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Generalized Continuous Frames for Operators

Chander Shekhar[1], Sunayana Bhati[2*] and G.S. Rathore[3]

Abstract. In this note, the notion of generalized continuous K-frame in a Hilbert space is defined. Examples have been given to exhibit the existence of generalized continuous K-frames. A necessary and sufficient condition for the existence of a generalized continuous K-frame in terms of its frame operator is obtained and a characterization of a generalized continuous K-frame for $\mathcal{H}$ with respect to μ is given. Also, a sufficient condition for a generalized continuous K-frame is given. Further, among other results, we prove that generalized continuous K-frames are invariant under a linear homeomorphism. Finally, keeping in mind the importance of perturbation theory in various branches of applied mathematics, we study perturbation of K-frames and obtain conditions for the stability of generalized continuous K-frames.

1. Introduction

Frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [12] who used frames as a tool in the study of non-harmonic Fourier series. Daubechies, Grossmann and Meyer [10], reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$. As we know frames are more flexible tools to convey information than bases and so they are suitable replacement for bases in a Hilbert space $\mathcal{H}$. Finding a representation of $x \in \mathcal{H}$ as a linear combination of vectors of a frame, is the main goal of discrete frame theory. But in case of a continuous frame, which is a natural generalization of the discrete case, this property of frame is not straightforward. However, one of the applications of frames is in wavelet theory. In fact,

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* Corresponding author.

the practical implementation of the wavelet transform in signal processing requires the selection of a discrete set of points in the transformed space. Keeping applications in mind, various generalizations of frames were introduced and studied namely Frames of subspaces in Hilbert spaces were first introduced and studied by Casazza and Kutyniok [7] and then in [4, 20]; Pseudo frames were introduced by Li and Ogawa [21]; Oblique frames were first introduced and studied by Eldar [13] and then by Christensen and Eldar [9]; Outer frames were introduced and studied by Aldourbi, Cabrelli and Molter [1] and Bounded quasi-projectors were studied by Fornasier [14]. Sun [25] introduced a more general concept called G-frames and pointed out that most of the above generalizations of frames may be regarded as a special cases of G-frames and many of their basic properties can be derived within this more general setup. Rahimi [22] studied Multipliers of generalized frames in Hilbert spaces and Rahimi and Balazs [23] studies Multipliers for p-Bessel sequences in Banach spaces.

Another generalization of frames was proposed by Kaiser [18] and independently by Ali Tawreque, Antoine and Gazeau [2] who named it as continuous frames while Kaiser used the terminology generalized frames. Recently, Gabardo and Han [15] studied continuous frames and use the terminology (Ω, μ)-frame. Discrete and continuous frames arise in many applications in both pure and applied mathematics and, in particular, they play important roles in digital signal processing and scientific computations. Alizadeh, Faroughi and Rahmani [3] studied continuous $K-G$-frames in Hilbert spaces. Continuous frames were further studied in [5, 11, 19]. For a nice introduction to frames an interested reader may refer to [8] and references therein.

In this paper, we define the notion of generalized continuous K-frame in a Hilbert space and give various characterisations of generalized continuous K-frames in terms of its frame operators. Also, we give a sufficient condition for the existence of generalized continuous K-frames. Further, we obtain a condition under which a generalized tight continuous K-frame for $\mathcal{H}$ yields a generalized continuous for $\mathcal{H}$. Finally, we study perturbation of K-frames and obtain various conditions for the stability of generalized continuous K-frames.

2. Preliminaries

Through this paper, $\mathcal{H}$ denotes seperable Hilbert spaces. L. Găvruţa [16, 17] recently introduced a frame with respect to a bounded linear operator K in a Hilbert space $\mathcal{H}$, which is called K-frame, to reconstruct the elements in the range of K (range of K is denoted by $R(K)$). In fact, they gave the following definition:

Definition 2.1. A system $\{f_k\} \subset \mathcal{H}$ is called K-frame for $\mathcal{H}$ if there exists two positive constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle x, f_k\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}. \tag{2.1}$$

We call A,B the lower frame bound and the upper frame bound for K-frame $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ respectively. If only the upper inequality in (2.1) is satisfied, then $\{f_k\}_{k\in\mathbb{N}}$ is called Bessel sequence.

