Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 2 (2020), 107-117
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2019.69719.273

On the Monotone Mappings in CAT(0) Spaces

Davood Afkhami Taba[1] and Hossein Dehghan[2*]

Abstract. In this paper, we first introduce a monotone mapping and its resolvent in general metric spaces. Then, we give two new iterative methods by combining the resolvent method with Halpern's iterative method and viscosity approximation method for finding a fixed point of monotone mappings and a solution of variational inequalities. We prove convergence theorems of the proposed iterations in CAT(0) metric spaces.

1. Introduction and preliminaries

Let (X, d) be a metric space. Berg and Nikolaev [6] introduced the concept of quasilinearization in metric spaces. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right), \tag{1.1}$$

for all $a, b, c, d \in X$. It can be easily seen that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle,$$

and

$$\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle,$$

for all $a, b, c, d, x \in X$.

A metric space (X, d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and

2010 *Mathematics Subject Classification.* 47H09, 47H06, 47H10.

Key words and phrases. Monotone mapping, Nonexpansive mapping, Variational inequality, Fixed point, CAT(0) metric space.

Received: 08 August 2017, Accepted: 13 September 2017.

* Corresponding author.

basic properties, we refer the reader to standard texts such as [5, 7]. Complete CAT(0) spaces are often called Hadamard spaces. Let $x, y \in X$ and $\lambda \in [0,1]$. We write $\lambda x \oplus (1-\lambda)y$ for the unique point z in the geodesic segment joining x to y such that

$$d(z,x) = (1-\lambda)d(x,y), \qquad d(z,y) = \lambda d(x,y). \tag{1.2}$$

We also denote by $[x,y]$ the geodesic segment joining x to y, that is, $[x,y] = \{\lambda x \oplus (1-\lambda)y : \lambda \in [0,1]\}$. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$. The metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d),$$

for all $a, b, c, d \in X$. It is known [6, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Define the relation $\sim$ on $X \times X$ as follows:

$$\overrightarrow{xy} \sim \overrightarrow{zt} \quad \Leftrightarrow \quad \langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = \langle \overrightarrow{ab}, \overrightarrow{zt} \rangle \quad (\forall a, b \in X).$$

The equivalent class of $\overrightarrow{xy}$ will be denoted by $[\overrightarrow{xy}]$. The metric space X is said to satisfy the $(\mathcal{S})$ property if for any $(x,y) \in X \times X$ there exists $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$ (see also [14, Definition 2.7]). It is obvious that, for example, any Hilbert space enjoys the $\mathcal{S}$ property (let $y_x := 2x - y$ then $[\overrightarrow{xy}] = [\overrightarrow{y_x x}] = [y - x]$). Moreover, it is not hard to check that any symmetric Hadamard manifold satisfies the $(\mathcal{S})$ property ($exp_x^{-1}y$ acts in the role of $[\overrightarrow{xy}]$ in Hadamard manifolds (see [13, p. 3455]).

The concept of Δ-convergence, introduced by Lim [18] in 1976, was shown by Kirk and Panyanak [16] in CAT(0) spaces to be very similar to the weak convergence in Hilbert space setting. Next, we give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [10, Proposition 7] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\} \subset X$ is said to Δ-converges to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

On the other hand, Kakavandi and Amini [13] introduced the concept of ω-convergence as follows. A sequence $\{x_n\} \subset X$ is said to ω-converges to $x \in X$ if $\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$ for each $y \in X$. Note that it is equivalent to $\lim_{n\to\infty} \langle \overrightarrow{ax_n}, \overrightarrow{bc} \rangle = \langle \overrightarrow{ax}, \overrightarrow{bc} \rangle$ for each $a, b, c \in X$. Kakavandi [14, Lemma 2.8] proved that ω-convergence and Δ-convergence are equivalent in CAT(0) spaces with $(\mathcal{S})$ property.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. It is known that for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x, u) = \inf_{y \in C} d(x, y).$$

The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C. It follows from [7, Proposition 2.4] that P_C is nonexpansive. Dehghan and Rooin [9] obtained the following characterization of a metric projection in CAT(0) metric spaces.

Theorem 1.1 ([9, Theorem 2.2])**.** *Let C be a nonempty convex subset of a Hadamard space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle \overrightarrow{ux}, \overrightarrow{yu} \rangle \geq 0, \quad \text{for all } y \in C.$$

We need the following lemmas in the sequel.

