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Common Fixed Point Results on Complex-Valued S-Metric Spaces

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Abstract. Banach's contraction principle has been improved and extensively studied on several generalized metric spaces. Recently, complex-valued S-metric spaces have been introduced and studied for this purpose. In this paper, we investigate some generalized fixed point results on a complete complex valued S-metric space. To do this, we prove some common fixed point (resp. fixed point) theorems using different techniques by means of new generalized contractive conditions and the notion of the closed ball. Our results generalize and improve some known fixed point results. We provide some illustrative examples to show the validity of our definitions and fixed point theorems.

1. Introduction

During the last several decades, Banach's contraction principle has been improved and studied by some authors on metric and several generalized metric spaces. In 1977, Rhoades studied some comparisons of known contractive mappings and proved new fixed point theorems [27]. Also he introduced a new contractive mapping called as a Rhoades' mapping. In 1994, Dien proved a common fixed point theorem for the pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space [6]. In 1998, Liu, Xu and Cho gave necessary and sufficient conditions for the existence of fixed and common fixed points of self-mappings of metric spaces [12]. They defined the notion of an L-mapping to give a fixed point theorem for a Rhoades' mapping. The present authors defined Rhoades' condition on S-metric

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spaces and proved some fixed point theorems [15]. Some new contractive mappings were studied on S-metric spaces and investigated their relationships with the Rhoades' condition [16]. It was generalized and extended known fixed point theorems in the literature using S-metric spaces [17]. New generalized fixed point results have been obtained on several generalized metric spaces such as ordered S-metric spaces, C^*-algebra-valued S-metric spaces (see [3, 4, 10, 11, 26, 29, 30]).

In 2011, Azam, Fisher and Khan introduced complex-valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions [5]. In 2014, Ahmad, Azam and Saejung improved the conditions of contractive mappings from the whole space to a closed ball and established common fixed point theorems [1]. In 2014, Öztürk established common fixed point theorems for two pairs of weakly compatible self-mappings of a complex-valued metric space [22]. Also, Öztürk and Kaplan proved common fixed point theorems for two Banach pairs of mappings with f-contraction [23]. In 2015, some coupled common fixed point theorems were obtained on a complex-valued G_b-metric space [24]. In 2014, Mlaiki introduced the notion of a complex-valued S-metric space and showed the existence and the uniqueness of a common fixed point of two self-mappings on a complex valued S-metric space [13]. In [2], some fixed point theorems were studied for new type generalized contractive mappings involving C-class function in complex-valued G_b-metric spaces. Similar studies have been extensively studied on various generalized complex valued metric spaces (see [7–9, 25]).

In this paper, we investigate some common fixed point theorems on complex valued S-metric spaces. In Section 2 we recall some definitions and lemmas which are needed in the sequel. In Section 3 we obtain a new generalization of the well known Banach's contraction principle using the notion of a complex-valued S-metric space. In Section 4 we introduce new notions on complex-valued S-metric spaces. In Section 5 we give a new common fixed point result on a complete complex-valued S-metric space. In Section 6 we define the notions of an open ball and a closed ball on a complex-valued S-metric space and give some applications of common fixed point theory in view of the closed balls. In the whole paper we give some examples to show the validity of our definitions and fixed point theorems.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and z_1, $z_2 \in \mathbb{C}$. The partial order $\precsim$ is defined on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \le \mathrm{Re}(z_2),\ Im(z_1) \le Im(z_2),$$

and

$$z_1 \prec z_2 \text{ if and only if } \mathrm{Re}(z_1) < \mathrm{Re}(z_2),\ Im(z_1) < Im(z_2).$$

Also we write $z_1 \precsim z_2$ if one of the following conditions hold:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $Im(z_1) < Im(z_2)$,
(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $Im(z_1) = Im(z_2)$.

Note that

$$0 \precsim z_1 \precnsim z_2 \quad \Rightarrow \quad |z_1| < |z_2|,$$

and

$$z_1 \precsim z_2,\ z_2 \prec z_3 \quad \Rightarrow \quad z_1 \prec z_3.$$

Now we recall some known definitions and lemmas as seen in the references.

Definition 2.1 ([32])**.** The "*max*" function is defined for the partial order relation $\precsim$ as follow:

(i) $\max\{z_1, z_2\} = z_2 \quad \Leftrightarrow \quad z_1 \precsim z_2$.
(ii) $z_1 \precsim \max\{z_2, z_3\} \quad \Rightarrow \quad z_1 \precsim z_2$ or $z_1 \precsim z_3$.
(iii) $\max\{z_1, z_2\} = z_2 \quad \Leftrightarrow \quad z_1 \precsim z_2$ or $|z_1| < |z_2|$.

Lemma 2.2 ([32])**.** *Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\precsim$ be defined on $\mathbb{C}$. Then the following statements are satisfied*:

(i) *If $z_1 \precsim \max\{z_2, z_3\}$ then $z_1 \precsim z_2$ if $z_3 \precsim z_2$,*
(ii) *If $z_1 \precsim \max\{z_2, z_3, z_4\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4\} \precsim z_2$,*
(iii) *If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4, z_5\} \precsim z_2$, and so on.*

Definition 2.3 ([13])**.** Let X be a nonempty set. A complex-valued S-metric on X is a function $\mathcal{S}_C : X \times X \times X \to \mathbb{C}$ that satisfies the following conditions for all $x, y, z, t \in X$:

(**CS1**) $0 \precsim \mathcal{S}_C(x, y, z)$,
(**CS2**) $\mathcal{S}_C(x, y, z) = 0$ if and only if $x = y = z$,
(**CS3**) $\mathcal{S}_C(x, y, z) \precsim \mathcal{S}_C(x, x, t) + \mathcal{S}_C(y, y, t) + \mathcal{S}_C(z, z, t)$.

The pair $(X, \mathcal{S}_C)$ is called a complex-valued S-metric space.

Example 2.4. Let $X = \mathbb{C}$ and the function $\mathcal{S}_C : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined by:

$$\mathcal{S}_C(z_1, z_2, z_3) = |\mathrm{Re}(z_1) - \mathrm{Re}(z_3)| + |\mathrm{Re}(z_2) - \mathrm{Re}(z_3)| + i(|Im(z_1) - Im(z_3)| + |Im(z_2) - Im(z_3)|,$$

for all $z_1, z_2, z_3 \in \mathbb{C}$. Then, it is easy to see that the function $\mathcal{S}_C$ is a complex-valued S-metric on $\mathbb{C}$.

We use the following definitions and lemmas in the next sections.

Definition 2.5 ([13])**.** Let $(X, \mathcal{S}_C)$ be a complex-valued S -metric space. Then

(i) A sequence $\{x_n\}$ in X converges to x if and only if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n \geq n_0$, we have $\mathcal{S}_C(x_n, x_n, x) \prec \varepsilon$ and it is denoted by $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$ there exists a natural number n_0 such that for all $n, m \geq n_0$, we have $\mathcal{S}_C(x_n, x_n, x_m) \prec \varepsilon$.

(iii) A complex-valued S-metric space $(X, \mathcal{S}_C)$ is called complete, if every Cauchy sequence is convergent.

Definition 2.6 ([13])**.** Two families of self-mappings $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^n$ are said to be pairwise commuting if the following three conditions hold:

(i) $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \ldots, m\}$,

(ii) $g_k g_l = g_l g_k$ for all $k, l \in \{1, 2, \ldots, n\}$,

(iii) $f_i g_k = g_k f_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

Lemma 2.7 ([13])**.** *Let $(X, \mathcal{S}_C)$ be a complex-valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|\mathcal{S}_C(x_n, x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.8 ([13])**.** *Let $(X, \mathcal{S}_C)$ be a complex-valued S -metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\mathcal{S}_C(x_n, x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 2.9 ([13])**.** *If $(X, \mathcal{S}_C)$ be a complex-valued S -metric space then*

$$\mathcal{S}_C(x, x, y) = \mathcal{S}_C(y, y, x),$$

for all $x, y \in X$.

