Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 2 (2020), 37-53
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.86797.440

Some Fixed Point Theorems in Generalized Metric Spaces Endowed with Vector-valued Metrics and Application in Linear and Nonlinear Matrix Equations

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Abstract. Let $\mathcal{X}$ be a partially ordered set and d be a generalized metric on $\mathcal{X}$. We obtain some results in coupled and coupled coincidence of g-monotone functions on $\mathcal{X}$, where g is a function from $\mathcal{X}$ into itself. Moreover, we show that a nonexpansive mapping on a partially ordered Hilbert space has a fixed point lying in the unit ball of the Hilbert space. Some applications for linear and nonlinear matrix equations are given.

1. Introduction

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V} : \theta \preceq v\}$, where θ is the zero-vector of $\mathcal{V}$, satisfies the usual properties

(i) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$;
(ii) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$;
(iii) $\alpha\mathcal{V}_+ \subset \mathcal{V}_+$, for $\alpha \geq 0$.

Let $\mathcal{X}$ be a nonempty set. A mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{V}$ is called a vector-valued metric on X, if the following properties are satisfied:

(i) $d(x, y) \succeq \theta$ for each $x, y \in \mathcal{X}$, if $d(x, y) = \theta$, then $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

The pair $(\mathcal{X}, d)$ is called the vector-valued metric space. Similarly, we can define a generalized normed space.

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Fixed points, Coupled fixed point, Coupled coincidence fixed Point, Generalized metric.
Received: 24 May 2018, Accepted: 30 July 2018.
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A set $\mathcal{X}$ equipped with a vector-valued metric d is called a generalized metric space and denoted by $(\mathcal{X}, d)$. By $M_{m,m}(\mathbb{R}^+)$, we mean the set of all $m \times m$ matrixes with positive elements. We denote by I the identity $m \times m$ matrix. Let $A \in M_{m,m}(\mathbb{R}^+)$, A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$ (for more details see [10]).

Let $\alpha, \beta \in \mathbb{R}^m$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c \in \mathbb{R}$. Note that $\alpha \leq \beta$ (resp. $\alpha < \beta$) means $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$, and also $\alpha \leq c$ (resp. $\alpha < c$) means $\alpha_i \leq c$ (resp. $\alpha_i < c$) for $1 \leq i \leq m$, respectively. We can define addition and multiplication on $\mathbb{R}^m$ as follows:

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_m + \beta_m),$$

and

$$\alpha \cdot \beta = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_m \beta_m).$$

In this paper, we need the following equivalent statements:

(i) A is convergent towards zero;
(ii) $A^n \to 0$ as $n \to \infty$;
(iii) The eigenvalues of A are located in the open unit disc, that is, $|\lambda| < 1$, for each $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;
(iv) The matrix $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ;$$

(v) $A^n q^T \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$, where q^T is the transpose of q.

The above statements are the classical results in matrix analysis (for more details see [1, 5, 9]). Denote , by $\mathcal{ZM}$ the set of all matrices $A \in M_{m,m}(\mathbb{R}^+)$ such that A^n converges to zero. Let $(\mathcal{X}, d)$ be a generalized metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping. For a given $A \in \mathcal{ZM}$, we call the function mapping T is an A-nonexpansive if $d(T(x), T(y)) \leq Ad(x, y)$ for all $x, y \in X$ and T to be said to be $\mathcal{ZM}$-nonexpansive if for any B in $\mathcal{ZM}$, T is a B-nonexpansive function.

Clearly, if $A \in \mathcal{ZM}$, then there exists a norm $||.||$ such that $\|A\| < 1$, so every $\mathcal{ZM}$-nonexpansive operator is nonexpansive, but the converse is not true, in general.

Fixed point theorems on spaces endowed with vector-valued metrics considered by Filip and Petruşel in [3] and some new results around this notion are obtained in [4].

Definition 1.1 ([2]). Let $(\mathcal{X}, \preceq)$ be a partially ordered set and let $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. The mapping F is said to be has the *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for every $x, y \in \mathcal{X}$,

(i) for each $x_1, x_2 \in \mathcal{X}$, if $x_1 \preceq x_2$, then $F(x_1, y) \preceq F(x_2, y)$;

(ii) for each $y_1, y_2 \in \mathcal{X}$, if $y_1 \preceq y_2$, then $F(x, y_1) \succeq F(x, y_2)$.

Let $(\mathcal{X}, \preceq)$ be a partially ordered set and d be a metric on $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. The product space $\mathcal{X} \times \mathcal{X}$ is endowed with the following partial order:

$$\text{for,} \qquad (x,y),(u,v) \in \mathcal{X} \times \mathcal{X}, \qquad (u,v) \leq (x,y) \quad \Leftrightarrow \quad x \geq u, y \leq v.$$

Definition 1.2 ([2])**.** Let $(\mathcal{X}, \preceq)$ be a partially ordered set and let $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a mapping. An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled fixed point of the mapping F, if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.3. An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called

(i) a coupled coincidence point of mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called a coupled point of coincidence.

(ii) a common coupled fixed point of mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.4. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be two self mappings. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for all $x_1, x_2 \in \mathcal{X}$, $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in \mathcal{X}$, and for all $y_1, y_2 \in \mathcal{X}$, $gy_1 \succeq gy_2$ implies $F(x, y_1) \preceq F(x, y_2)$ for all $x \in \mathcal{X}$.

Definition 1.5. Let $\mathcal{X}$ be a non-empty set. We say that the mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ are commutative if $g(F(x, y)) = F(gx, gy)$, for all $x, y \in \mathcal{X}$.

Bhaskar and Lakshmikantham in [2], studied the existence of coupled fixed points for continuous mapping with the mixed monotone property $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, where $(\mathcal{X}, \preceq)$ is a partially ordered set. The existence of coupled fixed point for a mapping with the mixed monotone property $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, where $(\mathcal{X}, d)$ is a complete generalized metric space, is considered in [7].