Găvruţa [16] also proved the following result:

Theorem 2.2 ([16]). *Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ and $K \in B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *$\{f_k\}_{k\in\mathbb{N}}$ is an atomic system for K;*
(ii) *$\{f_k\}_{k\in\mathbb{N}}$ is a K-frame for $\mathcal{H}$;*
(iii) *there exists a Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that*

$$Kx = \sum_{k\in\mathbb{N}} \langle x, g_k\rangle f_k, \quad \forall x \in \mathcal{H}.$$

We call the Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ as the K-dual frame of the K-frame $\{f_k\}_{k\in\mathbb{N}}$.

Theorem 2.3 ([6]). *Let $\mathcal{H}$ be a Hilbert space and $S, K \in B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *$R(K) \subseteq R(S)$.*
(ii) *$\lambda KK^* \leq SS^*$for some $\lambda > 0$.*
(iii) *$K = SQ$ for some $Q \in B(\mathcal{H})$.*

Let $B_{\mathcal{H}}$ be the collection of all Bessel sequences in a Hilbert space $\mathcal{H}$. Let I be an at most countable index set. The following definition of a generalized continuous frame introduced and studied in [11] .

Definition 2.4. Let $\mathcal{H}$ be a complex Hilbert space, $K \in B(\mathcal{H})$ and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to B_{\mathcal{H}}; \omega \to \{f_i(\omega)\}_{i\in I}$ is called a *generalized continuous frame* with respect to (Ω, μ) if:

(i) F is weakly measurable, i.e., for all $f \in \mathcal{H}, i \in I, \omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω;
(ii) there exist positive constants A, B such that

$$A\|f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.2}$$

The positive constants A and B are called generalized continuous frame bounds. F is called *A-tight generalized continuous frame* if condition (i) holds and

$$A\|f\|^2 = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2, \quad f \in \mathcal{H}.$$

The mapping F is called *Bessel* if the upper inequality in (2.2) holds. In this case, B is called the Bessel bound. If the cardinality of I is one, F is a continuous frame and if, further, μ is a counting measure and $\Omega := \mathbb{N}$, F is called a discrete frame.

3. Main Results

We began this section with the following definition:

Definition 3.1. Let $\mathcal{H}$ be a complex Hilbert space, $K \in B(\mathcal{H})$ and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to B_{\mathcal{H}}; \omega \to \{f_i(\omega)\}_{i\in I}$ is called a generalized continuous K-frame with respect to (Ω, μ) if:

(i) F is weakly measurable, i.e., for all $f \in \mathcal{H}, i \in I, \omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω;

(ii) there exist positive constants A and B such that

$$A\|K^*f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{3.1}$$

The positive constants A and B are called generalized continuous K-frame bounds. F is called A-tight generalized continuous K-frame if condition (i) holds and

$$A\|K^*f\|^2 = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2, \quad \text{for all } f \in \mathcal{H}.$$

The mapping F is called Bessel if the upper inequality in (3.1) holds. In this case, B is called the Bessel bound. If the cardinality of I is one, F is a continuous K-frame and if, further, μ is a counting measure and $\Omega := \mathbb{N}$, F is called a discrete K-frame. Let λ be a counting measure and F be Bessel with bound B.

The analysis operator associated with F is defined by

$$U_F : \mathcal{H} \to L^2(\Omega \times I, \mu \times \lambda), \qquad U_F f(\omega, i) = \langle f, f_i(\omega)\rangle, \tag{3.2}$$

and the synthesis operator associated with F is defined as

(3.3)

$$U_F^* : L^2(\Omega \times I, \mu \times \lambda) \to \mathcal{H}, \qquad U_F^*\phi = \int_\Omega \sum_{i\in I} \phi(\omega, i) f_i(\omega) d\mu(\omega).$$

The frame operator for generalized continuous K-frame is defined as

$$S_F f = \int_\Omega \sum_{i\in I} \langle f, f_i(\omega)\rangle f_i(\omega) d\mu(\omega), \quad \text{for all } f \in \mathcal{H}.$$

Next, we give examples to exhibit the existence of generalized continuous K-frames.

