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Joint and Generalized Spectral Radius of Upper Triangular Matrices with Entries in a Unital Banach Algebra

Hamideh Mohammadzadehkan[1*], Ali Ebadian[2], and Kazem Haghnejad Azar[3]

Abstract. In this paper, we discuss some properties of joint spectral radius(jsr) and generalized spectral radius(gsr) for a finite set of upper triangular matrices with entries in a Banach algebra and represent relation between geometric and joint/generalized spectral radius. Some of these are in scalar matrices, but some are different. For example for a bounded set of scalar matrices,Σ, $r_*(\Sigma)=\hat{r}(\Sigma)$, but for a bounded set of upper triangular matrices with entries in a Banach algebra(Σ), $r_*(\Sigma)\neq\hat{r}(\Sigma)$. We investigate when the set is defective or not and equivalent properties for having a norm equal to jsr, too.

1. Introduction

Throughout this paper, we assume that A is a unital complex Banach algebra. For $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$, n-tuples of elements of A, write $a\cdot b=(a_1,\dots,a_n)\cdot(b_1,\dots,b_n)=a_1b_1+\dots+a_nb_n$. If $a\cdot b=1$, declare that a is left inverse of b and b is a right inverse of a. The *joint spectrum* $\sigma(a)$ for n-tuples $a=(a_1,\dots,a_n)$ will be a set of n-tuples $\lambda=(\lambda_1,\dots,\lambda_n)$ of the complex numbers, for which it is not true that the n-tuples $a-\lambda=(a_1-\lambda_1,\dots,a_n-\lambda_n)$ has left and right inverses in A. The geometric joint spectral radius $r(a_1,\dots,a_n)$ of $(a_1,\dots,a_n)$ is defined by

$$r(a_1,\dots,a_n)=\max\{|\lambda_j|:(\lambda_1,\dots,\lambda_n)\in\sigma(a_1,\dots,a_n)\}.$$

Note that $r(a_1,\dots,a_n)$ is the radius of the smallest l^∞ ball in $\mathbb{C}^n$ containing $\sigma(a_1,\dots,a_n)$. Easy examples (see [6]) show that $\sigma(a_1,\dots,a_n)$

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may be empty; in that case we put $r(a_1, \ldots, a_n) = -\infty$. For the Banach algebra A assume that $M_m(A)$ is the algebra of $m \times m$ matrices with entries in A. We denote the subalgebra of $M_m(A)$ which contains all upper triangular matrices by $U_m(A)$; i.e.

$$U_m(A) = \{T = (T(ij)) \mid T(ij) = 0, \text{whenever } i > j\},$$

and by $M_{m,k}(A)$, we mean all $m \times k$ matrices with entries in A. The algebra $M_m(A)$ with many norms is a Banach algebra, for example

$$\|T\| = \sup_{1 \le i \le m} \left[\sum_{j=1}^{m} \|T(ij)\|_A\right],$$

which is a sub-multiplicative norm. Let $\Sigma = \{T_1, \ldots, T_n\}$ be a bounded set in $U_m(A)$. For $k \ge 1$, Σ^k is the set of all products of matrices in Σ of length k,

$$\Sigma^k = \{T_1 T_2 \ldots T_k : T_i \in \Sigma, i = 1, \ldots, k\}.$$

The spectral radius of a matrix T denoting by $r(T)$ is defined by $r(T) = \lim_{k\to\infty} \left\|T^k\right\|^{1/k}$, it is also equal to $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$. The joint and generalized spectral radius is defined as below:

(i) The limit

$$\hat{r}(\Sigma) := \limsup_{k\to\infty} [\hat{r}_k(\Sigma)]^{1/k} \left(= \lim_{k\to\infty} [\hat{r}_k(\Sigma)]^{1/k} = \inf_{k\ge1} [\hat{r}_k(\Sigma)]^{1/k}\right), \tag{1.1}$$

where

$$\hat{r}_k(\Sigma) := \sup\left\{\|T\| : T \in \Sigma^k\right\},$$

is called the *joint spectral radius* of the matrices Σ.

