2 + 2(1-\lambda)\alpha_n\|p\|\|z_n - p\|,
\end{aligned}$$

and hence

$$(1-\alpha_n)^2\|z_n - u_n\|^2 \leq \alpha_n\|z_n - p\|^2 + 2\alpha_n\lambda\|z_n - p\|^2 + 2(1-\lambda)\alpha_n\|p\|\|z_n - p\|.$$

So, we have $\|z_n - u_n\| \to 0, \text{ as } n \to \infty$. Since $\|y_n - u_n\| \leq \|z_n - y_n\| + \|z_n - u_n\|$, it follows that $\|y_n - u_n\| \to 0, \text{ as } n \to \infty$. Hence,

$$d(u_n, Su_n) \to 0, \text{ as } n \to \infty. \tag{3.8}$$

Let $p \in F$. From (3.5) and the fact that $Sp = \{p\}$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \langle\alpha_n(\lambda z_n) + (1-\alpha_n)y_n - p, z_n - p\rangle \\
&= \alpha_n\lambda\langle z_n - p, z_n - p\rangle + (1-\alpha_n)\langle y_n - p, z_n - p\rangle \\
&\quad - (1-\lambda)\alpha_n\langle p, z_n - p\rangle \\
&\leq [1 - (1-\lambda)\alpha_n]\|z_n - p\|^2 - (1-\lambda)\alpha_n\langle p, z_n - p\rangle.
\end{aligned}$$

So,

$$\|z_n - p\|^2 \leq \langle p, p - z_n\rangle. \tag{3.9}$$

Since H is reflexive and $\{u_{n_k}\}$ is bounded, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ which converges weakly to $x^* \in K$. From (3.8) and Lemma 2.2, we obtain $x^* \in F(S)$. Without loss of generality, we can assume that $u_{n_k} \rightharpoonup x^*$. Let us show $x^* \in EP(f)$. It follows by (3.5) and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq f(y, u_n),$$

and hence

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - z_{n_k}}{r_{n_k}}\right\rangle \geq f(y, u_{n_k}).$$

Since $\frac{u_{n_k} - z_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup x^*$, it follows from $(A4)$ that $f(y, x^*) \leq 0$ for all $y \in K$. For t with $0 < t < 1$ and $y \in K$, let $y_t = ty + (1-t)x^*$. Since $y \in K$ and $x^* \in K$, we have $y_t \in K$ and hence $f(y_t, x^*) \leq 0$. So, from $(A1)$ and $(A4)$ we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x^*) \leq tf(y_t, y),$$

and hence $0 \leq f(y_t, y)$. From $(A3)$, we have $f(x^*, y) \geq 0$ for all $y \in K$ and hence $x^* \in EP(f)$. Therofore, $x^* \in F(S) \cap EP(f) = F$.

Since $z_{n_k} \rightharpoonup x^*$ as $k \to \infty$, it follows from (3.9) that $z_{n_k} \to x^*$ as $k \to \infty$. This proves the relatively compactness of $\{z_n\}$.

Step 3. We show that the sequence $\{z_n\}$ converges to $x^* \in F$. We claim that the net $\{z_n\}$ has a unique cluster point. From Step 2, the sequence $\{z_n\}$ has a cluster point. Now suppose that $x^* \in K$ and $x^{**} \in E$ are two cluster points of $\{z_n\}$. Let $\{z_{n_k}\}$ and $\{z_{n_p}\}$ be two subsequences of $\{z_n\}$ such that $z_{n_k} \to x^*$, as $k \to \infty$ and $z_{n_p} \to x^{**}$.

Following the same arguments as in Step 2, it follows that $x^*, x^{**} \in F$, and the following estimates hold:

$$\|z_{n_k} - x^{**}\|^2 \leq \langle x^{**}, x^{**} - z_{nk}\rangle, \tag{3.10}$$

and

$$\|z_{n_p} - x^*\|^2 \leq \langle x^*, x^* - z_{n_p}\rangle. \tag{3.11}$$

Letting $k \to \infty$ and $p \to \infty$ in (3.10) and (3.11) gives

$$\|x^* - x^{**}\|^2 \leq \langle x^{**}, x^{**} - x^*\rangle \tag{3.12}$$

and

$$\|x^{**} - x^*\|^2 \leq \langle x^*, x^* - x^{**}\rangle. \tag{3.13}$$

Adding up (3.12) and (3.13) yields

$$2\|x^* - x^{**}\|^2 \leq \|x^* - x^{**}\|^2,$$

which implies that $x^* = x^{**}$.