Example 3.2. Let $\mathcal{H} = \ell^2(\mathbb{N}), I = \mathbb{N}, \Omega = (0,1)$, $K \in B(\mathcal{H})$, μ be the Lebesgue measure and $\{e_k\}$ be an ONB for $\mathcal{H}$. For $x \in \Omega$, define $\{f_k(x)\}_{k\in\mathbb{N}}$ as $f_k(x) = \sqrt{x}Ke_k$. Then, for any $x \in \Omega$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle f, f_k(x)\rangle|^2 &= \sum_{k=1}^{\infty} |\langle f, \sqrt{x}Ke_k\rangle|^2 \\ &= x\sum_{k=1}^{\infty} |\langle K^* f, e_k\rangle|^2 \\ &= x\|K^* f\|^2, \quad \text{for all } f \in \mathcal{H}.\end{aligned}$$

Hence, the sequence $\{f_k(x)\}_{k\in\mathbb{N}}$ is a tight K-frame for $\mathcal{H}$ with frame bounds $A_x = B_x = x$. Also, $\{\{f_k(x)\}_{k\in\mathbb{N}} : x \in \Omega\}$ is a generalized continuous K-frame for $\mathcal{H}$.

Example 3.3. Let $\mathcal{H} = \ell^2(\mathbb{N}), I = \mathbb{N}, \Omega = (0,1)$, $K \in B(\mathcal{H})$, μ be the Lebesgue measure and $\{e_k\}$ be ONB for $\mathcal{H}$. For $x \in \Omega$, define $\{f_k(x)\}_{k\in\mathbb{N}}$ as $f_k(x) = \sqrt{x}(e_k + e_{k+1})$ and $K : \mathcal{H} \to \mathcal{H}$ by

$$Kf = \sum_{k=1}^{\infty} \langle f, e_k\rangle (e_k + e_{k+1}).$$

Clearly

$$K^* f = \sum_{k=1}^{\infty} \langle f, e_k + e_{k+1}\rangle e_k.$$

Then, for any $x \in \Omega$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle f, f_k(x)\rangle|^2 &= \sum_{k=1}^{\infty} |\langle f, \sqrt{x}(e_k + e_{k+1})\rangle|^2 \\ &= x\sum_{k=1}^{\infty} |\langle f, e_k + e_{k+1}\rangle|^2 \\ &= x\|K^* f\|^2, \quad \text{for all } f \in \mathcal{H}.\end{aligned}$$

Hence, the sequence $\{f_k(x)\}_{k\in\mathbb{N}}$ is tight K-frame for $\mathcal{H}$ with frame bounds $A_x = B_x = x$. Also, we have

$$\int_\Omega \sum_{k\in\mathbb{N}} |\langle f, f_k(x)\rangle|^2 = \frac{1}{2}\|K^* f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

Therefore $\{\{f_k(x)\}_{k\in\mathbb{N}} : x \in \Omega\}$ is a generalized continuous K-frame for $\mathcal{H}$.

In the following result, we give a necessary and sufficient condition for a generalized continuous K-frame by using the synthesis operator.

Theorem 3.4. *Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space and $K \in B(\mathcal{H})$. A vector-valued mapping $F : \Omega \to B_{\mathcal{H}}; \omega \to \{f_i(\omega)\}_{i\in I}$ is called a generalized continuous K-frame with respect to (Ω, μ) if and only if the operator T_F^* as defined in (3.3) is well defined, bounded and $R(K) \subset R(T_F^*)$.*

Proof. Let $F = \{\{f_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ be generalized continuous K-frame for $\mathcal{H}$ with respect to μ. Then

$$A\|K^* f\|^2 \le \|T_F f\|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

This gives $\|T_F^*\| < \sqrt{B}$ and $AKK^* \le T_F^* T_F$. Then, by Theorem 2.2, there exists an operator $Q \in B(\mathcal{H}, L^2(\Omega\times I, \mu\times\lambda))$ such that $K = T_F^* Q$. Hence T_F^* is well defined, bounded and $R(K) \subset R(T_F^*)$. Conversely, since T_F^* is well defined and bounded, $F = \{\{f_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ is a Bessel family in $\mathcal{H}$. Also, $R(K) \subset R(T_F^*)$ so $AKK^* \le T_F^* T_F$. Hence

$$\begin{aligned} A\|K^* f\|^2 &\le \|T_F f\|^2 \\ &= \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2. \end{aligned}$$

which implies $F = \{\{f_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$. □

Next, we give another characterization of a generalized continuous K-frame for $\mathcal{H}$ with respect to μ.