In this paper, we consider the existence and uniqueness of coupled fixed points for mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, under some contractive conditions, where $(\mathcal{X}, d)$ is a complete generalized metric space.

2. Main Results

We say that $\mathcal{X}$ satisfies in condition (NDI) if $\mathcal{X}$ has the following properties:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n.

(ii) if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.

Theorem 2.1. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $(\mathcal{X}, d)$ be a complete generalized metric space which satisfies the condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, and let $g : \mathcal{X} \to \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$. Suppose that $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfies the following condition*

$$d(F(x, y), F(u, v)) \leq Ad(gx, gu) + Bd(gy, gv), \tag{2.1}$$

where $A = (a_{ij}), B = (b_{ij})$ are in $M_{m\times m}(\mathbb{R}^+), (A + B) \in \mathcal{ZM}$, A and B are nonzero matrices in $\mathcal{ZM}$. Furthermore, assume that F and g satisfy the following conditions

(i) *$F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$,*
(ii) *$g(\mathcal{X})$ is a complete subspace of $\mathcal{X}$,*
(iii) *F satisfies the mixed g-monotone property.*

If there exist $x_0, y_0 \in \mathcal{X}$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then F and g has a unique coupled coincidence fixed point.

Proof. Let $x_0, y_0 \in \mathcal{X}$ be such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$. Since $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, we can choose $x_2, y_2 \in \mathcal{X}$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since F satisfying the mixed g-monotone property, we have $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_2 \preceq gy_1 \preceq gy_0$. By continuing this process, we can construct two sequences (x_n) and (y_n) in $\mathcal{X}$ such that $gx_n = F(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n) \preceq gy_n = F(y_{n-1}, x_{n-1})$. Further, for $n = 1, 2, ...$, by (2.1), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq Ad(gx_{n-1}, gx_n) + Bd(gy_{n-1}, gy_n), \end{aligned}$$

and similarly,

$$\begin{aligned} d(gy_n, gy_{n+1}) &= d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq Ad(gy_{n-1}, gy_n) + Bd(gx_{n-1}, gx_n). \end{aligned}$$

Therefore, by letting $d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$, we have

$$\begin{aligned} d_n &= d \\ &\leq f(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \\ &\leq Ad(gx_{n-1}, gx_n) + Bd(gy_{n-1}, gy_n) \\ &\quad + Ad(gy_{n-1}, gy_n) + Bd(gx_{n-1}, gx_n) \\ &\leq (A + B)(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \\ &\leq (A + B) d_{n-1}. \end{aligned}$$

If we set $C = A + B$, then for all $n \in N$, we have

$$0 \leq d_n \leq Cd_{n-1} \leq C^2 d_{n-2} \leq \cdots \leq C^n d_0. \tag{2.2}$$

If $d_0 = 0$ then (x_0, y_0) is a coupled fixed point of F. Now, let $d_0 > \theta$. For each $n \geq m$, we have

$$d(gx_n, gx_m) \leq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m-1}, gx_m),$$

and

$$d(gy_n, gy_m) \leq d(gy_n, gy_{n-1}) + d(gy_{n-1}, gy_{n-2}) + \cdots + d(gy_{m-1}, gy_m).$$

We have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\leq d_{n-1} + d_{n-2} + d_{n-3} + \cdots + d_m \\ &\leq \left(C^{n-1} + C^{n-2} + \cdots + C^m\right) d_0 \\ &\leq \left(C^{n-1} + C^{n-2} + \cdots + C^m + \cdots\right) d_0 \\ &\leq C^m (I - C)^{-1} d_0. \end{aligned}$$

So

$$d(gx_n, gx_{n+1}) \leq (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)),$$

and

$$d(gy_n, gy_{n+1}) \leq (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)).$$

Let $m, n \in N$ with $m > n$. Since

$$d(gx_n, gx_m) \leq \sum_{i=n}^{m-1} d(gx_i, gx_{i+1}),$$

thus,

$$d(gx_n, gx_m) \leq (I - A - B)^{-1} (A + B)^n (d(gx_0, gx_1) + d(gy_0, gy_1)),$$

which implies that $\{gx_n\}$ is a Cauchy sequence in $g(\mathcal{X})$, and similarly $\{gy_n\}$ is a Cauchy sequence in $g(\mathcal{X})$. Since $g(\mathcal{X})$ is a complete metric space, there exist $gx, gy \in g(\mathcal{X})$ such that $\lim_{n\to\infty} gx_n = gx$ and $\lim_{n\to\infty} gy_n = gy$. Also

$$\begin{aligned} d(F(x,y), gx) &\leq d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &= d(F(x,y), F(x_n, y_n) + d(gx_{n+1}, gx)) \\ &\leq Ad(gx_n, gx) + Bd(gy_n, gy) + d(gx_{n+1}, gx). \end{aligned}$$

Therefore, $d(F(x,y), gx) = \theta$, and so $F(x,y) = gx$. Similarly, $F(y,x) = gy$, that is (gx, gy) is a coupled coincidence fixed point of F and g. Now, if (gx', gy') is another coupled coincidence fixed point of F and g, then

$$d\left(gx', gx\right) = d\left(F\left(x', y'\right), F(x,y)\right) \leq Ad\left(gx', gx\right) + Bd\left(gy', gy\right),$$

and

$$d\left(gy', gy\right) = d\left(F\left(y', x'\right), F\left(y, x\right)\right) \leq Ad\left(gy', gy\right) + Bd\left(gx', gx\right).$$

Then

$$d\left(gx', gx\right) + d\left(gy', gy\right) \leq (A+B)\, d\left(gx', gx\right) + d\left(gy', gy\right).$$

It follows that $d(gx', gx) + d(gy', gy)(I - C) \leq \theta$. Since $C \neq I$,(2.8) implies that $d(gx', gx) + d(gy', gy) = \theta$. Hence, we have $(gx', gy') = (gx, gy)$. □

It is a worth notice that when the matrices A and B in Theorem 2.1 are equal, we have the following result.

