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Fixed Point Theorems on Complete Quasi Metric Spaces Via C-class and A-Class Functions

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Abstract. In this paper, we present some fixed point theorems for single valued mappings on K-complete, M-complete and Symth complete quasi metric spaces. Here, for contractive condition, we consider some altering distance functions together with functions belonging to C-class and A-class. At the same time, we will consider two different type M functions in contractive conditions because the quasi metric does not provide the symmetry property. Finally, we show that our main results includes many fixed point theorems presented on both complete metric and complete quasi metric spaces in the literature. We also provide an illustrative example to show importance of our results.

1. Introduction

After the introduction to the concept of quasi metric space by Wilson [23], besides the topological properties of it, fixed point theory studies have been made and rapidly developed on this space [1, 11, 12, 16, 18, 22]. In this paper, we point out some results for fixed points of single valued mappings on quasi metric spaces. To do this, we will consider a general contractive condition, by taking into account some recent concepts of real valued functions such as altering distance functions [17], C-class and A-class functions [3]. First, let us recall some fundamental concepts of quasi metric spaces.

Let X be any set. Then a function $d : X \times X \to [0, \infty)$ is said to be a quasi metric on X if it has the following properties: for all $x, y, z \in X$

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(d1) $d(x,x) = 0$;
(d2) $d(x,y) \leq d(x,z) + d(z,y)$;
(d3) $d(x,y) = d(y,x) = 0 \Rightarrow x = y$.

If a quasi metric d satisfies the additional condition

(d4) $d(x,y) = 0 \Rightarrow x = y$,

then d is said to be T_1-quasi metric. A quasi (resp. T_1-quasi) metric space is a pair (X,d) such that X is a nonempty set and d is a quasi (resp. T_1-quasi) metric on X.

Given a quasi metric space (X,d) and any real number $\varepsilon > 0$, the open ball and closed ball, respectively, of radius ε and center $x_0 \in X$ are the sets $B_d(x_0,\varepsilon) \subset X$ and $B_d[x_0,\varepsilon] \subset X$ defined by

$$B_d(x_0,\varepsilon) = \{y \in X : d(x_0,y) < \varepsilon\}$$

and

$$B_d[x_0,\varepsilon] = \{y \in X : d(x_0,y) \leq \varepsilon\}.$$

Every quasi metric d on X, generates a natural topology on X. This topology on X generated by the family of open balls $\{B_d(x,\varepsilon) : x \in X$ and $\varepsilon > 0\}$ as base is called the topology τ_d (or, the topology induced by the quasi metric d). If (X,d) is a quasi metric space, then τ_d is a T_0 topology and if (X,d) is a T_1-quasi metric space, then τ_d is a T_1 topology on X.

Let d be a quasi metric on a set X. Define the functions d^{-1} and d^s as follows: for all $x,y \in X$

$$d^{-1}(x,y) = d(y,x), \qquad d^s = \max\left\{d(x,y), d^{-1}(x,y)\right\}.$$

If d is a quasi metric on X, then d^{-1} is also a quasi metric and d^s is an ordinary metric on X.

Definition 1.1. Let $\{x_n\}$ be a sequence in the quasi metric space (X,d) and $x \in X$. Then $\{x_n\}$ is said to be convergent to x with respect to τ_d (or simply d-convergent) iff $\lim_{n\to\infty} d(x,x_n) = 0$.

Remark 1.2. It is clear that, a sequence $\{x_n\}$ in a quasi metric space (X,d) is both d and d^{-1}-convergent iff it is d^s-convergent.

There are several notions of Cauchy sequences in quasi-metric spaces. Reilly et al. [21] defined six such notions, three of them being as follows:

Definition 1.3. A sequence $\{x_n\}$ in a quasi metric space (X,d) is called

- left K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,k, n \geq k \geq n_0, \quad d(x_k,x_n) < \varepsilon,$$

- right K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n,k,\ n \geq k \geq n_0, \quad d(x_n,x_k) < \varepsilon,$$

- d^s-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k \geq n_0, \quad d(x_n, x_k) < \varepsilon.$$

Remark 1.4. It is obvious that a sequence in a quasi metric space is d^s-Cauchy if and only if it is both left K-Cauchy and right K-Cauchy.