Corollary 2.10 ([13])**.** *If f is a self-mapping on a complete complex valued S-metric space $(X, \mathcal{S}_C)$ that satisfies*

$$\mathcal{S}_C(f^n x, f^n x, f^n y) \precsim h\mathcal{S}_C(x, x, y),$$

for all $x, y \in X$ and h a nonnegative real number such that $h < 1$, then f has a unique fixed point in X.

If we take $n = 1$ in Corollary 2.10, then we have the Banach's contraction principle on a complex-valued S-metric space as seen in the following theorem:

Theorem 2.11. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S -metric space and f be a self-mapping of X satisfying*

$$\mathcal{S}_C(fx, fx, fy) \precsim h\mathcal{S}_C(x, x, y), \tag{2.1}$$

for some $h \in [0,1)$ and all $x, y \in X$. Then f has a unique fixed point in X.

Notice that there exists a self-mapping f which has a fixed point, but it does not satisfy Banach's contraction principle on complex-valued S-metric spaces as we have seen in the following example:

Example 2.12. Let $X = \mathbb{C}$ and the function $S : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined

$$\mathcal{S}_C(z_1, z_2, z_3) = |z_1 - z_3| + |z_1 + z_3 - 2z_2|,$$

for all $z_1, z_2, z_3 \in \mathbb{C}$. Then the function S is a complex-valued S-metric on $\mathbb{C}$. Let us consider

$$fz = 1 - z.$$

Then f is a self-mapping on the complete complex-valued S-metric space $[0,1]$. f has a fixed point $z = \frac{1}{2}$, but f does not satisfy the Banach's contraction principle.

Hence it is important to study some new fixed point theorems.

3. A New Generalized Fixed Point Theorem

In this section, we prove a new generalization of the well known Banach's contraction principle using the notion of a complex-valued S-metric space.

Let $(X, \mathcal{S}_C)$ be a complex-valued S-metric space and f be a self-mapping of X. There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that

$$\mathcal{S}_C(fx, fx, fy) \precsim a\mathcal{S}_C(x, x, y) + b \max \left\{ \begin{array}{c} \mathcal{S}_C(fx, fx, x), \mathcal{S}_C(fx, fx, y), \\ \mathcal{S}_C(fy, fy, y), \mathcal{S}_C(fy, fy, x) \end{array} \right\}, \tag{3.1}$$

for all $x, y \in X$.

Theorem 3.1. *Let $(X, \mathcal{S}_C)$ be a complete complex valued S-metric space and f be a self-mapping of X. If f satisfies the inequality (3.1), then f has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as follows:

$$f^n x_0 = x_n.$$

Suppose that $x_n \neq x_{n+1}$ for all n. From the inequality (3.1), we get

$$\begin{aligned} \mathcal{S}_C(x_n, x_n, x_{n+1}) &= \mathcal{S}_C(fx_{n-1}, fx_{n-1}, fx_n) \precsim a\mathcal{S}_C(x_{n-1}, x_{n-1}, x_n) \\ &\quad + b \max \left\{ \begin{array}{c} \mathcal{S}_C(x_n, x_n, x_{n-1}), \mathcal{S}_C(x_n, x_n, x_n), \\ \mathcal{S}_C(x_{n+1}, x_{n+1}, x_n), \mathcal{S}_C(x_{n+1}, x_{n+1}, x_{n-1}) \end{array} \right\} \\ &= a\mathcal{S}_C(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

$$+ b\max\left\{\begin{array}{c}\mathcal{S}_C(x_n,x_n,x_{n-1}),\\ \mathcal{S}_C(x_{n+1},x_{n+1},x_n),\\ \mathcal{S}_C(x_{n+1},x_{n+1},x_{n-1})\end{array}\right\}$$
$$= a\mathcal{S}_C(x_{n-1},x_{n-1},x_n)+b\alpha.$$

Then using the condition (**CS3**), we get

$$\begin{aligned}|\mathcal{S}_C(x_n,x_n,x_{n+1})| &\leq a\,|\mathcal{S}_C(x_{n-1},x_{n-1},x_n)| + b\,|\alpha| \\ &\leq a\,|\mathcal{S}_C(x_{n-1},x_{n-1},x_n)| + 2b\,|\mathcal{S}_C(x_{n+1},x_{n+1},x_n)| \\ &\quad + b\,|\mathcal{S}_C(x_{n-1},x_{n-1},x_n)|,\end{aligned}$$

and so using Lemma 2.9, we have

$$(1-2b)\,|\mathcal{S}_C(x_n,x_n,x_{n+1})| \leq (a+b)\,|\mathcal{S}_C(x_{n-1},x_{n-1},x_n)|,$$

which implies

$$|\mathcal{S}_C(x_n,x_n,x_{n+1})| \leq \frac{a+b}{1-2b}\,|\mathcal{S}_C(x_{n-1},x_{n-1},x_n)|. \tag{3.2}$$

Let $\beta = \frac{a+b}{1-2b}$. Since $a+3b<1$, $\beta<1$. Using the inequality (3.2), we obtain

$$|\mathcal{S}_C(x_n,x_n,x_{n+1})| \leq \beta^n\,|\mathcal{S}_C(x_0,x_0,x_1)|. \tag{3.3}$$

Now we prove that the sequence $\{x_n\}$ is Cauchy. For all $n,m\in\mathbb{N}$, $n<m$, using the inequality (3.3), we find

$$|\mathcal{S}_C(x_n,x_n,x_m)| \leq \frac{\beta^n}{1-\beta}\,|\mathcal{S}_C(x_0,x_0,x_1)|.$$

Therefore $|\mathcal{S}_C(x_n,x_n,x_m)|\to 0$ as $n,m\to\infty$. Consequently, $\{x_n\}$ is a Cauchy sequence. Using the completeness hypothesis, there exists $x\in X$ such that $\{x_n\}\to x$. Now, we show that x is a fixed point of f. Suppose that $fx\neq x$. Then using the inequality (3.1), we have

$$\begin{aligned}\mathcal{S}_C(x_n,x_n,fx) = \mathcal{S}_C(fx_{n-1},fx_{n-1},fx) &\precsim a\mathcal{S}_C(x_{n-1},x_{n-1},x) \\ &\quad + b\max\left\{\begin{array}{c}\mathcal{S}_C(x_n,x_n,x_{n-1}),\mathcal{S}_C(x_n,x_n,x),\\ \mathcal{S}_C(fx,fx,x),\mathcal{S}_C(fx,fx,x_{n-1})\end{array}\right\},\end{aligned}$$

and so taking limit for $n\to\infty$, we get

$$\mathcal{S}_C(x,x,fx) \precsim b\mathcal{S}_C(fx,fx,x),$$

and by Lemma 2.9, we obtain

$$|\mathcal{S}_C(fx,fx,x)| \leq b\,|\mathcal{S}_C(fx,fx,x)|,$$

which implies $fx=x$, that is, x is a fixed point of f. Now, we prove that the fixed point x is unique. Suppose that y is another fixed point of f such that $x\neq y$. Using the inequality (3.1) and Lemma 2.9, we get

$$\mathcal{S}_C(fx,fx,fy) = \mathcal{S}_C(x,x,y) \precsim a\mathcal{S}_C(x,x,y)$$

$$+ b \max \left\{ \begin{array}{c} \mathcal{S}_C(x,x,x), \mathcal{S}_C(x,x,y), \\ \mathcal{S}_C(y,y,y), \mathcal{S}_C(y,y,x) \end{array} \right\}$$
$$= (a+b)\mathcal{S}_C(x,x,y),$$

and so

$$|\mathcal{S}_C(x,x,y)| \leq (a+b)\,|\mathcal{S}_C(x,x,y)|\,,$$

which implies $x = y$ since $a + b < 1$. Consequently, x is the unique fixed point of f. □

Remark 3.2.