(ii) The limit

$$r_*(\Sigma) := \limsup_{k\to\infty} [r_k(\Sigma)]^{1/k} \left(= \sup_{k\ge1} [r_k(\Sigma)]^{1/k}\right), \tag{1.2}$$

where

$$r_k(\Sigma) := \sup\left\{r(T) : T \in \Sigma^k\right\},$$

is called the *generalized spectral radius* of the matrices Σ.

As for the single-matrix case, the joint spectral radius does not depend on the matrix norm used. Remember that two matrices norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are related by

$$\alpha \|T\|_1 \le \|T\|_2 \le \beta \|T\|_1,$$

for some $0 < \alpha < \beta$. One has

$$\sup_{T\in\Sigma^k} (\alpha \|T\|_1) \le \sup_{T\in\Sigma^k} (\|T\|_2) \le \sup_{T\in\Sigma^k} (\beta \|T\|_1),$$

then

$$\alpha \sup_{T\in\Sigma^k} (\|T\|_1) \le \sup_{T\in\Sigma^k} (\|T\|_2) \le \beta \sup_{T\in\Sigma^k} (\|T\|_1),$$

which $\sup_{T\in\Sigma^k}(\|T\|_1)$ is $\hat{r}_k(\Sigma)$ with $\|\cdot\|_1$ and $\sup_{T\in\Sigma^k}(\|T\|_2)$ is $\hat{r}_k(\Sigma)$ with $\|\cdot\|_2$. So

$$\alpha^{1/k}\left(\sup_{T\in\Sigma^k}\|T\|_1\right)^{1/k} \le \sup_{T\in\Sigma^k}(\|T\|_2)^{1/k} \le \beta^{1/k}\sup_{T\in\Sigma^k}(\|T\|_1)^{1/k},$$

and by letting k tend to infinity, we conclude that the joint spectral radius is well defined independently of the matrix norm used.

Suppose that $\Sigma^0 = \{I\}$ and $M = \bigcup_{k=0}^{\infty}\Sigma^k$. The set Σ is called product bounded if M is a bounded set. As has been noted by M. Berger and Y. Wang [2] the quantities $\hat{r}(\Sigma)$ and $r_*(\Sigma)$ for bounded set of complex or real matrices Σ coincide with each other:

$$\hat{r}(\Sigma) = r_*(\Sigma). \tag{1.3}$$

This fundamental formula has numerous applications in the theory of joint generalized spectral radius. Another important consequence of the Berger-Wang formula (1.3) is (see [7]):

$$r_k(\Sigma)^{1/k} \le r_*(\Sigma) = \hat{r}(\Sigma) \le \hat{r}_k(\Sigma)^{1/k}. \tag{1.4}$$

There is an important equality in [10, Proposition 1], too. Let $\mathfrak{N}$ be the family of equivalent norms on matrix algebra which the algebra is a normed algebra such that all norms are assumed multiplicative.Then

$$\hat{r}(\Sigma) = \inf_{N\in\mathfrak{N}}\sup_{T\in\Sigma} N(T). \tag{1.5}$$

In this paper, first by upper triangular matrix properties, we prove that the spectral radius of T is a maximum of the spectral radius of its entries. After, we compare geometric, joint and generalized spectral radius quantities. Then we obtain some inequalities between spectral radius in special condition. The main results of the first section are Theorems 2.6 and 3.6 which prove that joint and generalized spectral radius of finite set of upper triangular matrices with entries in Banach algebras are maximum of joint and generalized spectral radius of their entries. In the next section, we investigate necessary and sufficient condition for having a norm is equal to jsr for Σ. The main result of the section is Theorem 3.6 and Corollary 3.7.

2. JOINT AND GENERALIZED SPECTRAL RADIUS

Recall A is a unital Banach algebra and $U_m(A)$ is the set of all upper triangular matrices with entries in A. We start with the following results which are tools for our main aims.