Lemma 1.2 ([16])**.** *Every bounded sequence in a Hadamard space always has a Δ-convergent subsequence.*

Lemma 1.3 ([11])**.** *If C is a closed convex subset of a Hadamard space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 1.4 ([12, Lemma 2.5])**.** *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y), \tag{1.3}$$

is satisfied for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

Lemma 1.5 ([7, Proposition 2.2])**.** *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s). \tag{1.4}$$

Lemma 1.6 ([12, Lemma 2.4])**.** *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda)d(y, z). \tag{1.5}$$

Lemma 1.7 ([1, Lemma 3.3])**.** *Let X be a CAT(0) space, $a, b, c, d \in X$ and $\lambda \in [0, 1]$. Then*

$$d^2\left(\lambda a \oplus (1-\lambda)b, \lambda c \oplus (1-\lambda)d\right) \leq \lambda^2 d^2(a,c) + (1-\lambda)^2 d^2(b,d) + 2\lambda(1-\lambda)\langle \overrightarrow{ac}, \overrightarrow{bd}\rangle. \tag{1.6}$$

Lemma 1.8 ([21])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\delta_n\}$ is a sequence in $\mathbb{R}$ and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and*

$$\sum_{n=0}^{\infty} \gamma_n = \infty,$$

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or*

$$\sum_{n=0}^{\infty} \gamma_n |\delta_n| < \infty,$$

(iii)

$$\sum_{n=0}^{\infty} \sigma_n < \infty.$$

Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

We begin with presenting an appropriate definition of monotone mappings in metric spaces.
Let C be a nonempty subset of a metric space X. We recall that a mapping $T : C \to X$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in C$. A point $x \in C$ is called a fixed point of T if $x = Tx$. For the set-valued mapping T, a point $x \in C$ is called a fixed point of T if $x \in Tx$. In each cases, we denote by $F(T)$ the set of all fixed points of T. Kirk [17, Theorem 5.1] showed that every nonexpansive self-mapping defined on a bounded closed convex subset of a Hadamard space always has a fixed point. Also, $F(T)$ is closed and convex.

For a mapping $A : X \to 2^X$, we define its domain, range and graph respectively as follows:

$$D(A) = \{x \in X : Ax \neq \emptyset\}, \qquad R(A) = \{Az : z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\}.$$

To obtain an appropriate definition for a monotone mapping in metric spaces, we first recall its definition in Hilbert spaces. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. A mapping $A : H \to 2^H$ is called monotone if for each $x, y \in D(A)$, we have

$$\langle u - v, x - y \rangle \geq 0, \tag{2.1}$$

for all $u \in Ax$ and $v \in Ay$ (see [15] and references therein). Putting $T = I + A$, where I is the identity map on H, we see that $T - I$ is monotone.

Definition 2.1. Let X be a metric space X and $T : X \to 2^X$ be a mapping. Let us formally say that $''T - I''$ is monotone if for each $x, y \in D(T)$, we have

$$\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle \geq d^2(x, y), \tag{2.2}$$

for all $u \in Tx$ and $v \in Ty$.

Note that $T - I$ is just a symbol. The definition of a monotone mapping finds its origin in Hilbert spaces.

Example 2.2. Consider $\mathbb{R}^2$ with the usual Euclidean metric d. Let $X = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on an Euclidean straight line passing through the origin and $d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0})$ otherwise (see [19, p. 65]). We put

$$A = \{(t, s) : t + s = 1, t \in [0, 1]\}, \qquad B = \{(t, s) : t + 2s = 2, t \in [0, 1]\},$$

$C = A \cup B$ and define $T : C \to 2^X$ by

$$Tx = \begin{cases} \{x, p_x\} & x \in A, \\ x & x \in B, \end{cases}$$

where p_x is the intersection of B with the line passing from x and the origin. By elementary computations we see that $T - I$ is monotone.

