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Some Results on the Field of Values of Matrix Polynomials

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Abstract. In this paper, the notions of pseudofield of values and joint pseudofield of values of matrix polynomials are introduced and some of their algebraic and geometrical properties are studied. Moreover, the relationship between the pseudofield of values of a matrix polynomial and the pseudofield of values of its companion linearization is stated, and then some properties of the augmented field of values of basic A-factor block circulant matrices are investigated.

1. Introduction and Preliminaries

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. The field of values and numerical radius of $A \in \mathbb{M}_n$ are defined, respectively, as $W(A) := \left\{x^*Ax : x \in S^1\right\}$ and $r(A) := \max\{|z| : z \in W(A)\}$, where S^1 denotes the unit sphere in $\mathbb{C}^n$, i.e., $S^1 = \{x \in \mathbb{C}^n : x^*x = 1\}$. These concepts are useful in studying of matrices and operators; e.g., see [5] and its references. They have also many applications in quantum physics, numerical analysis, differential equations, systems theory, etc; e.g., see [3, 5–7] and references cited there.

Let $\varepsilon \geq 0$ and $A \in \mathbb{M}_n$. The ε-pseudospectrum (pseudospectrum for short) of A is defined and denoted by

$$\begin{aligned}\sigma_\varepsilon(A) &= \{z \in \mathbb{C} : z \in \sigma(A+E) \text{ for some } E \in \mathbb{M}_n \text{ with } \|E\| \leq \varepsilon\} \\ &= \bigcup_{E \in \mathbb{M}_n, \|E\| \leq \varepsilon} \sigma(A+E),\end{aligned}$$

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where $\sigma(\cdot)$ denotes the spectrum and $\|.\|$ is the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). Also, the ε-pseudofield of values of A is defined, e.g., see [7], as

$$W_\varepsilon(A) := \bigcup_{\Delta \in \mathbb{M}_n, \|\Delta\| \leq \varepsilon} W(A + \Delta). \tag{1.1}$$

It is easy to see that $W_\varepsilon(A) = W(A) + D(0, \varepsilon)$, where

$$D(0, \varepsilon) = \{\mu \in \mathbb{C} : |\mu| \leq \varepsilon\}.$$

Note that $\sigma_0(A) = \sigma(A) \subseteq W(A) = W_0(A)$, and $\sigma_\varepsilon(A) \subseteq W_\varepsilon(A)$. In [5, p. 103], the set $W(A) + D(0, \varepsilon)$ is called the augmented field of values of A. For more information and some properties of pseudofield of values and its generalizations of matrices, see [7]. The theory of pseudospectra provides an analytical and graphical alternative for investigating non-normal matrices, gives a quantitative estimate of departure from non-normality, and also gives some information about the stability; e.g., see [10] and its references.

The our main motivation concerns the study of perturbed matrix polynomials. Recall that matrix polynomials arise in many applications and their spectral analysis is very important when we study the linear systems of ordinary differential equations with constant coefficients; e.g., see [4]. Suppose

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{1.2}$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. The numbers m and n refer to the degree and the order of $P(\lambda)$, respectively. The matrix polynomial $P(\lambda)$, as in (1.2), is called a monic matrix polynomial if $A_m = I_n$, where I_n denotes the $n \times n$ identity matrix. It is said to be a selfadjoint matrix polynomial if all the coefficients A_i are Hermitian matrices. A scalar $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an eigenvector of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalue of $P(\lambda)$ is the spectrum of $P(\lambda)$, i.e.,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : \det P(\mu) = 0\}.$$

The (classical) field of values of $P(\lambda)$, as in (1.2), is defined and denoted by

$$W[P(\lambda)] := \{\mu \in \mathbb{C} : x^*P(\mu)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\},$$

which is a closed set and contains $\sigma[P(\lambda)]$. For the case $P(\lambda) = \lambda I_n - A$, where $A \in \mathbb{M}_n$, we have $W[P(\lambda)] = W(A)$. The field of values (or the numerical range) of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and is also related to the stability theory; e.g., see [4, 8] and references therein.

Let $P(\lambda)$ be a matrix polynomial as in (1.2). For a given $\varepsilon \geq 0$ and an ordered set of nonnegative weights $\mathbf{w} = \{\omega_0, \omega_1, \ldots, \omega_n\}$ with at least one nonzero element, we denote the associated compact convex set of perturbations of $P(\lambda)$ by:

$$\mathcal{B}(P, \varepsilon, \mathbf{w}) := \{P_\Delta(\lambda) : \|\Delta_j\| \leq \varepsilon\omega_j, j = 0, 1, \ldots, m\}, \tag{1.3}$$

where $P_\Delta(\lambda) := (A_m + \Delta_m)\lambda^m + \cdots + (A_1 + \Delta_1)\lambda + (A_0 + \Delta_0)$. The weighted ε-pseudospectrum of $P(\lambda)$ which was first introduced in [9], is

(1.4)
$$\begin{aligned}\sigma_{\varepsilon,\mathbf{w}}[P(\lambda)] &= \{\mu \in \mathbb{C} : \det P_\Delta(\mu) = 0 \text{ for some } P_\Delta(\lambda) \in \mathcal{B}(P, \varepsilon, \mathbf{w})\} \\ &= \bigcup_{P_\Delta(\lambda) \in \mathcal{B}(P,\varepsilon,\mathbf{w})} \sigma[P_\Delta(\lambda)].\end{aligned}$$

For the case $\omega_0 = 1, \omega_1 = 0$, and $P(\lambda) = \lambda I_n - A$, where $A \in \mathbb{M}_n$, we have $\sigma_\varepsilon[P(\lambda)] = \sigma_\varepsilon(A)$; this shows that the pseudospectrum of matrix polynomials is a generalization of the pseudospectrum of matrices.