Corollary 2.2. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $(\mathcal{X}, d)$ be a complete generalized metric space which satisfies condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ with $gx \preceq gu, gv \preceq gy$ the following condition is satisfied:*

$$d\left(F(x, y), F(u, v)\right) \leq \frac{A}{2}\left[d\left(gx, gu\right) + d\left(gy, gv\right)\right], \tag{2.3}$$

such that $A = (a_{ij}) \in M_{m\times m}(\mathbb{R}^{+})$, is a nonzero matrix in $\mathcal{ZM}$ convergese to zero. Let F and g satisfy the following conditions

(i) *$F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$,*
(ii) *$g(\mathcal{X})$ is a complete subspace of $\mathcal{X}$, and*
(iii) *F has the mixed g-monotone property.*

If there exist $x_0, y_0 \in \mathcal{X}$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then F and g have a unique coupled coincidence fixed point.

Proof. In Theorem 2.1, take $A = B = \frac{A}{2}$. □

Corollary 2.3. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $(\mathcal{X}, d)$ be a complete generalized metric space that satisfies the condition (NDI), and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ with the following condition:*

$$d\left(F(x, y), F(u, v)\right) \leq \frac{A}{2}\left[d(x, u) + d(y, v)\right], \tag{2.4}$$

where $A = (a_{ij}) \in M_{m\times m}(\mathbb{R}^{+})$, is a nonzero matrix in $\mathcal{ZM}$. Also,it is satisfied for some comparable pairs $x \preceq u, v \preceq y$ and F has the mixed monotone property, If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in \mathcal{X}$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. It follows from Corollary 2.2 by taking $g =$ identity map. □

Example 2.4. Let $\mathcal{X} = [0, 1] \times [0, 1]$. Define $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^2$ with

$$d\left((x_1, y_1), (x_2, y_2)\right) = \left(|x_1 - x_2|, |y_1 - y_2|\right).$$

Then $(\mathcal{X}, d)$ is a complete generalized metric space. Consider the mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ with $F(U, V) = \left(\frac{x+u}{3}, \frac{y+v}{3}\right)$, where $U = (x, y), V = (u, v)$. Then F satisfies the contractive condition (2.4), for $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$, that is,

$$d(F(x, y), F(u, v)) \leq \frac{A}{2}[d(x, u) + d(y, v)]. \tag{2.5}$$

Therefore, by Corollary 2.3, F has a unique coupled fixed point, which in this case is (0, 0).

Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, and let $T : \mathcal{X} \to \mathcal{X}$ be a nonexpansive potential operator such that there is a functional $J : \mathcal{X} \to \mathbb{R}$ with $J(0) = 0$ and $J' = T$. Consider the measure space (Ω, μ) $(\Omega = [0, 1])$ such that $\mu(\Omega) = 1$, and consider $L^2(\Omega, X)$ that is consists of all μ-strongly measurable functions $u : \Omega \to \mathcal{X}$ such that $\int_\Omega \|u(t)\|^2 d\mu < \infty$ with L^2-norm. For $r > 0$, define $B_r = \{x \in \mathcal{X} : \ \|x\| \leq r\}$ and $S_r = \{x \in \mathcal{X} : \ \|x\| = r\}$. An interesting question that arises here is: when a fixed point of T lies in the interior of B_r? Ricceri answered this question in [8].

Corollary 2.5. *Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a partially ordered real Hilbert space with (NDI) property and with generalized norm, let $T : \mathcal{X} \to \mathcal{X}$ be an $A/2$-nonexpansive potential operator and $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ such that $F(x, y) = T(x)$. If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then T has a fixed point x lying in the interior of B_r and (x, x) is a coupled fixed point of F.*

Proof. Since T is $A/2$-nonexpansive, so F satisfies (2.4) and T has a unique fixed point x lying in B_r (see [5] or [3, Theorem 1.3]). Thus Corollary 2.3 implies that F has a coupled fixed point $x, y \in \mathcal{X}$ such that $x = F(x, y)$ and $y = F(y, x)$. The uniqueness of fixed point for T caused that $x = y$. □

Theorem 2.6. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $(\mathcal{X}, d)$ be a complete generalized metric space, and for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \to X$ and $g : \mathcal{X} \to \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$, satisfy the following condition*

$$d(F(x, y), F(u, v)) \leq Ad(gx, gu) + Bd(gy, gv), \tag{2.6}$$

where $A = (a_{ij}), B = (b_{ij}) \in M_{m\times m}(\mathbb{R}^+)$, $\|A+B\| < 1$ where A and B are nonzero matrices in $\mathcal{ZM}$. Suppose that F has the mixed g-monotone property, $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, g is continuous and g commutes with F. Also assume that F is continuous or $\mathcal{X}$ satisfies in condition (NDI). If there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof. As in the proof of Theorem 2.1, we can construct two Cauchy sequences (gx_n) and (gy_n) in $\mathcal{X}$. Since $(\mathcal{X}, d)$ is complete, there exist $x, y \in \mathcal{X}$ such that (gx_{n+1}) converges to x and (gy_{n+1}) converges to y. Since g is continuous, we have (ggx_{n+1}) converges to gx and (ggy_{n+1}) converges to gy. But

$$ggx_{n+1} = g\left(F\left(x_n, y_n\right)\right) = F\left(gx_n, gy_n\right),$$

and

$$ggy_{n+1} = g\left(F\left(y_n, x_n\right)\right) = F\left(gy_n, gx_n\right).$$

We complete the proof in two cases: (1) Suppose that F is continuous, then we have $(F(gx_n, gy_n))$ converges to $F(x, y)$ and $(F(gy_n, gx_n))$ converges to $F(y, x)$. Thus (ggx_{n+1}) converges to $F(x, y)$ and (ggy_{n+1}) converges to $F(y, x)$. Therefore,

$$d\left(ggx_{n+1}, gx\right) \to \theta, \qquad d\left(ggx_{n+1}, F\left(x, y\right)\right) \to \theta.$$