1. Theorem 3.1 is a generalization of the Banach's contraction principle on a complete complex-valued S-metric space.
2. If we take the function $\mathcal{S}_C : X \times X \times X \to [0, \infty)$ in Theorem 3.1 then we get Theorem 1 given in [17] on page 233 on a complete S-metric space.
3. If we consider Example 1 given in [17] on page 236, then we see an example of a function that satisfies the inequality (3.1) but not satisfy the Banach's contraction principle.

4. Some Notions on Complex-Valued S-Metric Spaces

In this section, we introduce new concepts on a complex-valued S-metric space. We give the definitions of CS-weakly computing and CS -compatible mappings and investigate the relationships between them.

We begin the following definitions.

Definition 4.1. Let $(X, \mathcal{S}_C)$ be a complex-valued S-metric space and $f, g : X \to X$ be two mappings. Then f and g are called CS -weakly commuting if and only if

$$\mathcal{S}_C(fgx, fgx, gfx) \precsim \mathcal{S}_C(fx, fx, gx),$$

for all $x \in X$.

Definition 4.2. Let f and g be self-mappings of a complex-valued S-metric space $(X, \mathcal{S}_C)$. The mappings f and g are called CS -compatible if

$$\lim_{n\to\infty} \mathcal{S}_C(fgx_n, fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $x \in X$.

Notice that every CS-weakly commuting mappings are CS-compatible. If f and g are two mappings of X into X such that

$$\lim_{n\to\infty} \mathcal{S}_C(fx_n, fx_n, gx_n) = 0,$$

this implies

$$\lim_{n\to\infty} \mathcal{S}_C(fgx_n, fgx_n, gfx_n) = 0,$$

and so the mappings f and g are CS-compatible. But every CS -compatible mappings are not always CS-weakly commuting as seen in the following example:

Example 4.3. Let $X = \mathbb{C}$ and f, g be two self-mappings of $\mathbb{C}$ such that

$$fz = z^2,$$

and

$$gz = e^{it}z^2,$$

for some fixed $t \in \mathbb{R}$, $t \neq 2k\pi$, $k \in \mathbb{Z}$, respectively.

Let us consider the function $\mathcal{S}_C : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$\mathcal{S}_C(z_1, z_2, z_3) = |z_1 - z_3| + |z_2 - z_3|,$$

for all $z_1, z_2, z_3 \in \mathbb{C}$. Then $\mathcal{S}_C$ is a complex-valued S-metric which is called complex valued usual S-metric. It can be easily seen that the functions f and g are CS-compatible but they are not CS-weakly commuting.

Definition 4.4. Let (X, S) be a complex-valued S-metric space and $f : X \to X$ be a mapping. f is called CS-orbitally continuous if $x_0 \in X$ such that $x_0 = \lim\limits_{i\to\infty} f^{n_i}x$ for some $x \in X$, then $fx_0 = \lim\limits_{i\to\infty} ff^{n_i}x$.

Now, we define the concept of CS-weakly compatibility.

Definition 4.5. Let (X, S) be a complex-valued S-metric space and f, g be two self-mappings of X. Then the pair (f, g) is called CS-weakly compatible if $fgx = gfx$ whenever $fx = gx$ for all $x \in X$.

5. A Common Fixed Point Theorem

In this section, we obtain a new common fixed point theorem using the notions of CS-weakly compatibility and commuting pair for six self-mappings on a complete complex-valued S-metric space.

Theorem 5.1. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S-metric space and f, g, h, k, l, m be six self-mappings of X satisfying the following conditions:*

$$fg(X) \subset h(X),\ kl(X) \subset m(X) \tag{5.1}$$

and

$$\begin{aligned} \mathcal{S}_C(klx, klx, fgy) \precsim\ & a\mathcal{S}_C(hx, hx, my) \\ & + b[\mathcal{S}_C(hx, hx, klx) + \mathcal{S}_C(my, my, fgy)] \\ & + c[\mathcal{S}_C(hx, hx, fgy) + \mathcal{S}_C(my, my, klx)], \end{aligned} \tag{5.2}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 3c < 1$. Assume that (fg, m) , (kl, h) are CS-weakly compatible and the pairs (f, g), (f, m), (f, h) , (g, m), (g, h), (k, l), (k, h), (k, f), (k, g), (l, h), (l, f), (l, g), (m, k) and (m, l) are commuting pairs of mappings. Then f, g, h, k, l, m have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and a sequence $\{y_n\}$ in X be defined as follows:

$$y_{2n} = klx_{2n} = mx_{2n+1} \text{ and } y_{2n+1} = fgx_{2n+1} = hx_{2n+2}, \tag{5.3}$$

for all $n = 1, 2, 3...$ by the condition (5.1). Using the inequality (5.2) we have

$$\begin{aligned}
&\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1}) \\
&\quad = \mathcal{S}_C(klx_{2n}, klx_{2n}, fgx_{2n+1}) \precsim a\mathcal{S}_C(hx_{2n}, hx_{2n}, mx_{2n+1}) \\
&\qquad + b[\mathcal{S}_C(hx_{2n}, hx_{2n}, klx_{2n}) + \mathcal{S}_C(mx_{2n+1}, mx_{2n+1}, fgx_{2n+1})] \\
&\qquad + c[\mathcal{S}_C(hx_{2n}, hx_{2n}, fgx_{2n+1}) + \mathcal{S}_C(mx_{2n+1}, mx_{2n+1}, klx_{2n})] \\
&\quad = a\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n}) \\
&\qquad + b[\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n}) + \mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1})] \\
&\qquad + c[\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n+1}) + \mathcal{S}_C(y_{2n}, y_{2n}, y_{2n})].
\end{aligned} \tag{5.4}$$

Using the condition (**CS3**) and Lemma 2.9, we obtain

$$\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n+1}) \precsim 2\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n}) + \mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1}). \tag{5.5}$$

By the inequalities (5.4) and (5.5) we get

$$\begin{aligned}
\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1}) \precsim\ & (a + b + 2c)\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n}) \\
& + (b + c)\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1}),
\end{aligned}$$

which implies

$$\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1}) \precsim \frac{a + b + 2c}{1 - b - c}\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n}),$$

and so

$$|\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1})| \leq t\,|\mathcal{S}_C(y_{2n-1}, y_{2n-1}, y_{2n})|\,,$$

where $t = \dfrac{a + b + 2c}{1 - b - c} < 1$.

By a similar way as above we obtain

$$|\mathcal{S}_C(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq t\,|\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1})|\,.$$

Hence we get

$$\begin{aligned}
|\mathcal{S}_C(y_{2n+1}, y_{2n+1}, y_{2n+2})| &\leq t\,|\mathcal{S}_C(y_{2n}, y_{2n}, y_{2n+1})| \\
&\ \ \vdots \\
&\leq t^{n+1}\,|\mathcal{S}_C(y_0, y_0, y_1)|\,,
\end{aligned}$$

for $n = 1, 2, 3, \ldots$.