Step 4. Finally, we show that x^* is the minimum-norm element of F. Following the same arguments as in Step 3, it follows that

$$\|x^* - p\|^2 \leq \langle -p, x^* - p\rangle, \quad p \in F.$$

Equivalently,

$$\|x^*\|^2 \leq \langle p, x^*\rangle, \quad \forall p \in F.$$

This clear implies that

$$\|x^*\| \leq \|p\|, \quad \forall p \in F.$$

Therefore, x^* is the minimum-norm element of F. This completes the proof. □

We now apply Theorem 3.1 for solving variational inequality problems.

Theorem 3.2. *The sequence* $\{z_n\}$ *defined by* (3.5) *converges strongly to a unique solution of the following variational inequality*

$$\langle x^*, x^* - p\rangle \leq 0, \quad \forall p \in F. \tag{3.14}$$

Proof. It follows from (3.5) that,

$$z_n = -\frac{1-\alpha_n}{(1-\lambda)\alpha_n}(z_n - y_n).$$

Using Lemma 2.8, for any $p \in F$, we have

$$\langle z_n, z_n - p\rangle = -\frac{1-\alpha_n}{(1-\lambda)\alpha_n}\langle z_n - y_n, z_n - p\rangle \leq 0.$$

Letting $n \to \infty$, noting the fact that $z_n \to x^*$, we obtain

$$\langle x^*, x^* - p\rangle \leq 0. \tag{3.15}$$

Finally, we show the uniqueness of the solution of the variational inequality (3.14). Suppose both $x^* \in F$ and $x^{**} \in F$ are solutions to (3.14), then

$$\langle x^*, x^* - x^{**}\rangle \leq 0, \tag{3.16}$$

and

$$\langle x^{**}, x^{**} - x^*\rangle \leq 0. \tag{3.17}$$

Adding up (3.16) and (3.17) yields

$$\langle x^{**} - x^*, x^{**} - x^*\rangle \leq 0, \tag{3.18}$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. □

We now apply Theorems 3.1 and 3.2 for finding a common element of the set of fixed points of multivalued nonexpansive mappings and the set of solutions of equilibrium problems.

In what follows, we use the following iteration scheme: let K be a nompety, closed convex cone of a real Hilbert space H and $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values.

Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in K, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, & y_n \in Su_n, n \geq 0, \end{cases} \tag{3.19}$$

$$\|y_n - y_{n-1}\| \leq H(Su_n, Su_{n-1}), \quad \forall\, n \geq 1, \tag{3.20}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(iii) $\lim_{n\to\infty} \inf r_n > 0$ and $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ and $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

Remark 3.3. From y_{n-1}, the existence of y_n in (3.19) satisfying (3.20) is guranteed by Lemma 2.5.

Theorem 3.4. *Let K be a nonempty, closed convex cone of a real Hilbert space H. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4), let $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F := EP(f) \cap F(S) \neq \emptyset$ and $Sp = \{p\}$, for all $p \in F$. Then, $\{x_n\}$ and $\{u_n\}$ defined by (3.19) and (3.20) converge strongly to $x^* \in F$, where x^* is the minimum-norm element of F.*

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $p \in F$. Then from $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|, \quad \forall n \geq 0.$$

From (3.19) and the fact that $Sp = \{p\}$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n - p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\| + (1 - \alpha_n)\|y_n - p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\| + (1 - \alpha_n)H(Su_n, Sp) \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\| + (1 - \alpha_n)\|x_n - p\| \\
&= [1 - (1 - \lambda_n)\alpha_n]\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\|,
\end{aligned}$$

and so

$$\|x_{n+1} - p\| \leq \max\{\|x_n - p\|, \|p\|\}. \tag{3.21}$$

Hence, $\{x_n\}$ is bounded and so $\{y_n\}$.