It follows that

$$d\left(gx, F\left(x, y\right)\right) \preceq d\left(gx, ggx_{n+1}\right) + d\left(ggx_{n+1}, F\left(x, y\right)\right).$$

Therefore, $d(gx, F(x, y)) = \theta$ and $gx = F(x, y)$. Similarly, $gy = F(y, x)$. Hence, (x, y) is a coincidence coupled point of F and g.

(2) Suppose that $\mathcal{X}$ satisfies the condition (NDI). Then $gx_n \preceq x$ and $y \preceq gy_n$ for all $n \in N$. Hence

$$\begin{aligned} d\left(ggx_{n+1}, F\left(x, y\right)\right) &= d\left(F\left(gx_n, gy_n\right), F\left(x, y\right)\right) \\ &\preceq Ad\left(ggx_n, gx\right) + Bd\left(ggy_n, gy\right). \end{aligned}$$

Since (ggx_n) converges to gx and (ggy_n) converges to y, we get (ggx_n) converges to $F(x, y)$. Similarly, (ggy_n) converges to $F(y, x)$. By similar arguments as above, one can show that $gx = F(x, y)$ and $gy = F(y, x)$. Thus, the pair (x, y) is a coupled coincidence point of F and g. □

Corollary 2.7. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $(\mathcal{X}, d)$ be a complete generalized metric space, and, for all $x, y, u, v \in \mathcal{X}$, $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ with $gx \preceq gu$ and $gv \preceq gy$ satisfy the following condition*

$$d\left(F\left(x, y\right), F\left(u, v\right)\right) \preceq A\left[d\left(gx, gu\right) + d\left(gy, gv\right)\right], \tag{2.7}$$

such that $A = (a_{ij}) \in M_{m\times m}(\mathbb{R}^+)$, where A is a nonzero matrix in $\mathcal{ZM}$. Suppose that F has the mixed g-monotone property, $F(\mathcal{X} \times \mathcal{X}) \subset g(\mathcal{X})$, g is continuous and g commutes with F. Also, assume that either F is continuous or $\mathcal{X}$ has the condition (NDI).

If there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof. In Theorem 2.6, take $A = B = \frac{A}{2}$. □

Theorem 2.8. *In addition to the hypothesis of Theorem 2.1, suppose that for every* $(x,y),(x^*,y^*) \in \mathcal{X}\times\mathcal{X}$*, there exists* $(u,v) \in \mathcal{X}\times\mathcal{X}$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$*.*

If (x,y) *and* (x^*,y^*) *are coupled coincidence points of* F *and* g*, then* $F(x,y) = gx = gx^* = F(x^*,y^*)$ *and* $F(y,x) = gy = gy^* = F(y^*,x^*)$*. Moreover, if* F *and* g *commutes, then* F *and* g *have a unique common fixed point, that is, there exists a unique pair* $(x,y) \in \mathcal{X}\times\mathcal{X}$ *such that* $x = gx = F(x,y)$ *and* $y = gy = F(y,x)$*.*

Proof. Following the proof of Theorem 2.1, there exists $(x,y) \in \mathcal{X}\times\mathcal{X}$ such that $F(x,y) = gx = p$ and $F(y,x) = gy = q$. Thus the existence of a coupled coincidence point is confirmed. Now, let (x^*,y^*) be another coincidence point of F and g; that is, $F(x^*,y^*) = gx^*$ and $F(y^*,x^*) = gy^*$. By the additional assumption, there is $(u,v) \in \mathcal{X}\times\mathcal{X}$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$.

Let $u_0 = u, v_0 = v, x_0 = x, y_0 = y, x_0^* = x^*$ and $y_0^* = y^*$. Since $F(\mathcal{X}\times\mathcal{X}) \subseteq gX$, we can construct the sequences (gu_n), (gv_n), (gx_n), (gy_n), (gx_n^*), and (gy_n^*), such that $gu_{n+1} = F(u_n,v_n)$, $gv_{n+1} = F(v_n,u_n)$, $gx_{n+1} = F(x_n,y_n)$, $gy_{n+1} = F(y_n,x_n)$, $gx_{n+1}^* = F(x_n^*,y_n^*)$ and $gy_{n+1}^* = F(y_n^*,x_n^*)$. Since

$$(gx,gy) = (F(x,y),F(y,x)) = (gx_1,gy_1),$$

and

$$(F(u,v),F(v,u)) = (gu_1,gv_1),$$

are comparable, then $gx \preceq gu_1$ and $gv_1 \preceq gy$. One can show that $gx \preceq gu_n$, and $gv_n \preceq gy$ for all $n \in N$. From

$$d(gx,gu_{n+1}) = d(F(x,y),F(u_n,v_n)) \leq Ad(gx,gu_n) + Bd(gy,gv_n),$$

and

$$d(gy,gv_{n+1}) = d(F(v_n,u_n),F(y,x)) \leq Ad(gv_n,gy) + Bd(gu_n,gx),$$

we have

$$d(gx,gu_{n+1}) + d(gy,gv_{n+1}) \leq (A+B)(d(gx,gu_n) + d(gy,gv_n)).$$

Since

$$d(gx,gu_{n+1}) \leq d(gx,gu_{n+1}) + d(gy,gv_{n+1}),$$

we have

$$\begin{aligned} d(gu_{n+1},gx) &\leq (A+B)(d(gx,gu_n) + d(gy,gv_n)) \\ &\leq (A+B)^2(d(gx,gu_{n-1}) + d(gy,gv_{n-1})) \end{aligned}$$

$$\vdots$$
$$\leq (A+B)^{n+1}\left(d\left(gx,gu\right)+d\left(gy,gv\right)\right).$$