In this paper, we are going to introduce and study the notions of pseudofield of values and joint pseudofield of values of matrix polynomials. For this, in Section 2, we state definitions and some basic properties of pseudofield of values and also its relation with the augemented field of values of matrix polynomials. In Section 3, we introduce and characterize the notion of joint field of values of matrix polynomials, and then we establish some of its properties and also its relation with the pseudofield of values of matrix polynomials. In Section 4, we investigate the relationship between pseudofield of values of a matrix polynomial and the pseudofield of values of its companion linearization, and then we study some properties of the augmented field of values of basic A-factor block circulant matrices.

2. Definitions and Basic Properties

We begin this section by introducing the notion of ε-pseudofield of values of matrix polynomials.

Definition 2.1. Let $P(\lambda)$ be a matrix polynomial as in (1.2). For a given $\varepsilon \geq 0$ and an ordered set $\mathbf{w} = \{\omega_0, \omega_1, \ldots, \omega_m\}$ of nonnegative weights with at least one nonzero element, the weighted ε-pseudofield of values (pseudofield of values for short) of $P(\lambda)$ is defined and denoted by

$$W_{\varepsilon,\mathbf{w}}[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in W(P_\Delta(\mu)) \text{ for some } P_\Delta(\lambda) \in \mathcal{B}(P, \varepsilon, \mathbf{w})\},$$

where $\mathcal{B}(P, \varepsilon, \mathbf{w})$ is the set of all perturbations, as in (1.3), of $P(\lambda)$.

It is clear that:

$$W[P(\lambda)] = \bigcap_{\varepsilon \geq 0} W_{\varepsilon,\mathbf{w}}[P(\lambda)]. \tag{2.1}$$

Also, using Definition 2.1, we have the following observation which will be useful in our discusions:

$$W_{\varepsilon,\mathbf{w}}[P(\lambda)] = \bigcup_{P_\Delta(\lambda)\in\mathcal{B}(P,\varepsilon,\mathbf{w})} W[P_\Delta(\lambda)]. \tag{2.2}$$

In the following Theorem, we show that in relation (2.2), the union can be taken over all perturbed matrix polynomials $P_\Delta(\lambda) \in \mathcal{B}(P,\epsilon,\mathbf{w})$ with $\text{rank}(\Delta_i)$ at most one.

Theorem 2.2. *Let $P(\lambda)$ be a matrix polynomial as in (1.2). Then*

$$W_{\epsilon,\boldsymbol{w}}[P(\lambda)] = \bigcup_{P_\Delta(\lambda)\in\mathcal{B}^{(1)}(P,\epsilon,\boldsymbol{w})} W[P_\Delta(\lambda)],$$

where $\mathcal{B}^{(1)}(P,\epsilon,\boldsymbol{w}) := \{P_\Delta(\lambda) \in \mathcal{B}(P,\epsilon,\boldsymbol{w}) : \mathit{rank}(\Delta_j) \leq 1, j = 0,\dots,m\}$.

Proof. Let $\mu \in W_{\epsilon,\mathbf{w}}[P(\lambda)]$. Then by Definition 2.1, there exist a vector $x \in S^1$ and $(\Delta_0,\dots,\Delta_m) \in M_n^{m+1}$ such that $\|\Delta_j\| \leq \epsilon w_j$ for $j = 0,\dots,m$, and

$$(x^*(A_m + \Delta_m)x)\mu^m + \cdots + x^*(A_0 + \Delta_0)x = 0. \tag{2.3}$$

Now, by setting $\Delta_j^{'} = x(\Delta_j^* x)^*$ for $j = 0,\dots,m$, we see that $\|\Delta_j^{'}\| \leq \|x\|\|x^*\|\|\Delta_j\| \leq \epsilon\omega_j$ and $\text{rank}(\Delta_j^{'}) \leq 1$. Therefore,

$$P_{\Delta'}(\lambda) := (A_m + \Delta_m^{'})\lambda^m + \cdots + (A_1 + \Delta_1^{'})\lambda + (A_0 + \Delta_0^{'}) \in \mathcal{B}^{(1)}(P,\epsilon,\mathbf{w}).$$

Relation (2.3) shows that $0 \in W(P_{\Delta'}(\mu))$. Therefore,

$$W_{\epsilon,\mathbf{w}}[P(\lambda)] \subseteq \bigcup_{P_\Delta(\lambda)\in\mathcal{B}^{(1)}(P,\epsilon,\mathbf{w})} W[P_\Delta(\lambda)].$$

By relation (2.2), the opposite inclusion also holds. So the proof is complete. □

In the following proposition, we state some general properties of the ε-pseudofield of values of matrix polynomials.