Proposition 2.1. *Let $T \in U_m(A)$, then*

$$r(T) = \max_{1\le k\le m} r(T(kk)). \tag{2.1}$$

Proof. Let $T \in U_m(A)$, by Proposition 1.1 of [1],

$$\sigma(T) = \bigcup_{k=1}^{m}\sigma(T(kk)).$$

Then

$$\begin{aligned} r(T) &= \max\left\{|\lambda| : \lambda \in \bigcup_{k=1}^{m} \sigma(T(kk))\right\} \\ &= \max_{1\le k\le m} \max\{|\lambda| : \lambda \in \sigma(T(kk))\} \\ &= \max_{1\le k\le m} r(T(kk)). \end{aligned}$$

□

Theorem 2.2. *Assume an n-tuple $(T_1,\ldots,T_n)$ is in $U_m(A)$. Then*

$$\begin{aligned} \max_{1\le k\le m} r(T_1(kk),\ldots,T_n(kk)) &\le r(T_1,\ldots,T_n) \\ &\le \max_{1\le k\le m}\max_{1\le j\le n} r(T_j(kk)) \\ &\le r_*(\Sigma) \\ &\le \hat{r}(\Sigma). \end{aligned} \tag{2.2}$$

Proof. Let $(T_1,\ldots,T_n) \in (U_m(A))^n$, by Theorem 1 of [9] and Proposition 2.6 of [8], we have

$$\begin{aligned} &\max_{1\le k\le m} r(T_1(kk),\ldots,T_n(kk)) \\ &\quad\le \max_k \max_j r(T_j(kk)) \\ &\quad= \max_k \max_j \max\{|\lambda| : \lambda \in \sigma(T_j(kk))\} \\ &\quad= \max_k \max_j \{|\lambda_j| : \lambda_j \in \sigma(T_j(kk))\} \\ &\quad= \max_{1\le k\le m} \{|(\lambda_1,\ldots,\lambda_n)| : (\lambda_1,\ldots,\lambda_n) \in \sigma(T_1(kk))\times\cdots\times\sigma(T_n(kk))\} \\ &\quad= \max\left\{|(\lambda_1,\ldots,\lambda_n)| : (\lambda_1,\ldots,\lambda_n) \in \bigcup_{k=1}^{m} \sigma(T_1(kk))\times\cdots\times\sigma(T_n(kk))\right\} \\ &\quad= \max\{|(\lambda_1,\ldots,\lambda_n)| : (\lambda_1,\ldots,\lambda_n) \in \sigma(T_1,\ldots,T_n)\} \\ &\quad= r(T_1,\ldots,T_n). \end{aligned}$$

For the next inequality, by Theorem 1 of [9] and Proposition 2.1, we have:

$$\begin{aligned} r(T_1,\ldots,T_n) &\le \max_{1\le j\le n} r(T_j) \\ &= \max_j \max_k r(T_j(kk)), \end{aligned}$$

and

$$\begin{aligned} \max_j \max_k r(T_j(kk)) &= \max_{1\le j\le n} r(T_j) \\ &\le \left(\sup_{T\in\Sigma^k} r(T)\right)^{1/k} \\ &= r_k(\Sigma)^{1/k} \end{aligned}$$

$$\leq \limsup_{k\to\infty} \left[r_k(\Sigma)\right]^{1/k}$$
$$= r_*(\Sigma).$$

For the last inequality, note that $r(T) \leq \|T\|$ for any $T \in \Sigma^k$, then $r_k(\Sigma) \leq \hat{r}_k(\Sigma)$ and so $r_*(\Sigma) \leq \hat{r}(\Sigma)$. □

The following results are true for cases $A = \mathbb{R}$ or $\mathbb{C}$ for bounded set, but we extend them to Banach algebra for finite set. In the next Lemma, we show that $\hat{r}(\Sigma) \leq 1$ for a finite and product bounded set, Σ.