From (3.19) and (3.20), it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n - \alpha_{n-1}(\lambda_{n-1}x_{n-1}) \\
&\quad - (1 - \alpha_{n-1})y_{n-1}\| \\
&= \|\alpha_n\lambda_n(x_n - x_{n-1}) + \alpha_n(\lambda_n - \lambda_{n-1})x_{n-1} \\
&\quad + (\alpha_n - \alpha_{n-1})(\lambda_{n-1}x_{n-1}) + (1 - \alpha_n)(y_n - y_{n-1}) \\
&\quad + (\alpha_{n-1} - \alpha_n)y_{n-1}\| \\
&\leq \alpha_n\lambda_n\|x_n - x_{n-1}\| + (1 - \alpha_n)\|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\| + \|y_{n-1}\|)
\end{aligned}$$

$$
\begin{aligned}
&+\ \alpha_n|\lambda_n-\lambda_{n-1}|\|x_{n-1}\| \\
&\le \alpha_n\lambda_n\|x_n-x_{n-1}\|+(1-\alpha_n)H(Su_n,Su_{n-1}) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\|+\|y_{n-1}\|) \\
&\quad +\alpha_n|\lambda_n-\lambda_{n-1}|\|x_{n-1}\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|x_{n+1}-x_n\| \le{}& \alpha_n\lambda_n\|x_n-x_{n-1}\|+(1-\alpha_n)\|u_n-u_{n-1}\| \\
&+(|\alpha_n-\alpha_{n-1}|+\alpha_n|\lambda_n-\lambda_{n-1}|)M_1,
\end{aligned}
\tag{3.22}
$$

where $M_1>0$ and $\sup_n\{\|x_{n-1}\|+\|y_{n-1}\|\}\le M_1$.

On other hand, we have

$$
f(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \tag{3.23}
$$

and

$$
f(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0, \tag{3.24}
$$

Putting $y=u_{n+1}$ in (3.23) and $y=u_n$ in (3.24), we have

$$
f(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle \ge 0
$$

and

$$
f(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0.
$$

So, from $(A2)$, we have

$$
\left\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_n}\right\rangle \ge 0,
$$

and hence

$$
\left\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \ge 0.
$$

Without loss of generality, let assume that there exists a real number b such that $r_n>b>0$ for all $n\in\mathbb{N}$. Then, we have

$$
\begin{aligned}
\|u_{n+1}-u_n\|^2 &\le \left\langle u_{n+1}-u_n,x_{n+1}-x_n+\Big(1-\frac{r_n}{r_{n+1}}\Big)(u_{n+1}-x_{n+1})\right\rangle \\
&\le \|u_{n+1}-u_n\|\left\{\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|\right\},
\end{aligned}
$$

and hence

$$
\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\|+\frac{1}{b}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|.
$$

This implies that

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L, \tag{3.25}$$

where $L>0$ is such $\sup_n\{\|u_{n+1}-x_{n+1}\|\} \leq L$.

So, from (3.22) we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \alpha_n\lambda_n\|x_n-x_{n-1}\| + (1-\alpha_n)(\|x_n-x_{n-1}\| \\
&\quad + \frac{1}{b}|r_n-r_{n-1}|L) + (|\alpha_n-\alpha_{n-1}| + \alpha_n|\lambda_n-\lambda_{n-1}|)M_1 \\
&= [1-(1-\lambda_n)\alpha_n]\|x_n-x_{n-1}\| + (1-\alpha_n)\frac{1}{b}|r_n-r_{n-1}|L \\
&\quad + (|\alpha_n-\alpha_{n-1}| + \alpha_n|\lambda_n-\lambda_{n-1}|)M_1 \\
&= [1-(1-\lambda_n)\alpha_n]\|x_n-x_{n-1}\| + \frac{1}{b}|r_n-r_{n-1}|L \\
&\quad + (|\alpha_n-\alpha_{n-1}| + \alpha_n|\lambda_n-\lambda_{n-1}|)M_1.
\end{aligned}$$

Using Lemma 2.4, we deduce $\lim_{n\to+\infty}\|x_{n+1}-x_n\| \to 0$ and from (3.25) and $|r_n-r_{n-1}| \to 0$, we have

$$\lim_{n\to+\infty}\|u_{n+1}-u_n\| = 0.$$

Using (3.20) and the fact that $x_n = \alpha_{n-1}(\lambda_{n-1}x_{n-1}) + (1-\alpha_{n-1})y_{n-1}$, we have

$$\begin{aligned}
\|x_n-y_n\| &\leq \|x_n-y_{n-1}\| + \|y_{n-1}-y_n\| \\
&\leq \alpha_{n-1}\|\lambda_{n-1}x_{n-1}-y_{n-1}\| + \|u_{n-1}-u_n\|.
\end{aligned}$$