If every Cauchy sequence in a metric space is convergent, then the space is said to be complete. But, in a quasi-metric space, completeness cannot be uniquely defined. Altun et al. [2] classified nine different definitions of completeness, three of them are as follows:

Definition 1.5. A quasi metric space (X, d) is called

- left (right) $\mathcal{K}$-complete if every left (right) K-Cauchy sequence is d-convergent,
- left (right) M-complete if every left (right) K-Cauchy sequence is d^{-1}-convergent,
- left (right) *Smyth* complete if every left (right) K-Cauchy sequence is d^s-convergent.

Now, we will remember some classes of real valued functions which are introduced by Ansari [3] as C-class and A-class.

Definition 1.6. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties hold:

(ψ1) ψ is continuous and non-decreasing,
(ψ2) $\psi(t) = 0$ if and only if $t = 0$,
(ψ3) $\psi(t) \geq Mt^\mu$ for every $t > 0$ where $M, \mu > 0$ are constants.

Definition 1.7. A function $\varphi : [0, \infty) \to [0, \infty)$ is called an ultra altering distance function if the following properties hold:

(φ1) φ is continuous,
(φ2) $\varphi(t) > 0$ for $t > 0$.

Definition 1.8. A function $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is of C-class if the following properties hold:

($f1$) f is continuous,
($f2$) $f\ (s, t) \leq s$,
($f3$) $f(s, t) = s \Rightarrow s = 0$ or $t = 0$.

Definition 1.9. A function $h : [0, \infty) \to [0, \infty)$ is of A-class if it is continuous and $h(t) \geq t$ for all $t \in [0, \infty)$.

Considering the C-class and A-class functions in the contractive conditions, many authors obtained nice fixed point results for single valued mapings on different type of spaces such as metric space, b-metric space, generalized metric space etc. (see [4–10, 13, 14, 19, 20]). The aim of this paper is to present some fixed point results for single valued mappings on quasi metric spaces via C-class and A-class functions.

2. Fixed Point Theorems

There are many researches in fixed point theory including different contraction conditions on complete metric spaces (X,d). Many of them contain the maximum of some terms $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, and $\frac{1}{2}[d(x,Ty)+d(y,Tx)]$ to give flexibility to the mapping $T: X \to X$. Here, if d is a metric, it is not important to change the order under d in these terms. However, if d is a quasi metric, then it is quite important. In [22], considering the function

$$M_1(x,y) = \max\left\{d(x,y), d(Tx,x), d(Ty,y), \frac{1}{2}[d(x,Ty)+d(Tx,y)]\right\},$$

Simsek and Yalçın presented a fixed point result on quasi metric spaces using the simulation function. Here, taking into account the functions $M_1(x,y)$ and the following $M_2(x,y)$ in (some kind of complete) quasi metric spaces, we obtain some new results via C-class and A-class functions:

$$M_2(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(Tx,y)]\right\}.$$

Theorem 2.1. *Let (X,d) be a right $\mathcal{K}$-complete T_1-quasi metric space, $\psi:[0,\infty)\to[0,\infty)$ be an altering distance function, $\varphi:[0,\infty)\to[0,\infty)$ be an ultra altering distance function, $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a function of C-class and $h:[0,\infty)\to[0,\infty)$ be a function of A-class. Let $T: X\to X$ satisfies*

$$h\left[\psi(d(Tx,Ty))\right] \le f\left[\psi(M_1(x,y)), \varphi(M_1(x,y))\right], \tag{2.1}$$

for $x,y\in Y$ where Y is a closed subset of X invariant under T. Then T has a fixed point in Y.

Proof. For an arbitrary $x_0 \in Y$, we can construct a sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$ since Y is invariant under T. Replacing x with x_n and y with x_{n-1} in (2.1), we have

$$\begin{aligned} \psi(d(x_{n+1},x_n)) &\le h\left[\psi(d(x_{n+1},x_n))\right] \\ &\le f\left[\psi(M_1(x_n,x_{n-1})), \varphi(M_1(x_n,x_{n-1}))\right], \end{aligned} \tag{2.2}$$

where

$$M_1(x_n,x_{n-1}) = \max\left\{\begin{array}{c} d(x_n,x_{n-1}), d(x_{n+1},x_n), d(x_n,x_{n-1}), \\ \frac{1}{2}\left[d(x_n,x_n)+d(x_{n+1},x_{n-1})\right] \end{array}\right\}.$$