Remark 2.3. Let $T : X \to 2^X$ be a mapping and

$$T^{-1} = \{(u, x) : x \in D(T), u \in Tx\},$$

be its inverse. By the Cauchy-Schwarz inequality and (2.2) for each $x, y \in D(T)$, we have

$$d^2(x, y) \leq \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle \leq d(u, v)d(x, y),$$

for all $u \in Tx$ and $v \in Ty$, which implies that $T^{-1} : R(T) \to D(T)$ is a nonexpansive single-valued mapping. For each $\lambda \in (0, 1)$, define the mapping $J_\lambda^T : R(T) \to X$ by $J_\lambda^T x = (1 - \lambda)x \oplus \lambda T^{-1}x$. It follows from

(1.4) that J_λ^T is also a nonexpansive single-valued mapping. Considering (1.2), we see that $F(T) = F(T^{-1}) = F(J_\lambda^T)$.

We formally say that "$I - T$ is demiclosed at zero" if the conditions $\{x_n\} \subseteq C$ Δ- converges to x^* and $d(x_n, Tx_n) \to 0$ imply $x^* \in F(T)$.

Lemma 2.4 ([11])**.** *Let C be a nonempty closed convex subset of Hadamard space X and $T : C \to C$ be a nonexpansive mapping. Then, $I - T$ is demiclosed, i.e., if $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$, then $Tx = x$.*

Next, we introduce the combination of the resolvent iterative method and Halpern's iterative method and prove the strong convergence of the iterative algorithms. For more information about Halpern's iterative method see [8, 20].

Theorem 2.5. *Let C be a nonempty closed convex subset of a Hadamard space X which satisfies $(\mathcal{S})$ property. Let $T : X \to 2^X$ be a mapping such that $T - I$ is monotone, $F(T) \neq \emptyset$ and $\overline{D(T)} \subset C \subset R(T)$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(i) *$\beta_n \subset (\alpha, \beta)$ with $\alpha, \beta \in (0,1)$ and*

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$$

(ii) *$\alpha_n \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$,*

$$\sum_{n=1}^{\infty} \alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence $\{x_n\}$ defined by $u, x_0 \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n, \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{2.3}$$

converges strongly to $q = P_{F(T)}u$.

Proof. Let $p \in F(T)$. It follows from (1.5) that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n u \oplus (1-\alpha_n) y_n, p) \\ &\leq \alpha_n d(u,p) + (1-\alpha_n) d(y_n, p). \end{aligned} \tag{2.4}$$

Again, by (1.5) and nonexpansiveness of J_λ^T we have

$$
\begin{aligned}
d(y_n,p) &= d\left((1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n, p\right)\\
&\le (1-\beta_n)d(x_n,p) + \beta_n d(J_\lambda^T x_n, p)\\
&\le (1-\beta_n)d(x_n,p) + \beta_n d(x_n,p)\\
&= d(x_n,p).
\end{aligned}
\tag{2.5}
$$

Combining (2.4) and (2.5) we get

$$
\begin{aligned}
d(x_{n+1},p) &\le \alpha_n d(u,p) + (1-\alpha_n)d(y_n,p)\\
&\le \alpha_n d(u,p) + (1-\alpha_n)d(x_n,p)\\
&\le \max\{d(u,p), d(x_n,p)\}\\
&\ \ \vdots\\
&\le \max\{d(u,p), d(x_0,p)\}.
\end{aligned}
$$

Hence, $\{x_n\}$, $\{y_n\}$ and $\{J_\lambda^T x_n\}$ are bounded. It follows from (1.4), the distance-preserving property of the segment $[x_n, J_\lambda^T x_n]$ and nonexpansiveness of J_λ^T that

$$
\begin{aligned}
d(y_{n+1},y_n) &= d\left((1-\beta_{n+1})x_{n+1} \oplus \beta_{n+1}J_\lambda^T x_{n+1}, (1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n\right)\\
&\le d((1-\beta_{n+1})x_{n+1} \oplus \beta_{n+1}J_\lambda^T x_{n+1}, (1-\beta_{n+1})x_n \oplus \beta_{n+1}J_\lambda^T x_n)\\
&\quad + d\left((1-\beta_{n+1})x_n \oplus \beta_{n+1}J_\lambda^T x_n, (1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n\right)\\
&\le (1-\beta_{n+1})d(x_{n+1},x_n) + \beta_{n+1} d\left(J_\lambda^T x_{n+1}, J_\lambda^T x_n\right)\\
&\quad + |\beta_{n+1}-\beta_n| d(x_n, J_\lambda^T x_n)\\
&\le d(x_{n+1},x_n) + |\beta_{n+1}-\beta_n| M,
\end{aligned}
\tag{2.6}
$$