Theorem 3.5. *Let $F = \{\{f_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ with respect to μ. Let $G : \Omega \to B_{\mathcal{H}};\ \omega \to \{g_i(\omega)\}_{i\in I}$ be a weakly measurable function. Then, the following statements are equivalent:*

(i) *$G = \{\{g_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$ with respect to μ.*

(ii) *There exists $\lambda > 0$ such that*

$$\begin{aligned} &\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - g_i(\omega)\rangle|^2 \\ &\qquad \le \lambda \min\left\{ \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2, \int_\Omega \sum_{i\in I} |\langle f, g_i(\omega)\rangle|^2 \right\}. \end{aligned}$$

Proof. For all $f \in \mathcal{H}$, we have

$$\begin{aligned}
A_F\|K^*f\|^2 &\leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \\
&= \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) + g_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\leq 2\left(\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega \sum_{i\in I} |\leq f, g_i(\omega)\rangle|^2\right) \\
&\leq 2(M+1)\int_\Omega \sum_{i\in I} |\langle f, g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\leq 4(M+1)\left(\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega)\right. \\
&\quad \left.+ \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega)\right) \\
&\leq 4(M+1)^2 \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \\
&\leq 4(M+1)^2 B_f\|f\|^2.
\end{aligned}$$

Hence, we obtain

$$\frac{A_F}{2(M+1)}\|f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq 2(M+1)\|f\|^2.$$

Conversely, Let A_F, B_F and A_G, B_G be frame bounds of generalized continuous K-frames $F = \{\{f_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ and $G = \{\{g_i(\omega)\}_{i\in I}\}_{\omega\in\Omega}$ respectively. Then, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}
&\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\quad = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - |\langle f, g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\quad \leq 2\left(\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega \sum_{i\in I} |f, g_i(\omega)\rangle|^2 d\mu(\omega)\right) \\
&\quad \leq 2\left(\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) + B_G\|f\|^2\right) \\
&\quad \leq 2\left(1 + \frac{B_G}{A_F}\right)\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega).
\end{aligned}$$

Similarly, we can show that

$$\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \leq 2\left(1 + \frac{B_G}{A_F}\right) \int_\Omega \sum_{i\in I} |\langle f, g_i(\omega)\rangle|^2 d\mu(\omega).$$

□

In the following result, we give a sufficient condition for the existence of a generalized continuous K-frame.

Theorem 3.6. *Let $K, L \in B(\mathcal{H})$ and $F = \{F(\omega)\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ with best bounds A and B.*

(i) *If $Q : \mathcal{H} \to \mathcal{H}$ is a co-isometry such that $KQ = QK$, then $\{Qf_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame with same best bounds.*

(ii) *If $R(L) \subseteq R(K)$ is such that K has closed range, then $F = \{f_i(\omega)\}_{\omega\in\Omega}$ is also a generalized continuous L-frame for $\mathcal{H}$.*

Proof. (i) Since $F : \Omega \to \mathcal{H}$ is weakly measurable i.e. the map $\omega \to \langle f_i(\omega), x\rangle$ from Ω into $\mathbb{C}$ is measurable for all $x \in \mathcal{H}$. So the map $\omega \to \langle Qf_i(\omega), x\rangle$ from Ω into $\mathbb{C}$ is also measurable for all $x \in \mathcal{H}$. Now, for all $x \in \mathcal{H}$, we have

$$\int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 \leq B\|x\|^2.$$

Also, we have

$$\begin{aligned}\int_\Omega \sum_{i\in I} |\langle x, Qf_i\omega\rangle|^2 &\geq A\|K^*Q^*x\|^2 \\ &= A\|K^*x\|^2, \quad \text{for all } x \in \mathcal{H}.\end{aligned}$$

Hence, $\{Qf_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame with same best bounds.

(ii) Straight forward.