From $\alpha_n \to 0$, we have $\|x_n-y_n\| \to 0$. For $p \in F$, we have

$$\begin{aligned}
\|u_n-p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}p, x_n-p\rangle \\
&\leq \langle u_n-p, x_n-p\rangle \\
&= \frac{1}{2}(\|u_n-p\|^2 + \|x_n-p\|^2 - \|x_n-u_n\|^2)
\end{aligned}$$

and hence

$$\|u_n-p\|^2 \leq \|x_n-p\|^2 - \|x_n-u_n\|^2.$$

Therefore, from (3.19) and Lemma 2.5, we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \|\alpha_n((\lambda_n x_n)-p) + (1-\alpha_n)(y_n-p)\|^2 \\
&\leq (1-\alpha_n)^2\|y_n-p\|^2 + 2\alpha_n\langle(\lambda_n x_n)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)^2\|u_n-p\|^2 + 2\alpha_n\lambda_n\langle x_n-p, x_{n+1}-p\rangle \\
&\quad + 2(1-\lambda_n)\alpha_n\langle p, x_{n+1}-p\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)^2(\|x_n-p\|^2-\|x_n-u_n\|^2)\\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|\\
&\leq (1-2\alpha_n+\alpha_n^2)\|x_n-p\|^2-(1-\alpha_n)^2\|x_n-u_n\|^2\\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|\\
&\leq \|x_n-p\|^2+\alpha_n\|x_n-p\|^2-(1-\alpha_n)^2\|x_n-u_n\|^2\\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|,
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)^2\|x_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|x_n-p\|^2\\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|\\
&\quad +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|\\
&\leq \|x_{n+1}-x_n\|\{\|x_n-p\|+\|x_{n+1}-p\|\}+\alpha_n\|x_n-p\|^2\\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|\\
&\quad +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|.
\end{aligned}$$

So, we have $\|x_n-u_n\|\to 0$ as, $n\to\infty$. Since $\|y_n-u_n\|\leq\|x_n-y_n\|+\|x_n-u_n\|$, it follows that $\|y_n-u_n\|\to 0$. Hence,

$$\lim_{n\to\infty} d(u_n,Su_n)=0. \tag{3.26}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle\leq 0,\quad$ where $x^*=\lim_{n\to\infty} z_n$.
We choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that:

$$\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle=\lim_{k\to+\infty}\langle x^*,x^*-x_{n_k}\rangle.$$

Since H is reflexive and $\{u_{n_k}\}$ is bounded, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ which converges weakly to $a\in K$. From (3.26) and Lemma 2.2, we obtain $a\in F(S)$. Without loss of generality, we can assume that $u_{n_k}\rightharpoonup a$. By the same argument as in the proof of Theorem 3.1, we have $a\in F(S)\cap EP(f)=F$. Hence,

$$\begin{aligned}
\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle &= \lim_{k\to+\infty}\langle x^*,x^*-x_{n_k}\rangle\\
&=\langle x^*,x^*-a)\rangle\leq 0.
\end{aligned}$$

Finally, we show that $x_n\to x^*$. From (3.19) and Lemma 2.5, we get that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*,x_{n+1}-x^*\rangle\\
&=\alpha_n\lambda_n\langle x_n-x^*,x_{n+1}-x^*\rangle+(1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle\\
&\quad +(1-\alpha_n)\langle y_n-x^*,x_{n+1}-x^*\rangle\\
&\leq \alpha_n\lambda_n\langle x_n-x^*,x_{n+1}-x^*\rangle+(1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha_n)\|y_n - x^*\|\|x_{n+1} - x^*\| \\
\le\ & \alpha_n\lambda_n\langle x_n - x^*, x_{n+1} - x^*\rangle + (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\
&+ (1-\alpha_n)H(Su_n, Sx^*)\|x_{n+1} - x^*\| \\
\le\ & \alpha_n\lambda_n\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\
&+ (1-\alpha_n)\|u_n - x^*\|\|x_{n+1} - x^*\| \\
\le\ & [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&+ (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\
\le\ & \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&+ (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle,
\end{aligned}
$$

which implies that

$$\|x_{n+1} - x^*\|^2 \le [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle.$$

We can check that all the assumptions of Lemma 2.4 are satisfied. Therefore, we deduce $x_n \to x^*$. This completes the proof. □

Corollary 3.5. *Let K be a nonempty, closed convex cone of a real Hilbert space H, let $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F(S) \neq \emptyset$ and $Sp = \{p\}$, for all $p \in F(S)$ and $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \quad y_n \in Sx_n, n \ge 0, \tag{3.27}$$

$$\|y_n - y_{n-1}\| \le H(Sx_n, Sx_{n-1}), \quad \forall\, n \ge 1, \tag{3.28}$$

where $\{\alpha_n\} \subset (0,1)$, and $\{\lambda_n\} \subset (0,1)$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(iii) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in F(S)$.