Proposition 2.3. *Let $P(\lambda)$ be a matrix polynomial as in (1.2). Then the following assertions are true:*

(i) *$W_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$ is a closed set in $\mathbb{C}$ which contains $\sigma_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$;*
(ii) *$W_{\varepsilon,\boldsymbol{w}}[P(\lambda+\alpha)] = W_{\varepsilon,\boldsymbol{w}}[P(\lambda)] - \alpha$, where $\alpha \in \mathbb{C}$;*
(iii) *$W_{\varepsilon,\boldsymbol{w}}[\alpha P(\lambda)] = W_{\frac{\varepsilon}{|\alpha|},\boldsymbol{w}}[P(\lambda)]$, where $\alpha \in \mathbb{C}$ is nonzero;*

(iv) *If* $Q(\lambda) = \lambda^m P(\lambda^{-1}) := A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m$, *then*

$$W_{\varepsilon,\boldsymbol{w}'}[Q(\lambda)] \setminus \{0\} = \{\frac{1}{\mu} : \mu \in W_{\varepsilon,\boldsymbol{w}}[P(\lambda)], \mu \neq 0\},$$

where $\boldsymbol{w}' = \{\omega_m, \ldots, \omega_0\}$ *is the ordered reversal of* $\boldsymbol{w}$*;*

(v) $W_{\varepsilon,\boldsymbol{w}}[U^*P(\lambda)U] = W_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$*, where* $U \in \mathbb{M}_n$ *is unitary;*

(vi) *If* $P(\lambda)$ *is selfadjoint or all the coefficients* A_i *are real matrices, then* $W_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$ *is symmetric with respect to the real axis.*

Proof. To prove (i), the inclusion $\sigma_{\varepsilon,\mathbf{w}}[P(\lambda)] \subseteq W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ follows from relations (1.4) and (2.2), and the fact that for every $P_\Delta(\lambda) \in \mathcal{B}(P, \epsilon, \mathbf{w})$, $\sigma[P_\Delta(\lambda)] \subseteq W[P_\Delta(\lambda)]$. Also, by a routin way, we see that $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ is closed. The result in (ii) can be easily verify by Definition 2.1. Also, the result in (iii) follows from Definition 2.1 and this fact that for every $\Delta := (\Delta_0, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$, $(\alpha P)_\Delta(\lambda) = \alpha P_{\frac{\Delta}{\alpha}}(\lambda)$, where $\frac{\Delta}{\alpha} := (\frac{\Delta_0}{\alpha}, \ldots, \frac{\Delta_m}{\alpha}) \in \mathbb{M}_n^{m+1}$. Using Definition 2.1 and the fact that for every $\mu \in \mathbb{C} \setminus \{0\}$ and $\Delta := (\Delta_0, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$, $Q_{\Delta'}(\mu) = \mu^m P_\Delta(\frac{1}{\mu})$, where $\Delta' := (\Delta_m, \Delta_{m-1}, \ldots, \Delta_0) \in \mathbb{M}_n^{m+1}$, the result in (iv) holds. Since for every unitary matrix $U \in \mathbb{M}_n$ and for every $\Delta \in \mathbb{M}_n^{m+1}$, $(U^*PU)_\Delta(\lambda) = U^*P_{U\Delta U^*}(\lambda)U$, the result in (v) also holds. The result in (vi) follows from Definition 2.1 and this fact that for every $A \in \mathbb{M}_n$, $\|A\| = \|\overline{A}\| = \|A^*\|$. So the proof is complete. □

In the following proposition, we describe the pseudofield of values of matrix polynomials for some special weights.

Proposition 2.4. *Let* $P(\lambda)$ *be a matrix polynomial as in (1.2) and* $\boldsymbol{w} = \{\omega_0 = 1, \omega_1 = 0, \ldots, \omega_m = 0\}$*. Then*

$$W_{\varepsilon,\boldsymbol{w}}[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in W(P(\mu)) + D(0, \epsilon)\}.$$

Proof. Denote $Z := \{\mu \in \mathbb{C} : 0 \in W(P(\mu)) + D(0, \epsilon)\}$, and let $\mu \in W_{\epsilon,\mathbf{w}}[P(\lambda)]$. Then there exist a vector $x \in S^1$ and $\Delta := (\Delta_0, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$ such that $\|\Delta_j\| \leq \epsilon\omega_j$ for $j = 0, \ldots, m$, and $0 = x^*P_\Delta(\mu)x = x^*P(\mu)x + x^*\Delta_0 x$. Now, by setting $\xi := x^*\Delta_0 x$, we see that $|\xi| \leq \|\Delta_0\| \leq \epsilon$. So, $0 \in W(P(\mu)) + D(0, \epsilon)$, and hence, $\mu \in Z$. This shows that $W_{\varepsilon,\mathbf{w}}[P(\lambda)] \subseteq Z$.

Conversely, let $\mu \in Z$. So, there exist a vector $x \in S^1$ and $\xi \in D(0, \epsilon)$ such that $x^*P(\mu)x + \xi = 0$. By setting $\Delta_0 := \xi I_n$ and $\Delta_j = 0$ for $j = 1, \ldots, m$, we have $\|\Delta_j\| \leq \epsilon\omega_j$ for $j = 0, \ldots, m$, and $x^*P_\Delta(\mu)x = x^*P(\mu)x + \xi = 0$. Therefore, $\mu \in W_{\epsilon,\mathbf{w}}[P(\lambda)]$. So, $Z \subseteq W_{\varepsilon,\mathbf{w}}[P(\lambda)]$, and this completes the proof. □

We know that for every $A \in \mathbb{M}_n$, $W_\varepsilon(A) = W(A) + D(0, \varepsilon)$. In the following example, we investigate the relationship between $W_{\epsilon,\mathbf{w}}[P(\lambda)]$ and $W[P(\lambda)] + D(0, \epsilon)$.