□

Proposition 3.7. *Let $K \in B(\mathcal{H})$ and let $\{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized continuous frame for $\mathcal{H}$ with frame bounds A and B. Then $\{Kf_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame with frame bounds A and $B\|K\|^2$.*

Proof. Since $\{f_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous frame for $\mathcal{H}$ with frame bounds A and B, we have

$$A\|f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Thus, we get

$$\begin{aligned}\int_\Omega \sum_{i\in I} |\langle f, Kf_i(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega \sum_{i\in I} |\langle K^*f, f_i(\omega)\rangle|^2 d\mu(\omega)\\ &\geq A\|K^*f\|^2, \quad \forall f\in\mathcal{H}.\end{aligned}$$

Also, we have

$$\begin{aligned}\int_\Omega \sum_{i\in I} |\langle f, Kf_i(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega \sum_{i\in I} |\langle K^*f, f_i(\omega)\rangle|^2 d\mu(\omega)\\ &\leq B\|K^*f\|^2\\ &\leq B\|K\|^2\|f\|^2, \quad \forall f\in\mathcal{H}.\end{aligned}$$

□

Proposition 3.8. *Let $K, L \in B(\mathcal{H})$, $R(K) \subseteq R(L)$ and $\{f_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous frame for $\mathcal{H}$ with frame bounds A and B, then $\{Lf_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame.*

Proof. Straight forward. □

Next, we obtain a condition under which a generalized tight continuous K-frame for $\mathcal{H}$ yields a generalized continuous frame for $\mathcal{H}$.

Theorem 3.9. *Let $K \in B(\mathcal{H})$ and $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized tight continuous K-frame for $\mathcal{H}$. Then $F = \{f_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous frame if and only if K is surjective.*

Proof. Let $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized tight continuous K-frame for $\mathcal{H}$ with frame bound A. Then, we have

$$A\|K^*x\|^2 = \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2, \quad \forall x\in\mathcal{H}.$$

Since $F = \{f_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous frame for $\mathcal{H}$, there exists a constant $C > 0$ such that $C\|x\| \leq \|K^*x\|, \quad x \in \mathcal{H}$. Hence K is surjective.

Conversely, let K is surjective. Then there exists $C > 0$ and $D > 0$ such that

$$C\|x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 \leq D\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Hence $F = \{f_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous frame for $\mathcal{H}$. □

Next, we show that generalized continuous K-frames are invariant under a linear homeomorphism.

Theorem 3.10. *Let $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ with best bounds A_1, B_1 and $Q : \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism that commutes with K^*. Then $\{Q(F(\omega))\}_{\omega\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$ and its best bounds A_2, B_2 satisfy the inequalities*

$$A_1\|Q\|^{-2} \leq A_2 \leq A_1\|Q^{-1}\|^2,$$
$$B_1\|Q\|^{-2} \leq B_2 \leq B_1\|Q\|^2.$$

Proof. Clearly, for all $f \in \mathcal{H}$, $\omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω. Now, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega \sum_{i\in I} |\langle Q^*x, f_i(\omega)\rangle|^2 d\mu(\omega)\\ &\leq B_1\|Q\|^2\|x\|^2.\end{aligned}$$

Also

$$\begin{aligned}\|K^*x\|^2 &= \|K^*(QQ^{-1}x)\|^2\\ &\leq \|Q\|^2\|K^*(Q^{-1}x)\|^2\\ &\leq \frac{\|Q\|^2}{A_1}\int_\Omega \sum_{i\in I} |\langle Q^{-1}x, f_i(\omega)\rangle|^2 d\mu(\omega)\\ &= \frac{\|Q\|^2}{A_1}\int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 d\mu(\omega).\end{aligned}$$

This gives

$$A_1\|Q\|^{-2}\|x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 d\mu(\omega).$$

Therefore, we get

$$A_1\|Q\|^{-2}\|K^*x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 d\mu(\omega) \leq B_1\|x\|^2.$$

Now, for all $x \in \mathcal{H}$, we have

$$A_2\|K^*x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, Qf_i(\omega)\rangle|^2 d\mu(\omega) \leq B_2\|x\|^2.$$

This gives

$$A_2\|Q^{-1}\|^{-2} \leq A_1, \qquad B_1 \leq B_2\|Q\|^2.$$

Hence

$$A_1\|Q\|^{-2} \leq A_2 \leq A_1\|Q^{-1}\|^2, \qquad B_1\|Q\|^{-2} \leq B_2 \leq B_1\|Q\|^2.$$

□

4. Perturbation of Generalized Continuous K-frames

Perturbation theory is an important tool in many branches of applied mathematics. In this section, we study perturbation of generalized continuous K-frames. We begin with the following result that gives a sufficient condition for the perturbation of a generalized continuous K-frame.