Thus, gu_{n+1} converges to gx in $(\mathcal{X},d)$. Similarly, we may show that gv_{n+1} converges to gy in $(\mathcal{X},d)$. Analogously, we can show that gu_{n+1} converges to gx^* and gv_{n+1} converges to gy^* in $(\mathcal{X},d)$. Since (gu_{n+1}) converges to gx and gx^*, we get $gx=gx^*$. Also, since (gv_{n+1}) converges to gy and gy^*, we get $gy=gy^*$. Thus, if (x,y) and (x^*,y^*) are coupled coincidence points of F and g, then

$$F\left(x,y\right)=gx=gx^*=F\left(x^*,y^*\right),$$

and

$$F\left(y,x\right)=gy=gy^*=F\left(y^*,x^*\right).$$

Assume that F and g commute, then

$$gp=g\left(gx\right)=g\left(F\left(x,y\right)\right)=F\left(gx,gy\right)=F\left(p,q\right),$$

and

$$gq=g\left(gy\right)=g\left(F\left(y,x\right)\right)=F\left(gy,gx\right)=F\left(q,p\right).$$

Hence, the pair (p,q) is also a coupled coincidence point of F and g. Thus, we have $gp=gx$ and $gq=gy$. Hence $gp=p$ and $gq=q$. Therefore $p=gp=F\left(p,q\right)$ and $q=gq=F\left(q,p\right)$.

Thus (p,q) is a coupled common fixed point of F and g. To prove the uniqueness, let (s,t) be any coupled common fixed point of F and g. Then $s=gs=F\left(s,t\right)$ and $t=gt=F\left(t,s\right)$.

Since the pair (s,t) is a coupled coincidence point of F and g, we have $gs=gx$ and $gt=gy$. Thus $s=gs=gp=p$ and $t=gt=gq=q$. This shows that the coupled fixed point is unique. □

3. Application in Linear Matrix Equations

Consider the linear matrix equations of the type

$$X-A_1^*XA_1-\cdots-A_m^*XA_m=Q, \tag{3.1}$$

and

$$X+A_1^*XA_1+\cdots+A_m^*XA_m=Q, \tag{3.2}$$

where Q is a positive definite matrix and $A_1,\ldots,A_m$ are arbitrary $n\times n$ matrices. We denote the set of all $n\times n$ matrices, $n\times n$ Hermitian matrices and $n\times n$ positive definite matrices by $M\left(n\right)$, $H\left(n\right)$ and $P\left(n\right)$, respectively. Clearly, we have the chain $P\left(n\right)\subseteq H\left(n\right)\subseteq M\left(n\right)$. Consider the spectral norm, $\|A\| = \sqrt{\lambda^+ (A^*A)}$ where $\lambda^+ (A^*A)$ is the largest eigenvalue of A^*A. Define maps G and K on $H(n)$ by

$$G(X) = Q + \sum_{i=1}^{m} A_i^* X A_i \ , \qquad K(X) = Q - \sum_{i=1}^{m} A_i^* X A_i. \tag{3.3}$$

The fixed points of G are solutions of (3.1) and the fixed points of K are solutions of (3.2). Fixed point theorems for these functions are studied in [6]. Now, we extend the above equations as follows:

$$\begin{cases} X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q, \\ \\ Y - B_1^* Y B_1 - \cdots - B_t^* Y B_t = Q', \end{cases} \tag{3.4}$$

and

$$\begin{cases} X + A_1^* X A_1 + \cdots + A_m^* X A_m = Q, \\ \\ Y + B_1^* Y B_1 + \cdots + B_t^* Y B_t = Q', \end{cases} \tag{3.5}$$

where Q and Q' are positive definite matrices, $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are arbitrary $n \times n$ matrices. Define maps $F_1 : H(n) \times H(n) \times H(n) \times H(n) = H(n)^4 \to H(n) \times H(n)$ and $F_2 : H(n) \times H(n) \times H(n) \times H(n) \to H(n) \times H(n)$ as follows

(3.6)

$$F_1(U,V) = \left(Q + \sum_{i=1}^{m} A_i^* \left(X + X' \right) A_i, Q' + \sum_{i=1}^{t} B_i^* \left(Y + Y' \right) B_i \right),$$

and

(3.7)

$$F_2(U,V) = \left(Q - \sum_{i=1}^{m} A_i^* \left(X + X' \right) A_i, Q' - \sum_{i=1}^{t} B_i^* \left(Y + Y' \right) B_i \right),$$

where $U = (X, Y)$ and $V = (X', Y')$. We consider the trace norm $\| \cdot \|_1$ on $H(n)$ as $\|A\|_1 = \sum_{i=1}^{n} s_i(A)$, where $s_i(A)$'s are singular values of A. For $Q \in P(n)$ we define $\|A\|_{1,Q} = \|Q^{\frac{1}{2}} A Q^{\frac{1}{2}}\|_1$. Then $H(n)$ by this norm becomes a complete metric space for any positive definite Q. Let $\trianglelefteq$ be a partially order on $H(n)$, as defined in [6] (we use $\trianglelefteq$ and $\triangleleft$ instead of $\leq$ and $<$, respectively, on $H(n)$). We consider the partial order on $H(n) \times H(n)$ as follows

$$(X, Y) \preceq \left(X', Y' \right) \quad \Leftrightarrow \quad X \trianglelefteq X' \ \text{ and } \ Y \trianglelefteq Y'. \tag{3.8}$$