Example 2.5. (a) Let $P(\lambda) = \lambda^2 I_2 - A$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By setting $\mathbf{w} = \{\omega_0 = 1, \omega_1 = 0, \omega_2 = 0\}$ and $\varepsilon = \frac{1}{2}$, we see, by Proposition 2.4, that $W_{\varepsilon,\mathbf{w}}[P(\lambda)] = \sqrt{W_{\frac{1}{2}}(A)}$, and also a simple calculation shows that $W[P(\lambda)] + D(0, \epsilon) = \sqrt{W(A)} + D(0, \frac{1}{2})$. So, we see that these sets are not equal; see Figure 1.

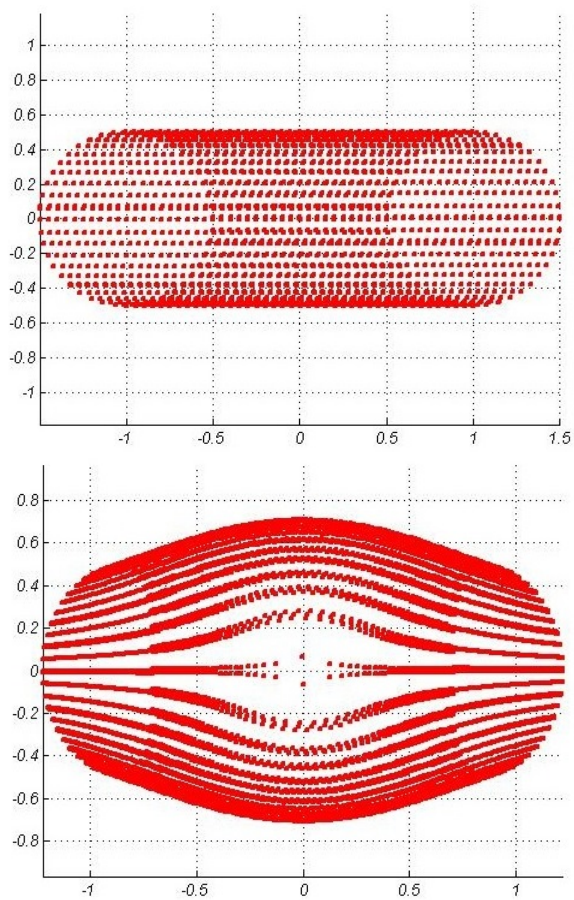

FIGURE 1. The above is $\sqrt{W(A)} + D(0, \frac{1}{2})$, and the below is $\sqrt{W_{\frac{1}{2}}(A)}$ (Example 2.5).

(b) Let $P(\lambda) = \lambda I_n - A$, where $A \in \mathbb{M}_n$. By setting $\omega_0 = 1$ and $\omega_1 = 0$, and using Proposition 2.4, we see that $W_{\varepsilon,\mathbf{w}}[P(\lambda)] = W[P(\lambda)] + D(0,\epsilon) = W(A) + D(0,\epsilon) = W_\varepsilon(A)$.

3. Some Algebraic and Geometrical Properties

We begin this section by introducing the notion of joint pseudofield of values of matrix polynomials. Let $P(\lambda)$ be a matrix polynomial as in (1.2). The joint field of values of $P(\lambda)$ is defined as the joint field of values of $(A_0, A_1, \ldots, A_m)$; namely,

$$\begin{aligned} JW[P(\lambda)] &= W(A_0, \ldots, A_m) \\ &= \left\{ (x^* A_0 x, \ldots, x^* A_m x) : x \in S^1 \right\}. \end{aligned}$$

By the argument as in (2.2), we define the weighted $\varepsilon-$joint pseudofield of values of $P(\lambda)$ as:

$$JW_{\varepsilon,\mathbf{w}}[P(\lambda)] = \bigcup_{P_\Delta(\lambda) \in \mathcal{B}(P,\varepsilon,\mathbf{w})} JW[P_\Delta(\lambda)]. \tag{3.1}$$

Since $JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$ is the range of the continuous function:

$$(x, (\Delta_0, \ldots, \Delta_m)) \longmapsto (x^*(A_0+\Delta_0)x, x^*(A_1+\Delta_1)x, \ldots, x^*(A_m+\Delta_m)x),$$

from the compact connected set

$$S^1 \times \left\{ (\Delta_0, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1} : \|\Delta_j\| \le \varepsilon\omega_j, j = 0, \ldots, m \right\},$$

to $\mathbb{C}^{m+1}$, we see that $JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$ is a compact and connected set in $\mathbb{C}^{m+1}$. Also, from relations (2.2) and (3.1), we see that $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ coincides with the following set:

$$\{\mu \in \mathbb{C} : a_m \mu^m + \cdots + a_0 = 0, (a_0, \ldots, a_m) \in JW_{\varepsilon,\mathbf{w}}[P(\lambda)]\}.$$