Theorem 4.1. *Let $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ and let $G : \Omega \to B_{\mathcal{H}}, \omega \to \{g_i(\omega)\}_{i\in I}$ be a weakly measurable function. Then $G = \{g_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$ if there exists $M > 0$ such that*

$$\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega)$$
$$\leq M \min\left\{\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega), \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right\}.$$

Converse part holds if K is a co-isometry.

Proof. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}
A_F\|K^*x\|^2 &\leq \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\\
&\leq 2\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle - \langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right.\\
&\quad \left.+ \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right)\\
&\leq 2\left(M\int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right)\\
&\leq (2M+1)\int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\\
&\leq (4M+1)\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle - \langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right.\\
&\quad \left.+ \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\right)\\
&\leq (4M+1)^2\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)
\end{aligned}$$

$$\leq (4M+1)^2 B_F \|x\|^2.$$

This gives

$$\frac{A_F}{2(M+1)}\|K^*x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega) \leq (2M+1)B_F\|x\|^2.$$

Hence $\{g_i(\omega)\}_{\omega\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$.
Conversely, let A_F,B_F and A_G, B_G be frame bounds for the generalized continuous K-frame $\{f_i(\omega)\}_{\omega\in\Omega}$ and $\{g_i(\omega)\}_{\omega\in\Omega}$ respectively. Then, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
&\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\quad\leq 2\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right) \\
&\quad\leq 2\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega) + B_G\|x\|^2\right) \\
&\quad\leq 2\left(1+\frac{B_G}{A_F}\right)\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega).
\end{aligned}$$

Similarly, we can show that

$$\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \leq 2\left(1+\frac{B_F}{A_G}\right)\int_\Omega |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega).$$

Hence, we get $M > 0$ such that

$$\begin{aligned}
&\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \\
&\quad\leq M \min\left\{\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega), \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right\},
\end{aligned}$$

where $M = \max\left\{2\left(1+\frac{B_G}{A_F}\right), 2\left(1+\frac{B_F}{A_G}\right)\right\}$. □

Next, we give a sufficient condition for the existence of a generalized continuous K-frame for $\mathcal{H}$.

Theorem 4.2. *Let $K \in B(H)$ be surjective with closed range and $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ with frame bounds A and B. Let $G : \Omega \to\ B_{\mathcal{H}}, \omega \to \{g_i(\omega)\}_{i\in I}$ be a weakly measurable function. Assume that there exists constants $\alpha, R, \mu \geq 0$ with*

$0 \leq \alpha + \frac{R+\mu\|K^\dagger\|}{A} < 1$, *where* $K^\dagger$ *is pseudo inverse of* K *and such that*

$$\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega)$$
$$\leq \alpha \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega) + R\|K^* x\|^2 + \mu\|x\|^2, \quad \forall x \in \mathcal{H}.$$

Then $G = \{g_i(\omega)\}_{\omega\in\Omega}$ *is generalized continuous* K*-frame for* $\mathcal{H}$ *with frame bounds*

$$A\left(1 - \sqrt{\alpha + \frac{R + \mu\|K^\dagger\|^2}{A}}\right)^2, \qquad B\left(1 + \sqrt{\alpha + \frac{\mu + R\|K\|^2}{B}}\right).$$

Proof. Using Minkowski inequality, we have

$$\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2} \leq \left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}$$
$$+ \left(\int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}, \quad \forall x \in \mathcal{H}.$$

This gives

$$\left(\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}$$
$$\leq \left(\alpha \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega) + R\|K^* x\|^2 + \mu\|x\|^2\right)^{1/2}$$
$$+ \left(\int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}, \quad \forall x \in \mathcal{H}.$$

Since $R(K)$ is closed, there exists pseudo inverse $K^\dagger$ of K such that

$$\|x\|^2 \leq \frac{\|K^\dagger\|}{A} \int_\Omega \sum_{i\in I} |\langle x, f_i(\omega)\rangle|^2 d\mu(\omega), \quad x \in R(K).$$