Now for all $m > n$ we have

$$\mathcal{S}_C(y_n, y_n, y_{n+m}) \precsim \frac{2t^n}{1-t}\mathcal{S}_C(y_0, y_0, y_1) + t^{n+m-1}\mathcal{S}_C(y_0, y_0, y_1),$$

and so

$$|\mathcal{S}_C(y_n, y_n, y_{n+m})| \leq \frac{2t^n}{1-t}|\mathcal{S}_C(y_0, y_0, y_1)| + t^{n+m-1}|\mathcal{S}_C(y_0, y_0, y_1)|,$$

which implies $|\mathcal{S}_C(y_n, y_n, y_{n+m})| \to 0$ as $n, m \to \infty$. Hence the sequence $\{y_n\}$ is a Cauchy sequence. Since $(X, \mathcal{S}_C)$ is a complete complex-valued S-metric space, there exists a point w in X such that

$$\lim_{n\to\infty} klx_{2n} = \lim_{n\to\infty} mx_{2n+1} = \lim_{n\to\infty} fgx_{2n+1} = \lim_{n\to\infty} hx_{2n+2} = w.$$

Also there exists a point $u \in X$ such that $hu = w$ since $fg(X) \subset h(X)$. Using the inequality (5.2) we have

$$\begin{aligned}
\mathcal{S}_C(klu, klu, w) &\precsim 2\mathcal{S}_C(klu, klu, fgx_{2n-1}) + \mathcal{S}_C(w, w, fgx_{2n-1}) \\
&\precsim 2(a\mathcal{S}_C(hu, hu, mx_{2n-1}) \\
&\quad + b[\mathcal{S}_C(hu, hu, klu) + \mathcal{S}_C(mx_{2n-1}, mx_{2n-1}, fgx_{2n-1})] \\
&\quad + c[\mathcal{S}_C(hu, hu, fgx_{2n-1}) + \mathcal{S}_C(mx_{2n-1}, mx_{2n-1}, klu)]) \\
&\quad + \mathcal{S}_C(w, w, fgx_{2n-1}).
\end{aligned}$$

Hence taking limit for $n \to \infty$ we obtain

$$\mathcal{S}_C(klu, klu, w) \precsim 2(b+c)\mathcal{S}_C(klu, klu, w),$$

and so

$$|\mathcal{S}_C(klu, klu, w)| \leq 2(b+c)|\mathcal{S}_C(klu, klu, w)|,$$

which is a contradiction since $2b + 2c < 1$. Therefore $klu = hu = w$.

There exists a point v in X such that $mv = w$ since $kl(X) \subset m(X)$. Using the inequality (5.2) we have

$$\begin{aligned}
\mathcal{S}_C(w, w, fgv) = \mathcal{S}_C(klu, klu, fgv) &\precsim a\mathcal{S}_C(hu, hu, mv) \\
&\quad + b[\mathcal{S}_C(hu, hu, klu) + \mathcal{S}_C(mv, mv, fgv)] \\
&\quad + c[\mathcal{S}_C(hu, hu, fgv) + \mathcal{S}_C(mv, mv, klu)] \\
&= a\mathcal{S}_C(w, w, w) \\
&\quad + b[\mathcal{S}_C(w, w, w) + \mathcal{S}_C(w, w, fgv)] \\
&\quad + c[\mathcal{S}_C(w, w, fgv) + \mathcal{S}_C(w, w, w)] \\
&= (b+c)\mathcal{S}_C(w, w, fgv),
\end{aligned}$$

and so

$$|\mathcal{S}_C(w, w, fgv)| \leq (b+c)|\mathcal{S}_C(w, w, fgv)|,$$

which is a contradiction since $b+c<1$. Then $fgv = mv = w = klu = hu$.

Since h and kl are CS-weakly compatible mappings of X, we have $klhu = hklu$ and so $klw = hw$. Now we prove that w is a fixed point of kl. Using the inequality (5.2), we have

$$\begin{aligned}\mathcal{S}_C(klw, klw, w) &= \mathcal{S}_C(klw, klw, fgv) \precsim a\mathcal{S}_C(hw, hw, mv)\\ &\quad + b[\mathcal{S}_C(hw, hw, klw) + \mathcal{S}_C(mv, mv, fgv)]\\ &\quad + c[\mathcal{S}_C(hw, hw, fgv) + \mathcal{S}_C(mv, mv, klw)]\\ &= a\mathcal{S}_C(w, w, klw)\\ &\quad + b[\mathcal{S}_C(klw, klw, klw) + \mathcal{S}_C(w, w, w)]\\ &\quad + c[\mathcal{S}_C(klw, klw, w) + \mathcal{S}_C(w, w, klw)]\\ &= (a+2c)\mathcal{S}_C(klw, klw, w),\end{aligned}$$

and so

$$|\mathcal{S}_C(klw, klw, w)| \leq (a+2c)\,|\mathcal{S}_C(klw, klw, w)|\,,$$

which is a contradiction since $a+2c<1$. Therefore $klw = w$ and $klw = hw = w$.

Similarly, m and fg are CS-weakly compatible mappings of X and we have $fgw = mw$. Now we show that w is a fixed point of fg. Using the inequality (5.2) we get

$$\begin{aligned}\mathcal{S}_C(w, w, fgw) &= \mathcal{S}_C(klw, klw, fgw) \precsim a\mathcal{S}_C(hw, hw, mw)\\ &\quad + b[\mathcal{S}_C(hw, hw, klw) + \mathcal{S}_C(mw, mw, fgw)]\\ &\quad + c[\mathcal{S}_C(hw, hw, fgw) + \mathcal{S}_C(mw, mw, klw)]\\ &= a\mathcal{S}_C(hw, hw, fgw)\\ &\quad + b[\mathcal{S}_C(w, w, w) + \mathcal{S}_C(fgw, fgw, fgw)]\\ &\quad + c[\mathcal{S}_C(w, w, fgw) + \mathcal{S}_C(fgw, fgw, w)]\\ &= (a+2c)\mathcal{S}_C(w, w, fgw),\end{aligned}$$

and so

$$|\mathcal{S}_C(w, w, fgw)| \leq (a+2c)\,|\mathcal{S}_C(w, w, fgw)|\,,$$

which is a contradiction since $a+2c<1$. Therefore $fgw = w$ and $fgw = mw = w$. Hence we obtain

$$klw = fgw = hw = mw = w.$$

Consequently, w is a common fixed point of the mappings kl, fg, h and m. Now, we prove that w is a unique common fixed point of the mappings kl, fg, h and m. Let w^* be also a common fixed point of kl, fg, h and m. Using the inequality (5.2)

$$\mathcal{S}_C(w, w, w^*) = \mathcal{S}_C(klw, klw, fgw^*)$$

$$\begin{aligned}&\precsim a\mathcal{S}_C(hw,hw,mw^*)\\&\quad+b[\mathcal{S}_C(hw,hw,klw)+\mathcal{S}_C(mw^*,mw^*,fgw^*)]\\&\quad+c[\mathcal{S}_C(hw,hw,fgw^*)+\mathcal{S}_C(mw^*,mw^*,klw)]\\&=(a+2c)\mathcal{S}_C(w,w,w^*),\end{aligned}$$

and so

$$|\mathcal{S}_C(w,w,w^*)|\leq(a+2c)\,|\mathcal{S}_C(w,w,w^*)|\,,$$

which is a contradiction since $a+2c<1$. Hence we obtain $w=w^*$.

Now, we show that w is the unique common fixed point of the six mappings f , g, k, l, h and m. Using the commuting conditions of the pair (f,g) we have

$$fw=f(fgw)=f(gfw)=fg(fw),$$

and

$$gw=g(fgw)=gf(gw)=fg(gw),$$

which implies that fw and gw are fixed points of the mapping fg.