Similarly, we can extend this for $H(n)^4$ as follows

$$(U, V) \,\widetilde{\preceq}\, \left(U', V' \right) \quad \Leftrightarrow \quad U \preceq U' \ \text{ and } \ V \preceq V'. \tag{3.9}$$

We define $\|\cdot\|_{(1,1),(Q,Q')} : H(n) \times H(n) \to \mathbb{R}^2$ by

$$\begin{aligned}\|(A,B)\|_{(1,1),(Q,Q')} &= \left(\|A\|_{1,Q}, \|B\|_{1,Q'}\right)\\ &= \left(\left\|Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\right\|_1, \left\|Q'^{\frac{1}{2}}BQ'^{\frac{1}{2}}\right\|_1\right).\end{aligned}$$

Clearly, $H(n) \times H(n)$ equipped with the above metric is a complete metric space for any positive definite Q and Q'.

Theorem 3.1. *Let $Q, Q' \in P(n)$ such that*

$$\sum_{i=1}^{m} A_i^* Q A_i \trianglelefteq Q, \quad \sum_{i=1}^{t} B_i^* Q' B_i \trianglelefteq Q',$$

$$\left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}\right\| < \frac{1}{2}$$

and

$$\left\|\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}}\right\| < \frac{1}{2}.$$

Then F_2 has a unique coupled fixed point in $H(n)$.

Proof. Let $(U,V), (U',V') \in H(n)^4$ such that $(U,V) \mathrel{\widetilde{\preceq}} (U',V')$, where

$$U = (X_1, Y_1), \quad V = \left(X_1', Y_1'\right), \quad U' = (X_2, Y_2),$$

and $V' = (X_2', Y_2')$. Take $U_0 = (0,0)$ and $V_0 = (Q, Q')$. Then

$$F_2(U_0, V_0) \preceq V_0, \qquad U_0 \preceq F_2(U_0, V_0).$$

Furthermore, for given $U = (X_1, Y_1), V = (X_1', Y_1'), U' = (X_2, Y_2)$ and $V' = (X_2', Y_2')$, we have

$$\begin{aligned}&\|F_2(U',V') - F_2(U,V)\|_{(1,1),(Q,Q')}\\ &= \left\|\left(\sum_{i=1}^{m} A_i^*\left(X_2 + X_2' - X_1 - X_1'\right)A_i, \sum_{i=1}^{t} B_i^*\left(Y_2 + Y_2' - Y_1 - Y_1'\right)B_i\right)\right\|_{(1,1),(Q,Q')}\\ &= \left(tr\left(\sum_{i=1}^{m} Q^{\frac{1}{2}} A_i^*\left(X_2 + X_2' - X_1 - X_1'\right)A_i Q^{\frac{1}{2}}\right),\right.\\ &\qquad \left. tr\left(\sum_{i=1}^{t} Q'^{\frac{1}{2}} B_i^*\left(Y_2 + Y_2' - Y_1 - Y_1'\right)B_i Q'^{\frac{1}{2}}\right)\right)\\ &= \left(tr\left(\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}} Q^{\frac{1}{2}}\left(X_2 + X_2' - X_1 - X_1'\right)Q^{\frac{1}{2}}\right),\right.\\ &\qquad \left. tr\left(\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}} Q'^{\frac{1}{2}}\left(Y_2 + Y_2' - Y_1 - Y_1'\right)Q'^{\frac{1}{2}}\right)\right)\\ &= \left(tr\left(\left(\sum_{i=1}^{m}\left(Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}\right)\left(Q^{\frac{1}{2}}\left(X_2 + X_2' - X_1 - X_1'\right)Q^{\frac{1}{2}}\right)\right)\right),\right.\end{aligned}$$

$$tr\left(\left(\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}}\right)\left(Q'^{\frac{1}{2}}\left(Y_2+Y_2'-Y_1-Y_1'\right)Q'^{\frac{1}{2}}\right)\right)\Bigg)$$

$$\leq \left(\left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}\right\| \left\|X_2+X_2'-X_1-X_1'\right\|_{1,Q},\right.$$

$$\left.\left\|\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}}\right\| \left\|Y_2+Y_2'-Y_1-Y_1'\right\|_{1,'Q}\right)$$

$$= \begin{bmatrix} \left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}\right\| & 0 \\ 0 & \left\|\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}}\right\| \end{bmatrix}$$

$$\times \left(\|X_2+X_2'-X_1-X_1'\|_{1,Q}, \|Y_2+Y_2'-Y_1-Y_1'\|_{1,'Q}\right).$$

In the above statements, we used Lemma 3.1 of [6]. Set

$$\alpha = \left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q A_i Q^{-\frac{1}{2}}\right\|,$$

and

$$\beta = \left\|\sum_{i=1}^{t} Q'^{-\frac{1}{2}} B_i^* Q' B_i Q'^{-\frac{1}{2}}\right\|.$$

Then

$$\|F_2(U',V') - F_2(U,V)\|_{(1,1),(Q,Q')},$$

$$\leq \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (\|X_2-X_1\|_{1,Q} + \|X_2'-X_1'\|_{1,Q}, \|Y_2-Y_1\|_{1,Q'} + \|Y_2'-Y_1'\|_{1,Q'})$$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} ((\|X_2-X_1\|_{1,Q}, \|Y_2-Y_1\|_{1,Q'}) + (\|X_2'-X_1'\|_{1,Q}, \|Y_2'-Y_1'\|_{1,Q'}))$$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \left(\|U'-U\|_{(1,1),(Q,Q')}, \|V-V'\|_{(1,1),(Q,Q')}\right).$$

Now apply Theorem 2.1. □

The above theorem says that, under the conditions of this theorem, the equation system (3.5) has a unique solution.