Thus, for each $x \in R(K)$, we get

$$A\left(1 - \sqrt{\alpha + \frac{R + \mu\|K^\dagger\|^2}{A}}\right)^2 \|K^* x\|^2 \leq \int_\Omega \sum_{i\in I} |\langle x, g_i(\omega)\rangle|^2 d\mu(\omega).$$

Also, we have

$$\begin{aligned}
&\left(\int_{\Omega}\sum_{i\in I}|\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}\\
&\quad\le\left(\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)-g_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}\\
&\qquad+\left(\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}\\
&\quad\le\left(\alpha\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)+R\|K^*x\|^2+\mu\|x\|^2\right)^{1/2}\\
&\qquad+\left(\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}\\
&\quad\le\left(\sqrt{\alpha B+R\|K\|^2+\mu}+\sqrt{B}\right)\|x\|,\quad \forall x\in R(K).
\end{aligned}$$

Hence, for each $x \in R(K)$, we have

$$\begin{aligned}
A\left(1-\sqrt{\alpha+\frac{R+\mu\|K^{\dagger}\|^2}{A}}\right)^2\|K^*x\|^2 &\le \int_{\Omega}\sum_{i\in I}|\langle x, g_i(\omega)\rangle|^2 d\mu(\omega)\\
&\le B\left(1+\sqrt{\alpha+\frac{\mu+R\|K\|^2}{B}}\right)^2\|x\|^2.
\end{aligned}$$

□

Finally, we give the following corollaries of Theorem 4.2 :

Corollary 4.3. *Let $F = \{f_i(\omega)\}_{\omega\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ with frame bounds A and B. Let $G : \Omega \to B_{\mathcal{H}}, \omega \to \{g_i(\omega)\}_{i\in I}$ be a weakly measurable function. Assume that there exists constants $\alpha, R \ge 0$ such that $0 \le \alpha + \frac{R}{A} < 1$ and*

$$\begin{aligned}
&\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)-g_i(\omega)\rangle|^2 d\mu(\omega)\\
&\quad\le \alpha\int_{\Omega}\sum_{i\in I}|\langle x, f_i(\omega)\rangle|^2 d\mu(\omega)+R\|K^*x\|^2,\quad \forall\ x\in\mathcal{H}.
\end{aligned}$$

Then $G = \{g_i(\omega)\}_{\omega\in\Omega}$ *is generalized continuous* K*-frame for* $\mathcal{H}$ *with frame bounds*

$$A\left(1-\sqrt{\alpha+\frac{R}{A}}\right)^2, \qquad B\left(1+\sqrt{\alpha+\frac{R\|K\|^2}{B}}\right)^2.$$

Proof. Take $\mu = 0$ in Theorem 4.2. □

Corollary 4.4. *Let* $F = \{f_i(\omega)\}_{\omega\in\Omega}$ *be a generalized continuous* K*-frame for* $\mathcal{H}$ *with frame bounds* A *and* B*. Let* $G : \Omega \to B_{\mathcal{H}}, \omega \to \{g_i(\omega)\}_{i\in I}$ *be a weakly measurable function. Assume that there exists constant* R *such that* $0 \le R < A$ *and*

$$\int_\Omega \sum_{i\in I} |\langle x, f_i(\omega) - g_i(\omega)\rangle|^2 d\mu(\omega) \le R\|K^*x\|^2, \quad \forall x \in \mathcal{H}.$$

Then $\{g_i(\omega)\}_{\omega\in\Omega}$ *is generalized continuous* K*-frame for* $\mathcal{H}$ *with frame bounds*

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2, \qquad B\left(1+\sqrt{\frac{R\|K\|^2}{B}}\right)^2.$$

Proof. Take $\alpha = 0$ and $\mu = 0$ in Theorem 4.2. □

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[1] Department of Mathematics Indraprastha college for Women, University of Delhi, Delhi 110054, India.

E-mail address: shekhar.hilbert@gmail.com

[2] Department of Mathematics and Statistics, University college of Science, M.L.S. University, Udaipur, Rajasthan, India.

E-mail address: bhatisunayana@gmail.com

[3] Department of Mathematics and Statistics, University college of Science, M.L.S. University, Udaipur, Rajasthan, India.

E-mail address: ghanshyamsrathore@yahoo.co.in