Since $d(x_n,x_n)=0$ and

$$\begin{aligned} \frac{1}{2}d(x_{n+1},x_{n-1}) &\le \frac{1}{2}\left[d(x_{n+1},x_n)+d(x_n,x_{n-1})\right] \\ &\le \max\left\{d(x_n,x_{n-1}), d(x_{n+1},x_n)\right\}, \end{aligned}$$

we get

$$M_1(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\}.$$

If for some n_0,

$$M_1(x_{n_0}, x_{n_0-1}) = d(x_{n_0+1}, x_{n_0})$$

then (2.2) becomes,

$$\begin{aligned}\psi(d(x_{n_0+1}, x_{n_0})) &\leq h\left[\psi(d(x_{n_0+1}, x_{n_0}))\right] \\ &\leq f\left[\psi(d(x_{n_0+1}, x_{n_0})), \varphi(d(x_{n_0+1}, x_{n_0}))\right].\end{aligned}$$

Since f is a function of C-class, by $(f2)$ we have

$$f\left[\psi(d(x_{n_0+1}, x_{n_0})), \varphi(d(x_{n_0+1}, x_{n_0}))\right] \leq \psi(d(x_{n_0+1}, x_{n_0})),$$

so that by $(f3)$

$$\psi(d(x_{n_0+1}, x_{n_0}) = 0 \quad \text{or} \quad \varphi(d(x_{n_0+1}, x_{n_0})) = 0.$$

Since ψ is an altering distance and φ is an ultra altering distance function, by $(\psi 2)$ and $(\varphi 2)$ we deduce $d(x_{n_0+1}, x_{n_0}) = 0$. Since X is a T_1-quasi metric space, $x_{n_0+1} = x_{n_0}$ and so x_{n_0} is the fixed point of T.

Now, let

$$\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n-1}),$$

for all $n > 0$. Therefore, $\{d(x_n, x_{n-1})\}$ is a decreasing sequence. Let $\lim_{n\to\infty} d(x_n, x_{n-1}) = r$. Hence, letting $n \to \infty$ in (2.2) we obtain that

$$\psi(r) \leq h\left[\psi(r)\right] \leq f\left[\psi(r), \varphi(r)\right].$$

Taking $(f3)$, $(\psi 2)$ and $(\varphi 2)$ into account, we can conclude that

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{2.3}$$

Now, we need to prove that the limit of $d(x_{n-1}, x_n)$ is also zero. Starting with the replacement of x by x_{n-1} of y by x_n in (2.1), we get

$$\begin{aligned}\psi(d(x_n, x_{n+1}) &\leq h\left[\psi(d(x_n, x_{n+1}))\right] \\ &\leq f\left[\psi(M_1(x_{n-1}, x_n)), \varphi(M_1(x_{n-1}, x_n))\right],\end{aligned} \tag{2.4}$$

where

$$M_1(x_{n-1}, x_n) = \max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_n, x_{n-1}), d(x_{n+1}, x_n), \\ \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)) \end{array}\right\}.$$

Since $\{d(x_n, x_{n-1})\}$ is a decreasing sequence we have by the triangular inequality,

$$M_1(x_{n-1}, x_n) \leq \max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_n, x_{n-1}), \\ \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \end{array}\right\}.$$

Now, if for some n_0,

$$\max\{d(x_{n_0-1}, x_{n_0}), d(x_{n_0}, x_{n_0-1}), d(x_{n_0}, x_{n_0+1})\} = d(x_{n_0}, x_{n_0+1}),$$

then (2.4) becomes,

$$\begin{aligned}\psi(d(x_{n_0}, x_{n_0+1}) &\leq h\left[\psi(d(x_{n_0}, x_{n_0+1}))\right] \\ &\leq f\left[\psi(M_1(x_{n_0-1}, x_{n_0})), \varphi(M_1(x_{n_0-1}, x_{n_0}))\right] \\ &\leq \psi(M_1(x_{n_0-1}, x_{n_0})),\end{aligned}$$

since f is a function of C-class. Now, we have

$$\psi(d(x_{n_0}, x_{n_0+1}) \leq \psi(M_1(x_{n_0-1}, x_{n_0})),$$

since ψ is nondecreasing. Therefore, we can say that

$$\psi(d(x_{n_0}, x_{n_0+1}) \leq \psi(M_1(x_{n_0-1}, x_{n_0})) \leq \psi(d(x_{n_0}, x_{n_0+1})),$$

which yields $d(x_{n_0}, x_{n_0+1}) = 0$ and so x_{n_0} is the fixed point.