Consequently, if $(0, 0, \ldots, 0) \in JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$, then $W_{\varepsilon,\mathbf{w}}[P(\lambda)] = \mathbb{C}$. In the following theorem, we characterize the joint pseudofield of values of matrix polynomials. For this, we denote the weighted ε-cell in $\mathbb{C}^{m+1}$ with center at $a = (a_0, a_1, \ldots, a_m) \in \mathbb{C}^{m+1}$ by

$$\begin{aligned} \mathcal{C}_\varepsilon(a_0, \ldots, a_m) &:= D(a_0, \varepsilon\omega_0) \times \cdots \times D(a_m, \varepsilon\omega_m) \\ &= \{(z_0, \ldots, z_m) : |z_i - a_i| \le \varepsilon\omega_i, i = 0, \ldots, m\}. \end{aligned}$$

Theorem 3.1. *Let $P(\lambda)$ be a matrix polynomial as in (1.2). Then*

$$JW_{\varepsilon,\boldsymbol{w}}[P(\lambda)] = W(A_0, \ldots, A_m) + \mathcal{C}_\varepsilon(0, \ldots, 0).$$

Proof. Let $(c_0, c_1, \ldots, c_m) \in JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$. So, by (3.1), there exist $P_\Delta(\lambda) := (A_m + \Delta_m)\lambda^m + \cdots + (A_1 + \Delta_1)\lambda + (A_0 + \Delta_0) \in \mathcal{B}(P, \varepsilon, \mathbf{w})$ and $x \in S^1$ such that

$$(c_0, c_1, \ldots, c_m) = (x^* A_0 x, \ldots, x^* A_m x) + (x^* \Delta_0 x, \ldots, x^* \Delta_m x).$$

Since $|x^*\Delta_j x| \leq \|\Delta_j\| \leq \varepsilon\omega_j$ for every $j = 0, 1, \ldots, m$, we see that $(c_0, c_1, \ldots, c_m) \in W(A_0, \ldots, A_m) + \mathcal{C}_\varepsilon(0, \ldots, 0)$.

Conversly, let $x \in S^1$ and $\xi_j \in \mathbb{C}$ for $j = 0, 1, \ldots, m$, be such that $|\xi_j| \leq \varepsilon\omega_j$. Then by setting $\Delta_j = \xi_j I_n$, we see that $P_\Delta(\lambda) \in \mathcal{B}(P, \varepsilon, \mathbf{w})$ and $(x^*A_0x+\xi_0, x^*A_1x+\xi_1, \ldots, x^*A_mx+\xi_m) \in JW[P_\Delta(\lambda)]$. This shows that $W(A_0, \ldots, A_m) + \mathcal{C}_\varepsilon(0, \ldots, 0) \subseteq JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$, and so, the proof is complete. □

Now, we show that every interior point of $JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$ produces an interior point of $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$.

Theorem 3.2. *Let $P(\lambda)$ be a matrix polynomial as in (1.2). If $a_m\mu^m + \cdots + a_1\mu + a_0 = 0$, where $\mu \in \mathbb{C}$ and $(a_0, a_1, \ldots, a_m) \in \mathrm{Int}(JW_{\varepsilon,\boldsymbol{w}}[P(\lambda)])$, then $\mu \in \mathrm{Int}(W_{\varepsilon,\boldsymbol{w}}[P(\lambda)])$. Here, $\mathrm{Int}(S)$ denotes the set of all interior points of $S \subseteq \mathbb{C}$.*

Proof. By hypothesis and Theorem 3.1(i), $\mu \in W_{\varepsilon,\mathbf{w}}[P(\lambda)]$. Moreover, by (3.1), there exist $P_\Delta(\lambda) \in \mathcal{B}(P, \varepsilon, \mathbf{w})$ and $x \in S^1$ such that $a_j = x^*(A_j + \Delta_j)x$ for $j = 0, 1, \ldots, m$, and $a_m\lambda^m + \cdots + a_1\lambda + a_0 = x^*P_\Delta(\lambda)x$. So, $x^*P_\Delta(\mu)x = 0$. Now, let $\lambda_1 = \mu$ and $\lambda_2, \ldots, \lambda_m$ be the roots of the equation $x^*P_\Delta(\lambda)x = 0$. Therefore,

$$x^*P_\Delta(\lambda)x = a_m(\lambda - \mu)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m).$$

Now, if $\mu \notin \mathrm{Int}(W_{\varepsilon,\mathbf{w}}[P(\lambda)])$, then there exists a sequence $\{\mu_t\}_{t=1}^{\infty} \in \mathbb{C} \setminus W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ such that $\mu_t \longrightarrow \mu$ as $t \longrightarrow \infty$. By setting

$$\begin{aligned} q_t(\lambda) :&= a_m(\lambda - \mu_t)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m) \\ &= a_m\lambda^m + b_{m-1,t}\lambda^{m-1} + \cdots + b_{1,t}\lambda + b_{0,t}, \end{aligned} \tag{3.2}$$

where $t \in \mathbb{N}$, we see that

$$\lim_{t \longrightarrow \infty} (b_{0,t}, \ldots, b_{m-1,t}, a_m) = (a_0, \ldots, a_m). \tag{3.3}$$

Since $(a_0, \ldots, a_m) \in \mathrm{Int}(JW_{\varepsilon,\mathbf{w}}[P(\lambda)])$, by (3.3), there exists $t_0 \in \mathbb{N}$ such that for every $t \in \mathbb{N}$ with $t \geq t_0$, $(b_{0,t}, \ldots, b_{m-1,t}, a_m) \in JW_{\varepsilon,\mathbf{w}}[P(\lambda)]$. So, by Theorem 3.1($i$) and the fact, see (3.2), that $q_t(\mu_t) = 0$, we see that $\mu_t \in W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ for all $t \geq t_0$, which is a contradiction. So, the proof is complete. □

Since $W[P(\lambda)]$ is not bounded in general (see [8]), the inclusion $W[P(\lambda)] \subseteq W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ (see relation (2.1)) shows that $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ need not be a bounded set. The following theorem is related to the boundedness of the ε-pseudofield of values of matrix polynomials.