Proof. Put $f(x,y) = 0$ for all $x, y \in K$ and $r_n = 1$, we get $u_n = x_n$ in Theorem 3.4. The proof follows from Theorem 3.4. □

Corollary 3.6. *Let K be a nonempty, closed convex cone of a real Hilbert space H, f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4) and $S : K \to K$ be a nonexpansive mapping with convex-values* *such that $F := EP(f) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$(3.29)\quad \begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in K, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n, & n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfy:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim\limits_{n\to\infty} \lambda_n = 1$;

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$ *and* $\sum\limits_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv) $\sum\limits_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Then, $\{x_n\}$ defined by (3.29) converges strongly to $x^* \in F(S)$.

Proof. Since every single-valued mapping can be viewed as a multivalued mapping, the proof follows from Theorem 3.4. □

Remark 3.7. In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, our results can be used to approximate fixed points of nonexpansive mappings from the closed unit ball to itself.

Corollary 3.8. *Let H be a real Hilbert space, B be the closed unit ball of H, f be a bifunction from $B \times B \to \mathbb{R}$ satisfying (A1)-(A4), $S : B \to CB(B)$ be a multivalued nonexpansive mapping with convex-values such that $F := EP(f) \cap F(S) \neq \emptyset$ and $Sp = \{p\}$, for all $p \in F$, and $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in B$ by:*

$$(3.30)\quad \begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in B, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, & y_n \in Su_n, n \geq 0, \end{cases}$$

$$(3.31)\quad \|y_n - y_{n-1}\| \leq H(Su_n, Su_{n-1}), \quad \forall\, n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfy:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim\limits_{n\to\infty} \lambda_n = 1$;

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$ *and* $\sum\limits_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Then $\{x_n\}$ *and* $\{u_n\}$ *defined by* (3.30) *and* (3.31). *converge strongly to* $x^* \in F$.

4. Application to Optimization Problem with Constraint

In this section, we study the problem of finding a minimizer of a proper convex function g defined in a real Hilbert space.

Proposition 4.1 ([24]). *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone mapping such that $K := D(A)$ is closed and convex. Assume that A is bounded on bounbded subset and hemi-continuous on K. Then, the bifunction $f(x, y) := \langle Ax, y - x\rangle$ satisfies conditions (A1)-(A4).*

The following basic results are well known.

Lemma 4.2. *Let E be a normed linear space, $g : E \to \mathbb{R}$ be a real valued differentiable convex function, and $dg : E \to E^*$ denote the differential map associated to g. Then the following holds. If g is bounded, then g is locally Lipschitzian, i.e., for every $x_0 \in K$ and $r > 0$, there exists $\gamma > 0$ such that g is γ-Lipschitzian on $B(x_0, r)$, i.e.,*

$$|g(x) - g(y)| \leq \gamma\|x - y\|, \quad \forall\, x, y \in B(x_0, r).$$

Lemma 4.3. *Let K be a nonempty, closed and convex subset of E and $g : K \to \mathbb{R}$ be a real valued differentiable convex function. Then x^* is a minimizer of g over K if and only if x^* solves the following variational inequality $\langle dg(x^*), x - x^*\rangle \geq 0$ for all $y \in K$.*

Remark 4.4. Let K be a nonempty, closed convex subset of H, and $g : K \to \mathbb{R}$ be a real valued differentiable convex function. It is well know that the differential map associated to g is monotone.

Lemma 4.5. *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $g : K \to \mathbb{R}$ be a real valued differentiable convex function. Assume that g is bounded. Then the differentiable map, $dg : K \to H$ is bounded.*

Proof. For $x_0 \in K$ and $r > 0$, let $B := B(x_0, r)$. We show that $dg(B)$ is bounded. By Lemm 4.2, there exists $\gamma > 0$ such that

$$|g(x) - g(y)| \leq \gamma\|x - y\|, \quad \forall\, x, y \in B. \tag{4.1}$$

Let $z^* \in dg(B)$ and $x^* \in B$ such that $z^* = dg(x^*)$. For $u \in E$, since B is open, then there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = dg(x^*)$, convexity of g and inequality (4.1), it follows

$$\langle z^*, tu\rangle \leq g(x^* + tu) - g(x^*)$$

$$\leq t\gamma\|u\|.$$

So, $\langle z^*, u\rangle \leq \gamma\|u\|, \forall\, u \in E$. Therefore, $\|z^*\| \leq \gamma$. Hence $dg(B)$ is bounded. □

Let K be a nonempty, closed convex cone of a real Hilbert space H, $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F(S) \neq \emptyset$ and $g : K \to \mathbb{R}$ be a real valued continuously differentiable convex function.
We introduce the following optimization problem:

$$(P) \quad \begin{cases} \min g(x) \\ x \in F(S). \end{cases}$$

Finding an optimal point in the fixed points set of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering.
We prove the following theorem.