Hence, for all $n > 0$ we can assume that

$$M_1(x_{n-1}, x_n) = \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\right\}.$$

Now, we will consider the following cases.

CASE 1: If $d(x_{n_i-1}, x_{n_i}) > d(x_{n_i}, x_{n_i-1})$, for only finite i, say for $1 \leq i \leq k$, then if $n > n_k$, we have $d(x_{n-1}, x_n) \leq d(x_n, x_{n-1})$ and from (2.3) we can conclude that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

CASE 2: If for infinitely many n_i we have $d(x_{n_i-1}, x_{n_i}) > d(x_{n_i}, x_{n_i-1})$, then we have

$$M_1(x_{n_i-1}, x_{n_i}) = d(x_{n_i-1}, x_{n_i}),$$

and (2.4) becomes,

$$\begin{aligned}\psi(d(x_{n_i}, x_{n_i+1}) &\leq h\left[\psi(d(x_{n_i}, x_{n_i+1}))\right] \\ &\leq f\left[\psi(d(x_{n_i-1}, x_{n_i})), \varphi(d(x_{n_i-1}, x_{n_i}))\right] \\ &\leq \psi(d(x_{n_i-1}, x_{n_i})),\end{aligned} \tag{2.5}$$

which implies $\{d(x_{n_i-1}, x_{n_i})\}$ is also a decreasing sequence. If we let

$$\lim_{i\to\infty} d(x_{n_i-1}, x_{n_i}) = r,$$

and take limit as $i \to \infty$ in (2.5) we obtain

$$\psi(r) \leq h\left[\psi(r)\right] \leq f\left[\psi(r), \varphi(r)\right],$$

which brings us to

$$\lim_{i\to\infty} d(x_{n_i-1}, x_{n_i}) = 0. \tag{2.6}$$

Let us relabel the elements of the sequence x_n which are not included in the subsequence x_{n_i} as x_{m_i} for $i \in \mathbb{N}$. Then we know that $d(x_{m_i-1}, x_{m_i}) \leq d(x_{m_i}, x_{m_i-1})$. If there are finitely many elements, then the limit does not change in (2.6) for x_n. If there are infinitely many elements let us take the limit as $i \to \infty$,

$$\lim_{i\to\infty} d(x_{m_i-1}, x_{m_i}) \le \lim_{i\to\infty} d(x_{m_i}, x_{m_i-1}) = 0.$$

Considering this equation with (2.6), the sequence $d(x_{n-1}, x_n)$ is a composition of two sequences whose limits are both zero, we can conclude that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.7}$$

Now, we need to prove that $\{x_n\}$ is a right K-Cauchy sequence. Assuming the contrary, there exists $k \in \mathbb{N}$ such that for $\varepsilon > 0$, we can find $n_k > m_k > k$ satisfying $d(x_{n_k}, x_{m_k}) \ge \varepsilon$ and $d(x_{n_k-1}, x_{m_k}) < \varepsilon$. We have

$$\begin{aligned}\varepsilon &\le d(x_{n_k}, x_{m_k})\\ &\le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k})\\ &\le d(x_{n_k}, x_{n_k-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (2.3), we conclude that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon.$$

To obtain the same result for $d(x_{n_k-1}, x_{m_k-1})$ we use the triangle inequality twice as follows;

$$\begin{aligned}d(x_{n_k}, x_{m_k}) &\le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})\\ &\le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k})\\ &\quad + d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}).\end{aligned}$$

By taking $k \to \infty$, in the above inequality and considering (2.3) and (2.6), we obtain that $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon$.