Similarly, fw and gw are common fixed points of the mappings kl, h and m using the hypothesis. Using the hypothesis, also by a similar way we obtain kw and lw are common fixed points of the mappings kl, fg, h and m. Consequently, by the uniqueness of the common fixed point, we get

$$hw=mw=fw=gw=kw=lw=w,$$

that is, f, g, k, l, h and m have a unique common fixed point w in X. □

6. Some Applications of Common Fixed Point Theory in View of Closed Ball

Let $(X,\mathcal{S}_C)$ be a complex-valued S-metric space. For $0\prec r$ and $x\in X$ the open ball $B_S^C(x,r)$ and closed ball $B_S^C[x,r]$ with center x and radius r are defined as follows, respectively:

$$B_S^C(x,r)=\{y\in X:\mathcal{S}_C(y,y,x)\prec r\},$$
$$B_S^C[x,r]=\{y\in X:\mathcal{S}_C(y,y,x)\precsim r\}.$$

A point $x\in X$ is called an interior point of a set $A\subseteq X$, if there exists $0\prec r\in\mathbb{C}$ such that

$$B_S^C(x,r)\subseteq A.$$

A point $x\in X$ is called a limit point of A whenever we have

$$B_S^C(x,r)\cap(A-\{x\})\neq\emptyset,$$

for every $0\prec r\in\mathbb{C}$.

A subset $A \subseteq X$ is said to be open if each element of A is an interior point of A.

Example 6.1. Let $X = \mathbb{C}$ and the complex-valued S-metric be defined by

$$\mathcal{S}_C(z_1, z_2, z_3) = |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|, \tag{6.1}$$

for all $z_1, z_2, z_3 \in \mathbb{C}$ (using the definition of S-metric generated by S-norm given in [**?**]). If we choose $z = x + iy$, $z_0 = 3 + 2i$ and $r = 30$ in $\mathbb{C}$ then we obtain

$$B_S^C[z_0, r] = \{z \in \mathbb{C} : \sqrt{(x-3)^2 + (y-2)^2} \leq 15\}.$$

Now we recall the notion of a complex-valued metric space.

Let X be a non-empty set and $d_C : X \times X \to \mathbb{C}$ be a mapping. Then d_c is called a complex-valued metric if

(i) $0 \precsim d_C(x, y)$ for all $x, y \in X$,
(ii) $d_C(x, y) = 0$ if and only if $x = y$,
(iii) $d_C(x, y) = d_C(y, x)$ for all $x, y \in X$,
(iv) $d_C(x, y) \precsim d_C(x, z) + d_C(z, y)$ for all $x, y, z \in X$,

and (X, d_C) is called a complex-valued metric space [5].

Now, we give relationships between complex-valued metric and complex-valued S-metric.

Let (X, d_C) be a complex-valued metric. Then the function $\mathcal{S}_{C_d}$: $X \times X \times X \to \mathbb{C}$ defined by

$$\mathcal{S}_{C_d}(x, y, z) = d_C(x, z) + d_C(y, z),$$

for all $x, y, z \in X$ is a complex-valued S-metric. We call this complex valued metric $\mathcal{S}_{C_d}$ as the complex-valued S-metric generated by d_C.

Example 6.2. Let $X = \mathbb{C}$ and

$$d_C(z_1, z_2) = \sqrt{\frac{(x_1 - x_2)^2}{9} + 4(y_1 - y_2)^2},$$

for all $z_1, z_2 \in \mathbb{C}$ where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then (X, d_C) is a complex-valued metric space. Therefore the complex-valued S-metric generated by d_C is defined by

$$\mathcal{S}_{C_d}(z_1, z_2, z_3) = d_C(z_1, z_3) + d_C(z_2, z_3), \tag{6.2}$$

for all $z_1, z_2, z_3 \in \mathbb{C}$ where $z_3 = (x_3, y_3)$.

The closed ball $B_S^C[z_0, r]$ in $\mathbb{C}$ is an ellipse given by

$$\begin{aligned} B_S^C[z_0, r] &= \{z \in \mathbb{C} : \mathcal{S}_{C_d}(z, z, z_0) \precsim r\} \\ &= \{z \in \mathbb{C} : 2d_C(z, z_0) \precsim r\}. \end{aligned}$$

If we choose $z = x + iy$, $z_0 = 2 + 3i$ and $r = 10$ then we obtain

$$B_S^C[z_0, r] = \left\{ z \in \mathbb{C} : \sqrt{\frac{(x-2)^2}{9} + 4(y-3)^2} \precsim 5 \right\}.$$

We give the following theorem on the closed ball $B_S^C[x, r]$.

Theorem 6.3. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S-metric space, $x_0 \in X$, $0 \prec r \in \mathbb{C}$ and a, b, c, d, e be five real numbers such that $a, b, c, d, e \geq 0$ and $a + b + c + 3d + 3e < 1$. Let $f, g : X \to X$ be two mappings satisfying*

$$\begin{aligned} \mathcal{S}_C(fx, fx, gy) \precsim\ & a\mathcal{S}_C(x, x, y) + b\frac{\mathcal{S}_C(fx, fx, x)\mathcal{S}_C(gy, gy, y)}{1 + \mathcal{S}_C(x, x, y)} \\ & + c\frac{\mathcal{S}_C(fx, fx, y)\mathcal{S}_C(gy, gy, x)}{1 + \mathcal{S}_C(x, x, y)} \\ & + d\frac{\mathcal{S}_C(fx, fx, x)\mathcal{S}_C(gy, gy, x)}{1 + \mathcal{S}_C(x, x, y)} \\ & + e\frac{\mathcal{S}_C(fx, fx, y)\mathcal{S}_C(gy, gy, y)}{1 + \mathcal{S}_C(x, x, y)}, \end{aligned} \tag{6.3}$$

for all $x, y \in B_S^C[x_0, r]$. If

$$|\mathcal{S}_C(fx_0, fx_0, x_0)| \leq \frac{1-h}{2} |r|, \tag{6.4}$$

where $h = \max\left\{ \frac{a+2d}{1-b-d}, \frac{a+2e}{1-b-e} \right\}$, then there exists a unique common fixed point $w \in B_S^C[x_0, r]$ of the self-mappings f and g.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as follows:

$$x_{2k+1} = fx_{2k} \text{ and } x_{2k+2} = gx_{2k+1},$$

where $k = 0, 1, 2, \ldots$. We show that $x_n \in B_S^C[x_0, r]$ for all $n \in \mathbb{N}$ by mathematical induction. Using the inequality (6.4) and $h < 1$ we get

$$|\mathcal{S}_C(fx_0, fx_0, x_0)| \leq |r|,$$

which implies that $x_1 \in B_S^C[x_0, r]$.

Let $x_2, \ldots, x_i \in B_S^C[x_0, r]$ for some $i \in \mathbb{N}$. If $i = 2k + 1$ where $k = 0, 1, 2, \ldots, \frac{i-1}{2}$ or $i = 2k + 2$ where $k = 0, 1, \ldots, \frac{i-2}{2}$, using the inequality (6.3) we have

$$\begin{aligned} & \mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k+2}) \\ & = \mathcal{S}_C(fx_{2k}, fx_{2k}, gx_{2k+1}) \precsim a\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1}) \\ & \quad + b\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k+1})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})} \end{aligned}$$

$$+ c\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k+1})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})}$$
$$+ d\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})}$$
$$+ e\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k+1})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k+1})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})},$$

and so

$$\begin{aligned}
&\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k+2})\\
&\quad \precsim a\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})\\
&\qquad + b\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k+1})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})}\\
&\qquad + d\frac{\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k})}{1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})},
\end{aligned}$$

which implies

$$\begin{aligned}
&|\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k+2})|\\
&\quad \le a\,|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|\\
&\qquad + b\frac{|\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})|\,|\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k+1})|}{|1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|}\\
&\qquad + d\frac{|\mathcal{S}_C(fx_{2k}, fx_{2k}, x_{2k})|\,|\mathcal{S}_C(gx_{2k+1}, gx_{2k+1}, x_{2k})|}{|1 + \mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|}\\
&\quad \le a\,|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|\\
&\qquad + b\frac{|\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k})|\,|\mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k+1})|}{|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|}\\
&\qquad + d\frac{|\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k})|\,|\mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k})|}{|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|}.
\end{aligned}$$