Theorem 4.6. *Let K be a nonempty, closed convex cone of a real Hilbert space H, $g : K \to \mathbb{R}$ be a real valued continuously differentiable convex and bounded function, and $S : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F(S) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Assume that (P) has the solution. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated iteratively from arbitrary $x_0 \in K$ by:*

(4.2)
$$\begin{cases} \langle dg(u_n), y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n, & y_n \in Su_n n \geq 0, \end{cases}$$

$$\|y_n - y_{n-1}\| \leq H(Su_n, Su_{n-1}), \quad \forall\, n \geq 1, \tag{4.3}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfy:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim\limits_{n\to\infty} \lambda_n = 1$;

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$ *and* $\sum\limits_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv) $\sum\limits_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum\limits_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to x^* solution of (P).

Proof. Let $f(x,y) := \langle dg(x), y - x\rangle$ for all $x, y \in K$. From the properties of g, Proposition 4.1, Remark 4.4 and Lemma 4.5, it follows that dg is monotone, continous and bounded on a bounbded subset on K. So, f satisfies $(A1)$-$(A4)$. Using the assumption that (P) has the solution and Lemma 4.3, we have x^* is a solution of (P) if and only if $x^* \in F(S) \cap EP(f)$. Then, the proof follows from Theorem 3.4. □

5. Application to the Split Feasibility Problem

In this section, we study the problem of finding a common element of the set of solutions of equilibrium problems and the set of solutions of the split feasibility problem.

The split feasibility problem (SFP) was first introdued by Censor and Elfving [4] in 1994. The SFP is to find

$$x \in K, \text{ such that } Ax \in Q, \tag{5.1}$$

where K is a nonempty, closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

The split feasibility problem arises in many fields in the real world, such as signal processing, image reconstruction, and medical care, for details see, [1, 25, 27] and the references therein. Let Ω be the solution set of the split feasibility problem.

The following lemma appears in [2].

Lemma 5.1. *Given $x^* \in H$, x^* solves SFP (5.1) if and only if x^* is the solution of the fixed point equation $x = P_K(I - \gamma A^*(I - P_Q)A)x$.*

The following proposition was also given in [10].

Proposition 5.2 ([10])**.** *Let K be a nonempty, closed and convex subset of a Hilbert space H_1 and Q be a nonempty, closed and convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ be a bounded linear operator. Let P_K and P_Q denote the orthogonal projections onto sets K, Q respectively. Let $0 < \gamma < \frac{2}{\rho}$, ρ be the spectral raduis of A^*A, and A^* be the adjoint of A. Then, the operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K.*

Theorem 5.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and $A^* : H_2 \to H_1$ be a adjoint operator of A. Let K be a nonempty, closed convex cone of H_1, Q be a nonempty, closed and convex subset of H_2 and f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4). Assume that $F := EP(f) \cap \Omega \neq \emptyset$. Let $0 < \gamma < \frac{2}{\rho}$, ρ be the spectral raduis of A^*A, and $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in K, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)P_K(I - \gamma A^*(I - P_Q)A)u_n, & n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} (1 - \lambda_n)\alpha_n = \infty$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where x^* is the minimum-norm element of a common element of the set of solutions of equilibrium problems and the set of solutions of the split feasibility problem.

Proof. From Lemma 5.1, we know $x \in \Omega$ if and only if $x = P_K(I - \gamma A^*(I - P_Q)A)x$. From Proposition 5.2, we have the operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K. Using, Corollary 3.6, we can obtain the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where x^* is the minimum-norm element of a common element of the set of solutions of equilibrium problems and the set of solutions of split feasibility problem. □

6. Conclusion

In this work, we introduce and analyze a new iterative method for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of multivalued nonexpansive mappings. This method can be applied in solving the relevant problem, such as optimization problem, the split feasiblity problem (SFF), and so on.

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