Replacing x by x_{n_k-1} and y by x_{m_k-1} in (2.1) we have

$$\begin{aligned}\psi(d(x_{n_k}, x_{m_k}) &\le h\left[\psi(d(x_{n_k}, x_{m_k}))\right]\\ &\le f\left[\psi(M_1(x_{n_k-1}, x_{m_k-1})), \varphi(M_1(x_{n_k-1}, x_{m_k-1}))\right],\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}M_1(x_{n_k-1}, x_{m_k-1}) &= \max\left\{\begin{array}{c} d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k}, x_{n_k-1}),\\ d(x_{m_k}, x_{m_k-1}),\\ \frac{1}{2}(d(x_{n_k-1}, x_{m_k}) + d(x_{n_k}, x_{m_k-1})) \end{array}\right\}\\ &\le \max\left\{\begin{array}{c} d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k}, x_{n_k-1}),\\ d(x_{m_k}, x_{m_k-1}),\\ \frac{1}{2}[d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k})\\ +d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1})] \end{array}\right\}.\end{aligned}$$

Here, we know

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon,$$

and

$$\begin{aligned}\lim_{k\to\infty} d(x_{n_k}, x_{n_k-1}) &= \lim_{k\to\infty} d(x_{m_k}, x_{m_k-1}) \\ &= \lim_{k\to\infty} d(x_{n_k-1}, x_{n_k}) \\ &= \lim_{k\to\infty} d(x_{m_k-1}, x_{m_k}) \\ &= 0.\end{aligned}$$

Thus we obtain that

$$\lim_{k\to\infty} M_1(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

By taking the limit in (2.8), we have

$$\psi(\varepsilon) \leq h\left[\psi(\varepsilon)\right] \leq f\left[\psi(\varepsilon), \varphi(\varepsilon)\right],$$

which is a contradiction. Thus $\{x_n\}$ is a right K-Cauchy sequence. Since (X, d) is right $\mathcal{K}$-complete, $\{x_n\}$ is d-convergent, i.e. there exists $x \in Y$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. We need to show that x is the fixed point of T. If we replace y by x_n in (2.1) we get

$$\begin{aligned}\psi(d(Tx, x_{n+1}) &\leq h\left[\psi(d(Tx, x_{n+1}))\right] \\ &\leq f\left[\psi(M_1(x, x_n)), \varphi(M_1(x, x_n))\right],\end{aligned} \tag{2.9}$$

where

$$M_1(x, x_n) = \max\left\{\begin{array}{c} d(x, x_n), d(Tx, x), d(x_{n+1}, x_n), \\ \frac{1}{2}(d(x, x_{n+1}) + d(Tx, x_n)), \end{array}\right\}.$$

Since

$$d(Tx, x_n) \leq d(Tx, x) + d(x, x_n)$$

by letting $n \to \infty$ we have $M_1(x, x_n) \to d(Tx, x)$ and (2.9) becomes

$$\psi(d(Tx, x) \leq h\left[\psi(d(Tx, x))\right] \leq f\left[\psi(d(Tx, x)), \varphi(d(Tx, x))\right],$$

which means $d(Tx, x) = 0$. Since X is a T_1-quasi metric space, we can conclude that x is a fixed point of T. Now, we prove that the fixed point is unique. Let $x = Tx$ and $y = Ty$ for $x, y \in Y$ then we know

$$\begin{aligned}M(x, y) &= \max\left\{d(x, y), d(Tx, x), d(Ty, y), \frac{1}{2}(d(x, Ty) + d(Tx, y))\right\} \\ &= \max\left\{d(x, y), d(x, x), d(y, y), \frac{1}{2}(d(x, y) + d(x, y))\right\} \\ &= d(x, y),\end{aligned}$$

and (2.1) becomes

$$\psi(d(x,y)) \leq h\left[\psi(d(x,y))\right] \leq f\left[\psi(d(x,y)), \varphi(d(x,y))\right],$$

which means $d(x,y)=0$ and that guarantees the uniqueness of the fixed point. □

By inserting $M_2(x,y)$ instead of $M_1(x,y)$ in the contraction condition, we can present the following result in a left M-complete quasi metric space.

Theorem 2.2. *Let (X,d) be a left* M*-complete T_1 quasi metric space, $\psi:[0,\infty)\to[0,\infty)$ be an altering distance function, $\varphi:[0,\infty)\to[0,\infty)$ be an ultra altering distance function, $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a function of C-class and $h:[0,\infty)\to[0,\infty)$ be a function of A-class. Let $T:X\to X$ satisfies*

$$h\left[\psi(d(Tx,Ty))\right] \leq f\left[\psi(M_2(x,y)), \varphi(M_2(x,y))\right], \tag{2.10}$$

for $x,y\in Y$ where Y is a closed subset of X invariant under T then T has a fixed point in Y.