Using Lemma 2.9 we have

$$\begin{aligned}
&|\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k+2})|\\
&\quad \le a\,|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})| + b\,|\mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k+1})|\\
&\qquad + d\,|\mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k})|.
\end{aligned}$$

Using the condition (**CS3**) and Lemma 2.9 we get

$$\mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k}) \precsim 2\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1}) + \mathcal{S}_C(x_{2k+2}, x_{2k+2}, x_{2k+1}),$$

and

$$|\mathcal{S}_C(x_{2k+1}, x_{2k+1}, x_{2k+2})| \le \frac{a + 2d}{1 - b - d}\,|\mathcal{S}_C(x_{2k}, x_{2k}, x_{2k+1})|. \tag{6.5}$$

By a similar way as above we obtain

$$|\mathcal{S}_C(x_{2k+2},x_{2k+2},x_{2k+3})| \leq \frac{a+2e}{1-b-e}\left|\mathcal{S}_C(x_{2k+1},x_{2k+1},x_{2k+2})\right|. \tag{6.6}$$

If we put $h=\max\left\{\frac{a+2d}{1-b-d},\frac{a+2e}{1-b-e}\right\}$ we have

$$|\mathcal{S}_C(x_i,x_i,x_{i+1})| \leq h^i\left|\mathcal{S}_C(x_0,x_0,x_1)\right|,$$

for all $i\in\mathbb{N}$. Let us consider

$$\begin{aligned}
&|\mathcal{S}_C(x_0,x_0,x_{i+1})| \\
&\quad\leq 2\left|\mathcal{S}_C(x_0,x_0,x_1)\right|+2\left|\mathcal{S}_C(x_1,x_1,x_2)\right|+\cdots+\left|\mathcal{S}_C(x_i,x_i,x_{i+1})\right| \\
&\quad\leq 2\left|\mathcal{S}_C(x_0,x_0,x_1)\right|(1+h+\cdots+h^{i-1})+h^i\left|\mathcal{S}_C(x_0,x_0,x_1)\right| \\
&\quad\leq 2\frac{1-h}{2}\left|r\right|(1+h+\cdots+h^{i-1})+h^i\frac{1-h}{2} \\
&\quad\leq |r|\,(1-h)(1+h+\cdots+h^i)\leq |r|\,,
\end{aligned}$$

which implies $x_{i+1}\in B_S^C[x_0,r]$. Hence $x_n\in B_S^C[x_0,r]$ and

$$|\mathcal{S}_C(x_n,x_n,x_{n+1})| \leq h^n\left|\mathcal{S}_C(x_0,x_0,x_1)\right|,$$

for all $n\in\mathbb{N}$.

If we take $m>n$ then we have

$$\begin{aligned}
|\mathcal{S}_C(x_n,x_n,x_m)| \leq{}& 2\left|\mathcal{S}_C(x_n,x_n,x_{n+1})\right|+2\left|\mathcal{S}_C(x_{n+1},x_{n+1},x_{n+2})\right| \\
&+\cdots+\left|\mathcal{S}_C(x_{m-1},x_{m-1},x_m)\right|\to 0,
\end{aligned}$$

as $m,n\to\infty$, which implies that the sequence $\{x_n\}$ is a Cauchy sequence in $B_S^C[x_0,r]$. Hence there exists a point $w\in B_S^C[x_0,r]$ with $\lim\limits_{n\to\infty}x_n=w$. Now we prove $fw=w$. Using the inequality (6.3) we have

$$\begin{aligned}
|\mathcal{S}_C(fw,fw,w)| &\leq 2\left|\mathcal{S}_C(w,w,x_{2k+2})\right|+\left|\mathcal{S}_C(x_{2k+2},x_{2k+2},fw)\right| \\
&= 2\left|\mathcal{S}_C(w,w,x_{2k+2})\right|+\left|\mathcal{S}_C(fw,fw,gx_{2k+1})\right| \\
&\precsim 2\left|\mathcal{S}_C(w,w,x_{2k+2})\right|+a\left|\mathcal{S}_C(w,w,x_{2k+1})\right| \\
&\quad+b\frac{\left|\mathcal{S}_C(fw,fw,w)\right|\left|\mathcal{S}_C(gx_{2k+1},gx_{2k+1},x_{2k+1})\right|}{\left|1+\mathcal{S}_C(w,w,x_{2k+1})\right|} \\
&\quad+c\frac{\left|\mathcal{S}_C(fw,fw,x_{2k+1})\right|\left|\mathcal{S}_C(gx_{2k+1},gx_{2k+1},w)\right|}{\left|1+\mathcal{S}_C(w,w,x_{2k+1})\right|} \\
&\quad+d\frac{\left|\mathcal{S}_C(fw,fw,w)\right|\left|\mathcal{S}_C(gx_{2k+1},gx_{2k+1},w)\right|}{\left|1+\mathcal{S}_C(w,w,x_{2k+1})\right|} \\
&\quad+e\frac{\left|\mathcal{S}_C(fw,fw,x_{2k+1})\right|\left|\mathcal{S}_C(gx_{2k+1},gx_{2k+1},x_{2k+1})\right|}{\left|1+\mathcal{S}_C(w,w,x_{2k+1})\right|},
\end{aligned}$$

which implies that this inequality converges 0 as $n \to \infty$. Therefore we obtain $|\mathcal{S}_C(fw, fw, w)| = 0$, that is, $fw = w$. By a similar way as above we show that $gw = w$.

Now we prove that the fixed point w is unique. Assume that $w^* \in B_S^C[x_0, r]$ is also a common fixed point of f and g. Then we have

$$\begin{aligned} |\mathcal{S}_C(w, w, w^*)| &= |\mathcal{S}_C(fw, fw, gw^*)| \\ &\leq a\,|\mathcal{S}_C(w, w, w^*)| \\ &\quad + b\frac{|\mathcal{S}_C(fw, fw, w)|\,|\mathcal{S}_C(gw^*, gw^*, w^*)|}{|1+\mathcal{S}_C(w, w, w^*)|} \\ &\quad + c\frac{|\mathcal{S}_C(fw, fw, w^*)|\,|\mathcal{S}_C(gw^*, gw^*, w)|}{|1+\mathcal{S}_C(w, w, w^*)|} \\ &\quad + d\frac{|\mathcal{S}_C(fw, fw, w)|\,|\mathcal{S}_C(gw^*, gw^*, w)|}{|1+\mathcal{S}_C(w, w, w^*)|} \\ &\quad + e\frac{|\mathcal{S}_C(fw, fw, w^*)|\,|\mathcal{S}_C(gw^*, gw^*, w^*)|}{|1+\mathcal{S}_C(w, w, w^*)|}. \end{aligned}$$

Hence we get

$$|\mathcal{S}_C(w, w, w^*)| \leq (a+c)\,|\mathcal{S}_C(w, w, w^*)|\,,$$

since $|1+\mathcal{S}_C(w, w, w^*)| > |\mathcal{S}_C(w, w, w^*)|$. Therefore $w = w^*$ as $a + c < 1$. Consequently, w is the unique common fixed point of f and g. Then the proof is completed. □

Notice that if we put $f = g$ in Theorem 6.3, then we have the following corollary.