where we put $M = \sup_n \left\{d\left(x_n, J_\lambda^T x_n\right)\right\}$. Now, we estimate $d(x_{n+2}, x_{n+1})$. Again, by (1.4) and the distance-preserving property of the segment $[x_n, J_\lambda^T x_n]$ we have

$$
\begin{aligned}
d(x_{n+2},x_{n+1}) &= d(\alpha_{n+1}u \oplus (1-\alpha_{n+1})y_{n+1}, \alpha_n u \oplus (1-\alpha_n)y_n)\\
&\le d(\alpha_{n+1}u \oplus (1-\alpha_{n+1})y_{n+1}, \alpha_{n+1}u \oplus (1-\alpha_{n+1})y_n)\\
&\quad + d(\alpha_{n+1}u \oplus (1-\alpha_{n+1})y_n, \alpha_n u \oplus (1-\alpha_n)y_n)\\
&\le \alpha_{n+1}d(u,u) + (1-\alpha_{n+1})d(y_{n+1},y_n)\\
&\quad + |\alpha_{n+1}-\alpha_n| d(u,y_n)\\
&\le (1-\alpha_{n+1})d(y_{n+1},y_n) + |\alpha_{n+1}-\alpha_n| L,
\end{aligned}
$$

where $L = \sup_n\{d(u,y_n)\}$. This together with (2.6) implies that

$$
\begin{aligned}
d(x_{n+2},x_{n+1}) &\le (1-\alpha_{n+1})d(y_{n+1},y_n) + |\alpha_{n+1}-\alpha_n| L\\
&\le (1-\alpha_{n+1})d(x_{n+1},x_n) + |\beta_{n+1}-\beta_n| M + |\alpha_{n+1}-\alpha_n| L.
\end{aligned}
$$

It follows from Lemma 1.8 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Using (1.2) we have

$$d(x_n, y_n) \le d(x_{n+1}, x_n) + d(x_{n+1}, y_n) = d(x_{n+1}, x_n) + \alpha_n d(u, y_n) \to 0, \quad (as\ n \to \infty). \tag{2.7}$$

By (2.5) we get

$$\begin{aligned} d(x_n, p) &\le d(x_n, y_n) + d(y_n, p) \\ &= d(x_n, y_n) + d(x_n, p). \end{aligned}$$

This together with (2.7) implies that

$$\lim_{n\to\infty} \sup d(x_n, p) = \lim_{n\to\infty} \sup d(y_n, p). \tag{2.8}$$

Using (1.3) we have

$$\begin{aligned} d^2(y_n, p) &= d^2\left((1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n, p\right) \\ &\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(J_\lambda^T x_n, p) - \beta_n(1-\beta_n)d^2(x_n, J_\lambda^T x_n) \\ &\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, J_\lambda^T x_n) \\ &= d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, J_\lambda^T x_n). \end{aligned}$$

Hence,

$$\begin{aligned} \alpha(1-\beta)d^2(x_n, J_\lambda^T x_n) &< \beta_n(1-\beta_n)d^2(x_n, J_\lambda^T x_n) \\ &\le d^2(x_n, p) - d^2(y_n, p). \end{aligned}$$

Therefore, by passing to the subsequence and using (2.8), we conclude that

$$\limsup_{n\to\infty} d^2\left(x_n, J_\lambda^T x_n\right) = 0. \tag{2.9}$$

Next, we show that

$$\limsup_{n\to\infty} \langle \overrightarrow{uq}, \overrightarrow{x_n q} \rangle \le 0,$$

where $q = P_{F(T)}u$. Since $\{x_n\}$ is bounded, Lemma 1.2 follows that $\{x_n\}$ has a Δ-convergent subsequence and so by the $(\mathcal{S})$ property it has a ω-convergent subsequence. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that ω-converges to x^* and

$$\limsup_{n\to\infty} \langle \overrightarrow{uq}, \overrightarrow{x_n q} \rangle = \lim_{k\to\infty} \langle \overrightarrow{uq}, \overrightarrow{x_{n_k} q} \rangle.$$

Using (2.9) and Theorem 2.4 (demiclosedness of J_λ^T), we get $x^* \in F(T)$. It follows from the definition of ω-convergence and Theorem 1.1 that

$$\limsup_{n\to\infty} \langle \overrightarrow{uq}, \overrightarrow{x_n q} \rangle = \lim_{k\to\infty} \langle \overrightarrow{uq}, \overrightarrow{x_{n_k} q} \rangle = \langle \overrightarrow{uq}, \overrightarrow{x^* q} \rangle \le 0.$$