Theorem 3.3. *Let $P(\lambda)$ be a matrix polynomial as in (1.2). Then $W_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$ is bounded if and only if $0 \notin W_{\varepsilon\omega_m}(A_m)$.*

Proof. At first, we assume that $0 \notin W_{\varepsilon\omega_m}(A_m)$, and we will show that $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ is bounded. Since $W_{\varepsilon\omega_m}(A_m)$ is a compact convex set, there exists a real number $\delta > 0$ such that $D(0,\delta) \cap W_{\varepsilon\omega_m}(A_m) = \emptyset$. By setting $M = \frac{N}{\delta} + 1$, where $N = \max\{r(A_i) + \varepsilon\omega_i : i = 0, \ldots, m\}$, we will show that:

$$W_{\varepsilon,\mathbf{w}}[P(\lambda)] \subseteq \{\mu \in \mathbb{C} : |\mu| \leq M\}.$$

For this, let $\mu \in W_{\varepsilon,\mathbf{w}}[P(\lambda)]$. Since $M \geq 1$, it is enough to assume that $|\mu| > 1$. There exist a vector $x \in S^1$ and $\Delta := (\Delta_0, \Delta_1, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$ such that $\|\Delta_j\| \leq \varepsilon\omega_j$ for $j = 0, \ldots, m$, and $x^* P_\Delta(\mu) x = 0$. Therefore, by the fact, see (1.1), that $x^*(A_m + \Delta_m)x \in W_{\varepsilon\omega_m}(A_m)$, we obtain

$$\begin{aligned} |\mu|^m \delta &\leq \sum_{i=0}^{m-1} |\mu|^i N \\ &= \frac{|\mu|^m - 1}{|\mu| - 1} N. \end{aligned}$$

So, $|\mu| \leq \frac{N}{\delta} + 1 = M$, and hence, the result holds.

Conversly, let $0 \in W_{\varepsilon\omega_m}(A_m)$. So, by (1.1) and [8, Theorem 2.3], one of the admissible perturbed matrix polynomials of $P(\lambda)$ has an unbounded field of values, and hence by (2.2), $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ is unbounded. This completes the proof. □

The following corollary follows from Theorem 3.3 by the fact that $W_0(A_m) = W(A_m)$.

Corollary 3.4. *Let $\omega_m = 0$, and $P(\lambda)$ be a matrix polynomial as in (1.2). Then $W_{\varepsilon,\boldsymbol{w}}[P(\lambda)]$ is bounded if and only if $0 \notin W(A_m)$.*

In the following theorem, we show that $W_{\varepsilon,\mathbf{w}}[P(\lambda)]$ for monic matrix polynomials is contained in a circular annulus.

Theorem 3.5. *Let $P(\lambda)$, as in (1.2), be a monic matrix polynomial and $\varepsilon\omega_m < 1$. Then*

$$W_{\varepsilon,\boldsymbol{w}}[P(\lambda)] \subseteq \{z \in \mathbb{C} : r_1 \leq |z| \leq 1 + r_2\},$$

where

$$r_1 := \frac{\max\{d(0, W(A_0)) - \varepsilon\omega_0, 0\}}{\max\{d(0, W(A_0)) - \varepsilon\omega_0, 0\} + \max\{r(A_i) + \varepsilon\omega_i : i = 1, \ldots, m\}},$$

and

$$r_2 := \frac{\max\{r(A_i) + \varepsilon\omega_i : i = 0, \ldots, m-1\}}{1 - \varepsilon\omega_m}.$$

Proof. Let $\mu \in W_{\varepsilon,\mathbf{w}}[P(\lambda)]$. Then, by Definition 2.1, there exist $\Delta := (\Delta_0, \Delta_1, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$ with $\|\Delta_j\| \le \varepsilon\omega_j$ for $j = 0, \ldots, m$, and a vector $x \in S^1$ such that $x^* P_\Delta(\mu)x = 0$. So,

$$x^*(I_n + \Delta_m)x\mu^m + x^*(A_{m-1} + \Delta_{m-1})x\mu^{m-1} + \cdots + x^*(A_0 + \Delta_0)x = 0. \tag{3.4}$$

Now, to prove the left inequality, since $r_1 \le 1$, it is enough to consider the case that $|\mu| < 1$. By (3.4) and using (1.1), we see that

$$\begin{aligned}\max\{d(0, W(A_0)) - \varepsilon\omega_0, 0\} &\le |x^*(A_0 + \Delta_0)x| \\ &\le \frac{|\mu|}{1 - |\mu|} \times \max_{i=1,\ldots,m} (r(A_i) + \varepsilon\omega_i).\end{aligned}$$

Hence, $r_1 \le |\mu|$.