Proof. The proof is similar to the first theorem. In this case, we replace x with x_{n-1} and y with x_n in (2.10)

$$\begin{aligned}\psi(d(x_n,x_{n+1}) &\leq h\left[\psi(d(x_n,x_{n+1}))\right] \\ &\leq f\left[\psi(M_2(x_{n-1},x_n)), \varphi(M_2(x_{n-1},x_n))\right],\end{aligned} \tag{2.11}$$

where

$$M_2(x_{n-1},x_n) = \max\left\{\begin{array}{c} d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ \frac{1}{2}(d(x_{n-1},x_{n+1})+d(x_n,x_n)), \end{array}\right\}.$$

Since

$$d(x_{n-1},x_{n+1}) \leq d(x_{n-1},x_n)+d(x_n,x_{n+1})$$

we have

$$M_2(x_{n-1},x_n) = \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}.$$

Using the same idea in the main theorem, we conclude that for all $n>0$

$$\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\} = d(x_{n-1},x_n),$$

means $\{d(x_{n-1},x_n)\}$ is a decreasing sequence and we have both

$$\lim_{n\to\infty} d(x_{n-1},x_n)=0, \tag{2.12}$$

and

$$\lim_{n\to\infty} d(x_n,x_{n-1})=0. \tag{2.13}$$

We should prove that $\{x_n\}$ is a left K-Cauchy sequence. Assuming the contrary, there exists $k\in\mathbb{N}$ such that for $\varepsilon>0$, we can find $n_k>m_k>k$

satisfying $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k}, x_{n_k-1}) < \varepsilon$. We use triangular inequality and sandwich theorem to have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon,$$
$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

We replace x by x_{m_k-1} and y by x_{n_k-1} in (2.10) to obtain

$$\begin{aligned}\psi(d(x_{m_k}, x_{n_k}) &\leq h\left[\psi(d(x_{m_k}, x_{n_k}))\right] \\ &\leq f\left[\psi(M_2(x_{m_k-1}, x_{n_k-1})), \varphi(M_2(x_{m_k-1}, x_{n_k-1}))\right],\end{aligned}$$

where $\lim_{k\to\infty} M_2(x_{m_k-1}, x_{n_k-1}) = \varepsilon$. Taking limit as $k \to \infty$ we have

$$\psi(\varepsilon) \leq h\left[\psi(\varepsilon)\right] \leq f\left[\psi(\varepsilon), \varphi(\varepsilon)\right],$$

which is a contradiction showing that $\{x_n\}$ is a left K-Cauchy sequence. Since (X, d) is left M-complete, $\{x_n\}$ is d^{-1}-convergent, i.e. there exists $x \in Y$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. To show that x is the fixed point of T, we replace x by x_n and y by x in (2.10) we get

$$\begin{aligned}\psi(d(x_{n+1}, Tx) &\leq h\left[\psi(d(x_{n+1}, Tx))\right] \\ &\leq f\left[\psi(M_2(x_n, x)), \varphi(M_2(x_n, x))\right],\end{aligned} \tag{2.14}$$

where

$$M_2(x_n, x) = \max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{1}{2}(d(x_n, Tx) + d(x_{n+1}, x))\right\}.$$

If we let $n \to \infty$ we have $M_2(x_n, x) \to d(x, Tx)$ and (2.14) becomes

$$\psi(d(x, Tx) \leq h\left[\psi(d(x, Tx))\right] \leq f\left[\psi(d(x, Tx)), \varphi(d(x, Tx))\right],$$

which means $d(x, Tx) = 0$. Since X is a T_1-quasi metric space, x is a fixed point of T. Uniqueness of the fixed point can be obtained identically. □

We can easily obtain similar results in the *Smyth* complete spaces since d^s-convergence requires both d and d^{-1}-convergence.