Furthermore,

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{uq},\overrightarrow{y_nq}\rangle &\leq \limsup_{n\to\infty}\langle\overrightarrow{uq},\overrightarrow{y_nx_n}\rangle + \limsup_{n\to\infty}\langle\overrightarrow{uq},\overrightarrow{x_nq}\rangle\\ &\leq \limsup_{n\to\infty} d(u,q)d(y_n,x_n) + \limsup_{n\to\infty}\langle\overrightarrow{uq},\overrightarrow{x_nq}\rangle\\ &\leq 0.\end{aligned}$$

Utilizing (1.6) we see that

$$\begin{aligned}d^2(x_{n+1},q) &= d^2(\alpha_n u \oplus (1-\alpha_n)y_n, q)\\ &\leq \alpha_n^2 d^2(u,q) + (1-\alpha_n)^2 d^2(y_n,q) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{uq},\overrightarrow{y_nq}\rangle\\ &\leq (1-\alpha_n)d^2(x_n,q) + \alpha_n\left(\alpha_n d^2(u,q) + 2(1-\alpha_n)\langle\overrightarrow{uq},\overrightarrow{y_nq}\rangle\right).\end{aligned}$$

Hence, from Lemma 1.8 we obtain the desired result. This completes the proof. □

Next, we introduce the combination of the resolvent method and the viscosity approximation method and prove the convergence of the iterative algorithm.
Let $T : X \to 2^X$ be a mapping such that $T-I$ is monotone and $\lambda \in (0,1)$. Define the sequence $\{z_n\}$ by $z_0 \in C$ and

$$\begin{cases} u_n = (1-\beta_n)z_n \oplus \beta_n J_\lambda^T z_n,\\ z_{n+1} = \alpha_n f(z_n) \oplus (1-\alpha_n)u_n, \quad n \geq 0,\end{cases} \tag{2.10}$$

where $f : C \to C$ is a contraction mapping from C into C with the contraction coefficient $c \in [0;1)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.

Theorem 2.6. *Let C be a nonempty closed convex subset of a Hadamard space X which satisfies the $(\mathcal{S})$ property. Let $T : X \to 2^X$ be a mapping such that $T - I$ is monotone, $F(T) \neq \emptyset$ and $\overline{D(T)} \subset C \subset R(T)$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(i) *$\beta_n \subset (\alpha,\beta)$ with $\alpha,\beta \in (0,1)$ and*

$$\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty;$$

(ii) *$\alpha_n \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$,*

$$\sum_{n=1}^{\infty}\alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty,$$

then the sequence $\{z_n\}$ defined by (2.10) converges strongly to an element $x^* \in F(T)$.

Proof. Since $P_{F(T)}$ is nonexpansive, then $P_{F(T)}f$ is a contraction and by the Banach Contraction Principle it has a unique fixed point, that is, $x^* = P_{F(T)}f(x^*)$ for some $x^* \in C$. Note that x^* is a unique solution of the variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* p} \rangle \leq 0, \quad (\forall p \in F(T)).$$

Replacing u by $f(x^*)$ in (2.3), from Theorem 2.5, the sequence $\{x_n\}$ converges strongly to $P_{F(T)}f(x^*) = x^*$. Using (1.4) we have

$$\begin{aligned} d(z_{n+1}, x_{n+1}) &= d(\alpha_n f(z_n) \oplus (1-\alpha_n)u_n, \alpha_n f(x^*) \oplus (1-\alpha_n)y_n) \\ &\leq \alpha_n d(f(z_n), f(x^*)) + (1-\alpha_n) d(u_n, y_n) \\ &\leq \alpha_n c d(z_n, x^*) \\ &\quad + (1-\alpha_n) d((1-\beta_n)z_n \oplus \beta_n J_\lambda^T z_n, (1-\beta_n)x_n \oplus \beta_n J_\lambda^T x_n) \\ &\leq \alpha_n c d(z_n, x^*) \\ &\quad + (1-\alpha_n)\left((1-\beta_n) d(z_n, x_n) + \beta_n d(J_\lambda^T z_n, J_\lambda^T x_n)\right) \\ &\leq \alpha_n c d(z_n, x^*) + (1-\alpha_n) d(z_n, x_n) \\ &\leq \alpha_n c d(z_n, x_n) + \alpha_n c d(x_n, x^*) + (1-\alpha_n) d(z_n, x_n) \\ &= (1-\alpha_n(1-c)) d(z_n, x_n) + \alpha_n (1-c) \frac{c}{1-c} d(x_n, x^*). \end{aligned}$$