4. Application in nonlinear matrix equations

In this section, we study the following class of nonlinear matrix equations system:

$$\begin{cases} X + A_1^*\mathcal{F}(X)A_1 + \cdots + A_m^*\mathcal{F}(X)A_m = Q, \\ Y + B_1^*\mathcal{G}(Y)B_1 + \cdots + B_t^*\mathcal{G}(Y)B_t = Q', \end{cases} \tag{4.1}$$

where Q and Q' are positive definite matrices, $A_1, \ldots, A_m$ and $B_1, \ldots, B_t$ are arbitrary $n \times n$ matrices and $\mathcal{F}$ and $\mathcal{G}$ are continuous maps, from $P(n) \times P(n)$ into $P(n)$, where $n \geq 3$. We define $\mathfrak{F} : H(n) \times H(n) \times H(n) \times H(n) = H(n)^4 \to H(n) \times H(n)$ as follows

(4.2)
$$\mathfrak{F}(U,V) = \left(Q - \sum_{i=1}^{m} A_i^* \mathcal{F}((X,Y)) A_i, Q' - \sum_{i=1}^{t} B_i^* \mathcal{G}\left((X',Y')\right) B_i \right),$$

for every $U = (X,Y), V = (X',Y') \in H(n) \times H(n)$. Clearly, if $\mathfrak{F}$ has a unique coupled fixed point then (4.1) has a unique solution. We consider the norm $\| \cdot \| : H(n) \times H(n) \to \mathbb{R}^2$ such that

$$\|(A,B)\|_{1,1} = (\|A\|_1, \|B\|_1), \quad A, B \in H(n). \tag{4.3}$$

Before considering the system (4.1), we take into account the result obtained in [6] for the following nonlinear matrix equation:

$$X + A_1^* \mathcal{F}(X) A_1 + \cdots + A_m^* \mathcal{F}(X) A_m = Q, \tag{4.4}$$

where Q and $A_1, \ldots, A_m$ are as above. By reviewing the proof of Theorem 4.1, we reach to a gap in the assumption. At first, note that the space of all $n \times n$, $M(n)$ is a unital Banach algebra with of unit I_n, where I_n is identity the matrix $n \times n$. Authors in [6] considered the Banach algebra $M(n)$ with two norms $\|A\| = \sqrt{\lambda^+(A^*A)}$ and $\|A\|_1 = \sum_{i=1}^n s_i(A)$ for $A \in M(n)$. It is known that all norms on a finite dimensional Banach algebra are equivalent. It is rutin in the Banach algebra theory that we assume the norm of identity element is equale to 1 (this holds when we use the spectral norm). Now, we use the other norm i.e., $\|I_n\|_1 = \sum_{i=1}^n s_i(I_n) = n$. We rewrite Theorem 4.1 and investigate it.

Let $Q \in P(n)$. Assume that there exists a positive number M for which $\sum_{j=1}^m A_j A_j^* < M \cdot I_n$ and such that for $X \leq Y$ we have

$$|tr(\mathcal{F}(Y) - \mathcal{F}(X))| \leq \frac{1}{M} |tr(Y - X)|. \tag{4.5}$$

Then (4.4) has a unique solution in $P(n)$.

In the proof of the above stated result, authors proved the following

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 \leq \left\| \sum_{i=1}^m A_j A_j^* \right\| \|\mathcal{F}(Y) - \mathcal{F}(X)\|_1.$$

We consider the proof with two norms $\| \cdot \|$ (spectral norm) and $\| \cdot \|_1$. For $\| \cdot \|_1$, by using of (4.5) the condition (1) of Theorem 2.1 does not hold. Because

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 \leq \left\| \sum_{i=1}^m A_j A_j^* \right\| \|\mathcal{F}(Y) - \mathcal{F}(X)\|_1$$

$$\leq M\|I_n\|_1|tr\left(\mathcal{F}\left(Y\right)-\mathcal{F}\left(X\right)\right)| \\ \leq \|I_n\|_1|tr\left(Y-X\right)| \\ = n|tr\left(Y-X\right)|.$$

Thus, the above statement does not satisfy condition (1) of Theorem 2.1 of [6]. If we use the spectral norm, then we obtain $\|\mathcal{G}\left(Y\right)-\mathcal{G}\left(X\right)\|_1 \leq |tr\left(Y-X\right)|$, again it dose not satisfy in condition (1), note that in condition (1), there is a $0 < c < 1$. Now consider the following conditions on M and (4.5):

(i) $M > 1$ and $|tr\left(\mathcal{F}\left(Y\right)-\mathcal{F}\left(X\right)\right)| \leq \frac{1}{(n+1)M}|tr\left(Y-X\right)|$.
(ii) $M > n$ and $|tr\left(\mathcal{F}\left(Y\right)-\mathcal{F}\left(X\right)\right)| \leq \frac{1}{M^2}|tr\left(Y-X\right)|$.