Theorem 2.3. *Let (X, d) be a right or left* Smyth *complete T_1-quasi metric space, $T : X \to X$ be a mapping, $\psi : [0, \infty) \to [0, \infty)$ be an altering distance function, $\varphi : [0, \infty) \to [0, \infty)$ be an ultra altering distance function, $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function of C-class and $h : [0, \infty) \to [0, \infty)$ be a function of A-class. Assume that one of the following conditions hold:*

- *(X, d) is right* Smyth *complete and for all $x, y \in Y$*

$$h\left[\psi(d(Tx, Ty))\right] \leq f\left[\psi(M_1(x, y)), \varphi(M_1(x, y))\right],$$

- (X,d) *is left* Smyth *complete and for all* $x, y \in Y$

$$h\left[\psi(d(Tx,Ty))\right] \le f\left[\psi(M_2(x,y)), \varphi(M_2(x,y))\right],$$

where Y *is a closed subset of* X *invariant under* T, *then* T *has a fixed point in* Y.

3. Conclusion

The metric case including altering, ultra altering distance functions is researched by many authors. The most similar theorem in complete metric spaces, which also includes functions of C-class and A-class without the maximum function is proven in [3].

Theorem 3.1 ([3])**.** *Let* T *be a self mapping defined on a complete metric space* (X,d) *satisfying the condition*

$$h\left[\psi(d(Tx,Ty))\right] \le f\left[\psi(d(x,y)), \varphi(d(x,y))\right],$$

for $x, y \in Y$ *where* Y *is a closed subset of* X *and invariant under* T, ψ *and* φ *are altering distance function and ultra altering distance function, respectively,* f *a function of* C*-class and* h *a function of* A*-class, then* T *has a fixed point in* Y.

If we consider X as a complete metric space, Y as the set X itself, $h(t) = t$ and $f(s,t) = s - t$ and replace $M_1(x,y)$ with $d(x,y)$ in the first theorem, we approach to the Theorem 2.1 in [15].

Theorem 3.2 ([15])**.** *Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a self mapping satisfying*

$$\psi[d(Tx,Ty)] \le \psi[d(x,y)] - \varphi[d(x,y)],$$

where $\psi, \varphi : [0,\infty) \to [0,\infty)$ *are altering distance functions. Then* T *has a unique fixed point in* X.

Assume that X is a complete metric space, Y as the set X itself, $h(t) = t$ and $f(s,t) = a(s)s$ where $a : \mathbb{R}^+ \to [0,1)$ is a decreasing function and replace $M_1(x,y)$ by $d(x,y)$ in the first theorem, we approach to the Theorem 3 in [23].

Theorem 3.3 ([17])**.** *Let* (X,d) *be a complete metric space,* T *be a self map of* X *and* φ *be an altering distance function. Let* a *be a decreasing function from* $\mathbb{R}^+$ *to* $[0,1)$ *such that*

$$\varphi[d(Tx,Ty)] \le a[d(x,y)]\varphi[d(x,y)],$$

where $x, y \in X$ *and* $x \ne y$. *Then* T *has a unique fixed point.*

Now, we present an illustrative example.

Example 3.4. Let $X = [0,1]$ and define a quasi metric on X as

$$d(x,y) = \begin{cases} y - x & , \quad x \leq y \\ \\ 1 & , \quad x > y. \end{cases}$$

Then (X,d) is a T_1-quasi metric space. Also, (X,d) is right $\mathcal{K}$-complete. Indeed, let $\{x_n\}$ be a right K-Cauchy sequence in X. Then for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all n,k, $n \geq k \geq n_0$, $d(x_n,x_k) < \varepsilon$. Choose $0 < \varepsilon < 1$, then by the definition of d, we have $x_k - x_n < \varepsilon$ or $x_n \leq x_k$ for $n \geq k \geq n_0$. Therefore, $\{x_n\}$ is nonincreasing sequence in $[0,1]$ for $n \geq n_0$. Hence there exists $\alpha \in [0,1]$ such that $\lim_{n\to\infty} x_n = \alpha$. In this case, we have $\lim_{n\to\infty} d(\alpha,x_n) = \lim_{n\to\infty}(x_n - \alpha) = 0$, that is, $\{x_n\}$, which is right K-Cauchy sequence, is d-convergent. Therefore, (X,d) is right $\mathcal{K}$-complete. Now, let $Y = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{4} & , \quad x \in Y \\ \\ 1 - x & , \quad \text{otherwise.} \end{cases}$$

Then Y is a closed subset of X and invariant under T. Moreover, since $d(Tx,Ty) = 0$ for all $x,y \in Y$, the contractive condition (2.1) is satisfied for $\psi(t) = t$, $\varphi(t) = \frac{t}{5}$, $f(s,t) = s - t$ and $h(t) = t$.

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