Corollary 6.4. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S-metric space, $x_0 \in X$, $0 \prec r \in \mathbb{C}$ and a, b, c, d, e be five real numbers such that $a, b, c, d, e \geq 0$ and $a + b + c + 3d + 3e < 1$. Let $f : X \to X$ be a mapping satisfying*

$$\begin{aligned} \mathcal{S}_C(fx, fx, fy) \precsim a\mathcal{S}_C(x, x, y) &+ b\frac{\mathcal{S}_C(fx, fx, x)\mathcal{S}_C(fy, fy, y)}{1+\mathcal{S}_C(x, x, y)} \\ &+ c\frac{\mathcal{S}_C(fx, fx, y)\mathcal{S}_C(fy, fy, x)}{1+\mathcal{S}_C(x, x, y)} \\ &+ d\frac{\mathcal{S}_C(fx, fx, x)\mathcal{S}_C(fy, fy, x)}{1+\mathcal{S}_C(x, x, y)} \\ &+ e\frac{\mathcal{S}_C(fx, fx, y)\mathcal{S}_C(fy, fy, y)}{1+\mathcal{S}_C(x, x, y)}, \end{aligned} \tag{6.7}$$

for all $x, y \in B_S^C[x_0, r]$. If

$$|\mathcal{S}_C(fx_0, fx_0, x_0)| \leq \frac{1-h}{2}\,|r|\,,$$

where $h=\max\left\{\frac{a+2d}{1-b-d},\frac{a+2e}{1-b-e}\right\}$, *then there exists a unique fixed point* $w\in B_S^C[x_0,r]$ *of the self-mapping* f.

Remark 6.5. If we choose $c=0$, $d=0$, $c=d=0$ and $c=d=e=0$ in Theorem 6.3, then we have similar corollaries. Also if we take $b=c=d=e=0$ in Corollary 6.4, then we obtain a new generalization of the classical Banach's contraction principle on the closed ball in a complex-valued S-metric space.

In the following example, we see that there exist a self-mapping satisfying the conditions of Corollary 6.4 on $\mathbb{C}$.

Example 6.6. Let $X=\mathbb{C}$ and the complex-valued S-metric on $\mathbb{C}$ be defined

$$\mathcal{S}_C(z_1,z_2,z_3)=\sqrt{\frac{(x_1-x_3)^2}{9}+4(y_1-y_3)^2}+\sqrt{\frac{(x_2-x_3)^2}{9}+4(y_2-y_3)^2},$$

for all $z_1,z_2,z_3\in\mathbb{C}$ where $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$ and $z_3=(x_3,y_3)$. Let $f:\mathbb{C}\to\mathbb{C}$ be given by

$$fz=z_0,$$

for all $z\in\mathbb{C}$ where z_0 is the center of the closed ball $B_S^C[z_0,r]$. If we put $a=\frac{1}{2}$, $b=c=d=e=0$ we obtain

$$\mathcal{S}_C(fz_1,fz_1,fz_2)=\mathcal{S}_C(z_0,z_0,z_0)=0\precsim\frac{1}{2}\mathcal{S}_C(z_1,z_1,z_2),$$

for all $z_1,z_2\in B_S^C[z_0,r]$. Then the inequality (6.7) is satisfied. Hence we have

$$h=\max\left\{\frac{a+2d}{1-b-d},\frac{a+2e}{1-b-e}\right\}=a=\frac{1}{2}$$

and

$$\left|\mathcal{S}_C(fz_0,fz_0,z_0)\right|=0\leq\frac{1}{4}\left|r\right|.$$

Consequently, Corollary 6.4 is satisfied and there exists a unique fixed point $z_0\in B_S^C[z_0,r]$ of the self-mapping f.

Now we give the following theorem using finitely many functions on the closed ball $B_S^C[x,r]$.

Theorem 6.7. *Let* $(X,\mathcal{S}_C)$ *be a complete complex-valued* S*-metric space,* $\{f_i\}_{1\leq i\leq m}$ *and* $\{g_j\}_{1\leq j\leq n}$ *are two finite pairwise commuting finite families of self-mappings of* X. *If the mapping* f *and* g, *where* $f=f_1f_2...f_m$ *and* $g=g_1g_2...g_n$ *satisfy the inequalities (6.3) and (6.4) in Theorem 6.3 then the component mappings of the families* $\{f_i\}_{1\leq i\leq m}$ *and* $\{g_j\}_{1\leq j\leq n}$ *have a unique common fixed point.*

Proof. Using Theorem 6.3, we see that the mappings f and g have a unique common fixed point w. Now we show that w is a common fixed point of all the component mappings of the families $\{f_i\}_{1\leq i\leq m}$ and $\{g_j\}_{1\leq j\leq n}$. In view of pairwise commutativity of the families $\{f_i\}_{1\leq i\leq m}$ and $\{g_j\}_{1\leq j\leq n}$ we get

$$f_k w = f_k f w = f f_k w \text{ and } f_k w = f_k g w = g f_k w,$$

for all $1 \leq k \leq m$, which implies that $f_k w$ is also a common fixed point of f and g. Using the uniqueness of the common fixed point we have $f_k w = w$ for all k. Hence w is a common fixed point of the family $\{f_i\}_{1\leq i\leq m}$.

Similarly, it can be seen that w is also common fixed point of the family $\{g_j\}_{1\leq j\leq n}$. ☐

Notice that if we take $f_1 = f_2 = ... = f_m = f$ and $g_1 = g_2 = ... = g_n = g$ in Theorem 6.3 we obtain the following corollary.

Corollary 6.8. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S-metric space, $x_0 \in X$, $0 \prec r \in \mathbb{C}$ and a, b, c, d, e be five real numbers such that $a, b, c, d, e \geq 0$ and $a + b + c + 3d + 3e < 1$. Let $f, g : X \to X$ be two mappings satisfying*

$$\begin{aligned}\mathcal{S}_C(f^m x, f^m x, g^n y) \precsim{}& a\mathcal{S}_C(x, x, y) + b\frac{\mathcal{S}_C(f^m x, f^m x, x)\mathcal{S}_C(g^n y, g^n y, y)}{1 + \mathcal{S}_C(x, x, y)}\\ &+ c\frac{\mathcal{S}_C(f^m x, f^m x, y)\mathcal{S}_C(g^n y, g^n y, x)}{1 + \mathcal{S}_C(x, x, y)}\\ &+ d\frac{\mathcal{S}_C(f^m x, f^m x, x)\mathcal{S}_C(g^n y, g^n y, x)}{1 + \mathcal{S}_C(x, x, y)}\\ &+ e\frac{\mathcal{S}_C(f^m x, f^m x, y)\mathcal{S}_C(g^n y, g^n y, y)}{1 + \mathcal{S}_C(x, x, y)},\end{aligned}$$

for all $x, y \in B^C_S[x_0, r]$ and

$$|\mathcal{S}_C(g^n x_0, g^n x_0, x_0)| \leq \frac{1-h}{2}|r|,$$

where $h = \max\left\{\dfrac{a+2d}{1-b-d}, \dfrac{a+2e}{1-b-e}\right\}$, then there exists a unique common fixed point $w \in B^C_S[x_0, r]$ of the self-mappings f and g.