To prove the right inequality, it is enough to consider the case that $|\mu| > 1$. By relations (3.4) and (1.1), we have:

$$\begin{aligned}|\mu|^m(1 - \varepsilon\omega_m) &\le \sum_{i=0}^{m-1} |\mu|^i |x^*(A_i + \Delta_i)x| \\ &\le \frac{|\mu|^m - 1}{|\mu| - 1} \times \max_{i=0,\ldots,m-1} (r(A_i) + \varepsilon\omega_i).\end{aligned}$$

So, $|\mu| \le 1 + r_2$. This completes the proof. □

4. On Pseudofield of Values of the Companion Linearization of Matrix Polynomials

Consider a matrix polynomial $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_0$ as in (1.2) in which $m \ge 2$. The companion linearization of $P(\lambda)$ is defined, e.g., see [4], as the following linear pencil $L(\lambda)$ of order mn:

$$L(\lambda) = \begin{pmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & \cdots & 0 & A_m \end{pmatrix} \lambda - \begin{pmatrix} 0 & I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_n \\ -A_0 & -A_1 & \cdots & \cdots & \cdots & -A_{m-1} \end{pmatrix}. \tag{4.1}$$

In the following theorem, we state the relationship between pseudofield of values of $P(\lambda)$ and the pseudofield of values of its companion linearization $L(\lambda)$.

Theorem 4.1. *Let $P(\lambda)$, as in (1.2), be a matrix polynomial with the companion linearization $L(\lambda)$ as in (4.1). Then*

$$W_{\varepsilon,\boldsymbol{w}}[P(\lambda)] \cup \{0\} \subseteq W_{\varepsilon,\boldsymbol{w}'}[L(\lambda)],$$

where $\boldsymbol{w}' = \left\{\omega_0' = \sqrt{m}\max\{\omega_0, \ldots, \omega_{m-1}\}, \omega_1' = \omega_m\right\}$.

Proof. For every $\mu \in \mathbb{C}$ and $x \in S^1$, we consider the following vector:

$$y = \frac{1}{\sqrt{1+|\mu|^2+|\mu|^4+\cdots+|\mu|^{2m-2}}}\begin{pmatrix} x \\ \mu x \\ \vdots \\ \mu^{m-1}x \end{pmatrix} \in \mathbb{C}^{mn}.$$

So, we have $y^*y = x^*x = 1$. Also, for every $\Delta := (\Delta_0, \ldots, \Delta_m) \in \mathbb{M}_n^{m+1}$, we consider the following two block matrices in $\mathbb{M}_{mn}$:

$$\widehat{\Delta}_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Delta_0 & -\Delta_1 & \cdots & -\Delta_{m-1} \end{pmatrix}, \qquad \widehat{\Delta}_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_m \end{pmatrix}.$$

Therefore, by the fact that $\max\left\{t_1+t_2+\cdots+t_m : t_i \in \mathbb{R}, \sum_{j=1}^m t_j^2 = 1\right\} = \sqrt{m}$, we have $\|\widehat{\Delta}_0\| \le \sqrt{m}\max\{\|\Delta_0\|, \ldots, \|\Delta_{m-1}\|\} \le \varepsilon\omega_0'$ and $\|\widehat{\Delta}_1\| = \|\Delta_m\| \le \varepsilon\omega_m$. Moreover, we have

$$y^*L_{\widehat{\Delta}}(\mu)y = \frac{\overline{\mu^{m-1}}}{1+|\mu|^2+\cdots+|\mu|^{2m-2}}x^*P_\Delta(\mu)x, \tag{4.2}$$

where $\widehat{\Delta} = (\widehat{\Delta}_0, \widehat{\Delta}_1)$. Now, let $\mu \in W_{\varepsilon,\mathbf{w}}[P(\lambda)] \cup \{0\}$. If $\mu = 0$, then by selecting any arbitrary vector $x \in S^1$ and $\Delta \in \mathbb{M}_n^{m+1}$, and using relation (4.2), we see that $y^*L_{\widehat{\Delta}}(0)y = 0$. This shows that $\mu = 0 \in W_{\varepsilon,\mathbf{w}'}[L(\lambda)]$. For the case $\mu \in W_{\varepsilon,\mathbf{w}}[P(\lambda)]$, there exist a vector $x \in S^1$ and $\Delta \in \mathbb{M}_n^{m+1}$ such that $x^*P_\Delta(\mu)x = 0$. So, by (4.2), we have $y^*L_{\widehat{\Delta}}(\mu)y = 0$, and hence, $\mu \in W_{\varepsilon,\mathbf{w}'}[L(\lambda)]$. This completes the proof. □

For the remainder of this section, using Theorem 4.1, we study pseudofield of values of the companion linearization of the matrix polynomial $P(\lambda) = \lambda^m I_n - A$, where $m \ge 2$ and $A \in \mathbb{M}_n$. By (4.1), the companion linearization of $P(\lambda)$ is $L(\lambda) = \lambda I_{mn} - \Pi_A$, where

$$\Pi_A = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \\ A & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{M}_{mn}. \tag{4.3}$$

The matrix Π_A, as in (4.3), is called the basic $A-$factor block circulant matrix. These matrices have important applications in vibration analysis and differential equations. For more details, see [2] and its references. To state the following theorem, if $S \subseteq \mathbb{C}$ and m is a positive integer, then the m-th root of S is denoted by $\sqrt[m]{S}$, i.e., $\sqrt[m]{S} := \{\mu \in \mathbb{C} : \mu^m \in S\}$.