We write the correction of Theorem 4.1 of [6] with our paper notations for partial orders as follows:

Theorem 4.1. *Let $Q \in P\left(n\right)$. Assume that there exists a positive number M for which $\sum_{j=1}^{m} A_j A_j^* \lhd M \cdot I_n$ and such that for $X \unlhd Y$ we have one of the condtions (i) or (ii) holds. Then (4.4) has a unique solution in $P\left(n\right)$.*

Now, we consider the nonlinear matrix equation system (4.1) as follows:

Theorem 4.2. *Let $Q, Q' \in P\left(n\right)$ and the following statements hold.*

(i) $\sum_{i=1}^{m} A_i^* \mathcal{F}\left(\left(X,Y\right)\right) A_i \unlhd Q$ *and* $\sum_{i=1}^{t} B_i^* \mathcal{G}\left(\left(X',Y'\right)\right) B_i \unlhd Q'$.
(ii) *There exist positive numbers M, M' such that*

$$\sum_{i=1}^{m} A_i^* A_i \lhd M \cdot I_n, \qquad \sum_{i=1}^{t} B_i^* B_i \lhd M' \cdot I_t,$$

and satisfy in one of the following conditions:

1. *$M, M' > 1$ and for every $\left(U,V\right),\left(U',V'\right) \in H\left(n\right)^4$,*

$$\left(|tr\left(\mathcal{F}\left(U\right)-\mathcal{F}\left(U'\right)\right)|, |tr\left(\mathcal{G}\left(V\right)-\mathcal{G}\left(V'\right)\right)|\right) \\ \leq \left(\frac{1}{\left(n+1\right)M}|tr\left(U-U'\right)|, \frac{1}{\left(t+1\right)M'}|tr\left(V-V'\right)|\right).$$

2. *$M > n$, $M' > t$ and for every $\left(U,V\right),\left(U',V'\right) \in H\left(n\right)^4$,*

$$\left(|tr\left(\mathcal{F}\left(U\right)-\mathcal{F}\left(U'\right)\right)|, |tr\left(\mathcal{G}\left(V\right)-\mathcal{G}\left(V'\right)\right)|\right) \\ \leq \left(\frac{1}{M^2}|tr\left(U-U'\right)|, \frac{1}{M'^2}|tr\left(V-V'\right)|\right).$$

Then $\mathfrak{F}$ has a unique coupled fixed point in $H\left(n\right)$.

Proof. Let $(U,V),(U',V')\in H(n)^4$ such that $(U,V)\,\widetilde{\preceq}\,(U',V')$, where $U=(X_1,Y_1),V=(X_1',Y_1'),U'=(X_2,Y_2)$ and $V'=(X_2',Y_2')$. Take $U_0=(0,0)$ and $V_0=(Q,Q')$. Then $\mathfrak{F}(U_0,V_0)\preceq V_0$ and $U_0\preceq\mathfrak{F}(U_0,V_0)$. Furthermore, for given $U=(X_1,Y_1),V=(X_1',Y_1'),U'=(X_2,Y_2)$ and $V'=(X_2',Y_2')$ we have

$$
\begin{aligned}
&\|\mathfrak{F}(U',V')-\mathfrak{F}(U,V)\|_{(1,1)}\\
&=\left\|\left(\sum_{i=1}^{m}A_i^*\left(\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\right)A_i,\sum_{i=1}^{t}B_i^*\left(\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\right)B_i\right)\right\|_{(1,1)}\\
&=\left(\sum_{i=1}^{m}tr\left(A_iA_i^*\left(\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\right)\right),\sum_{i=1}^{t}tr\left(B_i^*B_i\left(\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\right)\right)\right)\\
&=\left(tr\left(\sum_{i=1}^{m}A_i^*A_i\left(\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\right)\right),\right.\\
&\qquad\left.tr\left(\sum_{i=1}^{t}B_i^*B_i\left(\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\right)\right)\right)\\
&=\left(tr\left(\sum_{i=1}^{m}\left(A_i^*A_i\right)\left(\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\right)\right),\right.\\
&\qquad\left.tr\left(\left(\sum_{i=1}^{t}B_i^*B_i\right)\left(\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\right)\right)\right)\\
&\le\left(\left\|\sum_{i=1}^{m}A_i^*A_i\right\|\|\left(\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\|_1\right),\right.\\
&\qquad\left.\left\|\sum_{i=1}^{t}B_i^*B_i\right\|\|\left(\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\|_1\right)\right)\\
&=\begin{bmatrix}\left\|\sum_{i=1}^{m}A_i^*A_i\right\| & 0\\ 0 & \left\|\sum_{i=1}^{t}B_i^*B_i\right\|\end{bmatrix}\\
&\qquad\times\left(\|\mathcal{F}(X_2,Y_2)-\mathcal{F}(X_1,Y_1)\|_1,\|\mathcal{G}\left(X_2',Y_2'\right)-\mathcal{G}\left(X_1',Y_1'\right)\|_1\right).
\end{aligned}
$$

In the above statements, we used Lemma 3.1 of [6]. Suppose that (1) holds. Then,

$$
\begin{aligned}
&\|\mathfrak{F}\left(U',V'\right)-\mathfrak{F}(U,V)\|_{(1,1)}\\
&\quad\le\begin{bmatrix}M\|I_n\|_1 & 0\\ 0 & M'\|I_t\|_1\end{bmatrix}\left(\frac{1}{(n+1)M}\left(\|X_2-X_1\|_1+\|Y_2-Y_1\|_1\right),\right.\\
&\qquad\left.\frac{1}{(t+1)M'}\left(\|X_2'-X_1'\|_1+\|Y_2'-Y_1'\|_1\right)\right)\\
&\quad=\begin{bmatrix}\frac{1}{(n+1)} & 0\\ 0 & \frac{1}{(t+1)}\end{bmatrix}\left(\left(\|X_2-X_1\|_1+\|Y_2'-Y_1'\|_1\right),\right.
\end{aligned}
$$

$$\left(\|Y_2 - Y_1\|_1 + \|X'_2 - X'_1\|_1\right)\right)$$

$$= \begin{bmatrix} \frac{1}{(n+1)} & 0 \\ 0 & \frac{1}{(t+1)} \end{bmatrix} \left(\|U' - U\|_{(1,1)}, \|V - V'\|_{(1,1)}\right).$$

We obtain similar results for case (2), that provide the required conditions in Theorem 2.1. This completes the proof. □

Acknowledgment. The author wish to thank gratitude to the reviewers for their valuable comments and suggestions for the improvement of this paper.

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