Also by setting $m = n$ and $f = g = h$ in Corollary 6.8 we obtain the following corollary:

Corollary 6.9. *Let $(X, \mathcal{S}_C)$ be a complete complex-valued S-metric space, $x_0 \in X$, $0 \prec r \in \mathbb{C}$ and a, b, c, d, e be five real numbers such that*

$a,b,c,d,e \geq 0$ and $a+b+c+3d+3e<1$. Let $h : X \to X$ be a mapping satisfying

$$\begin{aligned}\mathcal{S}_C(h^n x, h^n x, h^n y) \precsim\, & a\mathcal{S}_C(x,x,y) + b\frac{\mathcal{S}_C(h^n x, h^n x, x)\mathcal{S}_C(h^n y, h^n y, y)}{1+\mathcal{S}_C(x,x,y)} \\ & + c\frac{\mathcal{S}_C(h^n x, h^n x, y)\mathcal{S}_C(h^n y, h^n y, x)}{1+\mathcal{S}_C(x,x,y)} \\ & + d\frac{\mathcal{S}_C(h^n x, h^n x, x)\mathcal{S}_C(h^n y, h^n y, x)}{1+\mathcal{S}_C(x,x,y)} \\ & + e\frac{\mathcal{S}_C(h^n x, h^n x, y)\mathcal{S}_C(h^n y, h^n y, y)}{1+\mathcal{S}_C(x,x,y)},\end{aligned}$$

for all $x, y \in B_S^C[x_0, r]$ and

$$|\mathcal{S}_C(h^n x_0, h^n x_0, x_0)| \leq \frac{1-\lambda}{2}|r|,$$

where $\lambda = \max\left\{\frac{a+2d}{1-b-d}, \frac{a+2e}{1-b-e}\right\}$, then there exists a unique fixed point $w \in B_S^C[x_0, r]$ of the self-mapping h.

7. Conclusions and Future Works

Recently, complex-valued S-metric spaces have been introduced and studied to improve the Banach's contraction principle and to generalize some metric spaces such as metric and S-metric spaces. In this paper, we have given some generalized common fixed point (resp. fixed point) results on a complete complex-valued S-metric space using different techniques by means of new generalized contractive conditions and the notion of the closed ball. Our results generalize and improve some known fixed point results. More recently, the fixed circle problem has been introduced and studied as a new direction of extensions (see [14, 18–21, 31]). As a future work, new fixed circle results can be investigated on a complex-valued S-metric space.

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References

1. J. Ahmad, A. Azam, and S. Saejung, *Common fixed point results for contractive mappings in complex-valued metric spaces*, Fixed Point Theory Appl., 67 (2014), 11 pp.
2. A.H. Ansari, Ö. Ege, and S. Radenović, *Some fixed point results on complex-valued G_b-metric spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 112 (2018), pp. 463-472.

3. A.H. Ansari, T. Dosenovic, S. Radenovic, and J.S. Ume, *C-class functions and common fixed point theorems satisfying φ-weakly contractive conditions*, Sahand Commun. Math. Anal.,(13) (2019), pp. 17-30.
4. A.H. Ansari and A. Razani, *Some fixed point theorems for C-class functions in b-metric spaces*, Sahand Commun. Math. Anal., 10 (2018), pp. 85-96.
5. A. Azam, B. Fisher, and M. Khan, *Common fixed point theorems in complex-valued metric spaces*, Numer. Funct. Anal. Optim., 32 (2011), pp. 243-253.
6. N.H. Dien, *Some remarks on common fixed point theorems*, J. Math. Anal. Appl., 187 (1994), pp. 76-90.
7. Ö. Ege, *Complex valued rectangular b-metric spaces and an application to linear equations*, J. Nonlinear Sci. Appl., 8 (2015), pp. 1014-1021.
8. Ö. Ege, *Complex valued G_b-metric spaces*, J. Comput. Anal. Appl., 21 (2016), pp. 363-368.
9. Ö. Ege, *Some fixed point theorems in complex-valued G_b-metric spaces*, J. Nonlinear Convex Anal., 18 (2017), pp. 1997-2005.
10. H. Faraji and K. Nourouzi, *Fixed and common fixed points for (ψ, φ)-weakly contractive mappings in b-metric spaces*, Sahand Commun. Math. Anal., 7 (2017), pp. 49-62.
11. C. Kalaivani and G. Kalpana, *Fixed point theorems in C^*-algebra-valued S-metric spaces with some applications*, Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar., 80 (2018), pp. 93-102.
12. Z. Liu, Y. Xu, and Y.J. Cho, *On Characterizations of Ffixed and common fixed points*, J. Math. Anal. Appl., 222 (1998), pp. 494-504.
13. N.M. Mlaiki, *Common fixed points in complex S-metric space*, Adv. Fixed Point Theory, 4 (2014), pp. 509-524.
14. N. Mlaiki, U. Ç elik, N. Taş, N.Y. Özgür, and A. Mukheimer, *Wardowski type contractions and the fixed-circle problem on S-metric spaces*, J. Math., 2018 (2018), 9 pp.
15. N.Y. Özgür and N. Taş, *Some fixed point theorems on S-metric spaces*, Mat. Vesnik, 69 (2017), pp. 39-52.
16. N.Y. Özgür and N. Taş, *Some new contractive mappings on S-metric spaces and their relationships with the mapping ($S25$)*, Math. Sci., 11 (2017), pp. 7-16.
17. N.Y. Özgür and N. Taş, *Some generalizations of fixed point theorems on S-metric spaces*, Essays in Mathematics and Its Applications in Honor of Vladimir Arnold, New York, Springer, 2016.

18. N.Y. Özgür and N. Taş, *Some fixed-circle theorems on metric spaces*, Bull. Malays. Math. Sci. Soc., 42 (2019), pp. 1433-1449.
19. N.Y. Özgür and N. Taş, *Some fixed-circle theorems and discontinuity at fixed circle*, AIP Conference Proceedings 1926, 020048 (2018).
20. N.Y. Özgür, N. Taş, and U. Çelik, *New fixed-circle results on S-metric spaces*, Bull. Math. Anal. Appl., 9 (2017), pp. 10-23.
21. N.Y. Özgür and N. Taş, *Fixed-circle problem on S-metric spaces with a geometric viewpoint*, Facta Universitatis. Series: Mathematics and Informatics, 34 (3) (2019), pp. 459-472.
22. M. Öztürk, *Common fixed point theorems satisfying contractive type conditions in complex-valued metric spaces*, Abstr. Appl. Anal., 2014 (2014), 7 pp.
23. M. Öztürk and N. Kaplan, *Common fixed points of f-contraction mappings in complex-valued metric spaces*, Math. Sci., 8 (2014), 7 pp.
24. M. Öztürk and N. Kaplan, *Some common coupled fixed points of mappings satisfying contractive conditions with rational expressions in complex-valued G_b-metric spaces*, Bangmod Int. J. Math. & Comp. Sci., 1 (2015), pp. 190-204.
25. N. Priyobarta, Y. Rohen, and N. Mlaiki, *Complex valued Sb-metric spaces*, J. Math. Anal., 8 (2017), pp. 13-24.
26. M.M. Rezaee, S. Sedghi, and K. S. Kim, *Coupled common fixed point results in ordered S-metric spaces*, Nonlinear Funct. Anal. Appl., 23 (2018), pp. 595-612.
27. B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226 (1977), pp. 257-290.
28. S. Sedghi, N. Shobe, and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, 64 (2012), pp. 258-266.
29. R.J. Shahkoohi and Z. Bagheri, *Rational Geraghty contractive mappings and fixed point theorems in ordered b_2-metric spaces*, Sahand Commun. Math. Anal., 13 (1) (2019), pp. 179-212.
30. H. Shayanpour and A. Nematizadeh, *Common fixed point theory in modified intuitionistic probabilistic metric spaces with common property $(E.A.)$*, Sahand Commun. Math. Anal., 13 (1) (2019), pp. 31-50.
31. N. Taş, N.Y. Özgür, and N. Mlaiki, *New types of F_c-contractions and the fixed-circle problem*, Mathematics, 6 (2018), 9 pp.
32. R.K. Verma and H.K. Pathak, *Common fixed point theorems using property $(E.A)$ in complex-valued metric spaces*, Thai J. Math., 11 (2013), pp. 347-355.

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