Theorem 4.2. *Let $A \in \mathbb{M}_n$, and Π_A be the basic $A-$factor block circulant matrix as in (4.3). Then*

$$conv\left(\sqrt[m]{W_\varepsilon(A)}\right) \subseteq W_{\varepsilon\sqrt{m}}(\Pi_A).$$

Proof. Consider the matrix polynomial $P(\lambda) = \lambda^m I_n - A$, and so, its companion linearization is $L(\lambda) = \lambda I_{mn} - \Pi_A$. By setting $\mathbf{w} = \{\omega_0 = 1, \omega_1 = \omega_2 = \cdots = \omega_m = 0\}$ and $\mathbf{w}' = \left\{\omega_0' = \sqrt{m}, \omega_1' = 0\right\}$, and also using Proposition 2.4 and relation (1.1), we have

$$\begin{aligned} W_{\varepsilon,\mathbf{w}}[P(\lambda)] &= \{\mu \in \mathbb{C} : 0 \in W(\mu^m I_n - A) - D(0,\epsilon)\} \\ &= \{\mu \in \mathbb{C} : \mu^m \in W(A) + D(0,\epsilon) = W_\epsilon(A)\} \\ &= \sqrt[m]{W_\varepsilon(A)}. \end{aligned}$$

Also, it is clear that

$$W_{\varepsilon,\mathbf{w}'}[L(\lambda)] = W_{\varepsilon\sqrt{m}}(\Pi_A).$$

So, by Theorem 4.1, we have

$$\sqrt[m]{W_\varepsilon(A)} \cup \{0\} = W_{\varepsilon,\mathbf{w}}[P(\lambda)] \cup \{0\} \subseteq W_{\varepsilon,\mathbf{w}'}[L(\lambda)] = W_{\sqrt{m}\varepsilon}(\Pi_A).$$

Since $m \geq 2$,

$$conv\left(\sqrt[m]{W_\varepsilon(A)} \cup \{0\}\right) = conv(\sqrt[m]{W_\varepsilon(A)}).$$

This completes the proof. □

The final example shows that the set equality in Theorem 4.2 does not hold in general.

Example 4.3. Let $P(\lambda) = \lambda^2 I_2 - A$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover, let $\mathbf{w} = \{\omega_0 = 1\ ,\ \omega_1 = 0\ ,\ \omega_2 = 0\}$, and $\omega' = \{\omega_0' = \sqrt{2}, \omega_1' = 0\}$, and

$\varepsilon = \frac{1}{2}$. We know, by Proposition 2.4, that $W_{\varepsilon,\mathbf{w}}[P(\lambda)] = \sqrt{W_{\frac{1}{2}}(A)}$ and $W_{\varepsilon,\mathbf{w}'}[L(\lambda)] = W_{\frac{\sqrt{2}}{2}}(\Pi_A)$. Since A is unitary, by [1, Theorem 3.3], Π_A is also a unitary matrix. Hence, $W_{\frac{\sqrt{2}}{2}}(\Pi_A) = conv(\{1\ ,\ -1\ ,\ i\ ,\ -i\}) + D(0, \frac{\sqrt{2}}{2})$. By comparing $W_{\frac{\sqrt{2}}{2}}(\Pi_A)$ and $\sqrt{W_{\frac{1}{2}}(A)}$ as in Figure 2, we see that these sets are not equal.

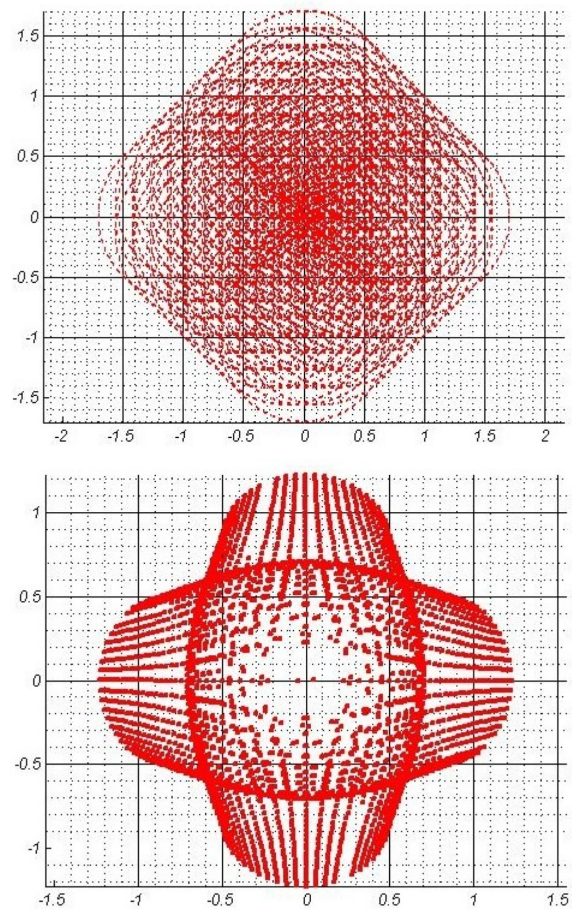

FIGURE 2. The above is $W_{\frac{\sqrt{2}}{2}}(\Pi_A)$, and the below is $\sqrt{W_{\frac{1}{2}}(A)}$ (Example 4.3).

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