Hence, from Lemma 1.8 we have $\lim_{n\to\infty} d(z_n, x_n) = 0$. Consequently, we obtain

$$\lim_{n\to\infty} d(z_n, x^*) \leq \lim_{n\to\infty} d(z_n, x_n) + \lim_{n\to\infty} d(x_n, x^*) = 0.$$

This completes the proof. □

Acknowledgment. The authors would like to thank Bandar Abbas Branch, Islamic Azad University for the Financial support of this research, which is based on a research project contract.

References

1. S. Alizadeha, H. Dehghan, and F. Moradlou, *Δ-convergence theorems for inverse-strongly monotone mappings in CAT(0) spaces*, Fixed Point Theory, 19(1) (2018), pp. 45-56.
2. M. Asadi, *Fixed points and common fixed points of mappings on CAT(0) spaces*, Fixed Point Theory, 14(1) (2013), pp. 29-38.
3. M. Asadi, *Some notes on fixed point sets in CAT(0) spaces*, Adv. Fixed Point Theory, 4(3) (2014), pp. 395-401.
4. M. Asadi, S.M. Vaezpour, and H. Soleimani, *α-Nonexpansive Mappings on CAT(0) spaces*, World Applied Sciences Journal, 11(10) (2010), pp. 1303-1306.
5. W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, in: DMV Seminar Band, vol. 25, Birkhäuser, Basel, (1995)

6. I.D. Berg and I.G. Nikolaev, *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedicata 133 (2008), pp. 195-218.
7. M. Bridson and A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Springer-Verlag, Berlin, Heidelberg, New York, (1999).
8. A. Cuntavepanit and B. Panyanak, *Strong convergence of modified Halpern iterations in CAT(0) spaces.* Fixed Point Theory Appl. 869458, (2011) doi:10.1155/2011/869458
9. H. Dehghan and J. Rooin, *A characterization of metric projection in CAT(0) spaces*, arXiv:1311.4174v1, 2013.
10. S. Dhompongsa, W.A. Kirk, and B. Sims, *Fixed points of uniformly lipschitzian mappings*, Nonlinear Anal., 65 (2006), pp. 762-772.
11. S. Dhompongsa, W.A. Kirk, and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear and Convex Anal., 8 (2007), pp. 35-45.
12. S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., 56 (2008), pp. 2572-2579.
13. B.A. Kakavandi and M. Amini, *Duality and subdifferential for convex functions on complete CAT(0) metric spaces*, Nonlinear Anal., 73 (2010), pp. 3450-3455.
14. B.A. Kakavandi, *Weak topologies in complete CAT(0) metric spaces*, Proc. Amer. Math. Soc., 141 (2013), pp. 1029-1039.
15. J.K. Kim and T.M. Tuyen, *New iterative methods for finding a common zero of a finite family of monotone operators in Hilbert spaces*, Bull. Korean Math. Soc., 54 (2017), pp. 1347-1359.
16. W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., 68 (2008), pp. 3689-3696.
17. W.A. Kirk, *Geodesic geometry and fixed point theory, in Seminar of Mathematical Analysis* (Malaga/Seville, 2002/2003), vol. 64 of Colecc. Abierta, pp. 195-225, University of Seville, Secretary Publication, Seville, Spain, (2003).
18. T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., 60 (1976), pp. 179-182.
19. A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, IRMA Lectures in Mathematics and Theoretical Physics, 6. European Mathematical Society (EMS), Zürich, 2005.
20. S. Ranjbar, *Strong convergence of a composite Halpern type iteration for a family of nonexpansive mappings in CAT(0) spaces*, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 3 (2017).
21. H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., 66 (2002), pp. 240-256.

[1]Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, P.O.Box 79158-93144, Bandar Abbas, Iran.

E-mail address: afkhami420@yahoo.com

[2] Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Gava Zang, P.O.Box 45137-66731, Zanjan, Iran.

E-mail address: hossein.dehgan@gmail.com