Lemma 2.3. *Let Σ be a finite and product bounded set of upper triangular matrices with entries in Banach algebra A, then $\hat{r}(\Sigma) \leq 1$.*

Proof. For every $x \in A^m$ define $\|x\| := \sup\{\|Tx\| : T \in M\}$; that is a norm, because if $\|x\| = 0$, so for any $T \in M$, $\|Tx\| = 0$ and so for I, $\|Ix\| = 0$ and $x = 0$. Other properties of the norm are trivially. Then for any $T \in \Sigma$ we have $\|x\| \geq \|Tx\|$ and so $\frac{\|Tx\|}{\|x\|} \leq 1$ and

$$\sup\{\|T\| : T \in \Sigma\} = \sup\left\{\sup\left\{\frac{\|Tx\|}{\|x\|} : x \in A^m\right\} : T \in \Sigma\right\} \leq 1.$$

The norm on Σ is sub-multiplicative, then for any k

$$\hat{r}_k(\Sigma) = \sup\left\{\|T\| : T \in \Sigma^k\right\} \leq 1.$$

Therefore, $\hat{r}(\Sigma) \leq 1$. □

Lemma 1 in [3], which compares spectral radius of linear combination and joint spectral radius of T_i's, is true for a finite set of upper triangular matrices with entries in Banach algebra A. We give the prove for this case in the following proposition.

Proposition 2.4. *Let $\Sigma = \{T_1, \ldots, T_n\}$ be a finite set of upper triangular matrices with entries in Banach algebra A and any $\alpha_i \geq 0$ satisfying $\sum_{i=1}^n \alpha_i = 1$. Then*

$$r\left(\sum_{i=1}^n \alpha_i T_i\right) \leq \hat{r}(\Sigma). \tag{2.3}$$

Proof. Fix some $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ and an integer $k \geq 1$. Then

$$\left\|\left(\sum_{i=1}^n \alpha_i T_i\right)^k\right\| = \left\|\sum_{i_l \in \{1,\ldots,n\}}^n \alpha_{i_1}\ldots\alpha_{i_k} T_{i_1}\ldots T_{i_k}\right\|$$
$$\leq \sum_{i_l \in \{1,\ldots,n\}}^n \alpha_{i_1}\ldots\alpha_{i_k} \|T_{i_1}\ldots T_{i_k}\|$$
$$\leq \max_{i_l \in \{1,\ldots,n\}} \|T_{i_1}\ldots T_{i_k}\|.$$

Thus

$$\begin{aligned} r\left(\sum_{i=1}^{n}\alpha_i T_i\right) &= \lim_{k\to\infty}\left\|\left(\sum_{i=1}^{n}\alpha_i T_i\right)^k\right\|^{1/k} \\ &= \limsup_{k\to\infty}\left\|\left(\sum_{i=1}^{n}\alpha_i T_i\right)^k\right\|^{1/k} \\ &\le \limsup_{k\to\infty}\max_{i_l\in\{1,\dots,n\}}\|T_{i_1}\dots T_{i_k}\|^{1/k} \\ &= \hat{r}(\Sigma). \end{aligned}$$

□

An immediate result is given by:

Corollary 2.5. *Let $\Sigma = \{T_1, \dots, T_n\}$ be a finite set of upper triangular matrices with entries in Banach algebra A, then*

$$\frac{1}{n} r\left(\sum_{i=1}^{n} T_i\right) \le \hat{r}(\Sigma).$$

Suppose that $T \in \Sigma^k$ is an upper trianglular matrix, we can rewrite T with block matrices $T^{(1)}, \dots, T^{(l)}$ in diagonal entries and $\Sigma^{(j)} := \left\{T^{(j)} : T \in \Sigma\right\}$:

$$T = \begin{pmatrix} T^{(1)} & \dots & * \\ & \ddots & \\ 0 & & T^{(l)} \end{pmatrix}.$$

In the following, we prove useful equalities between joint and generalized spectral radius and maximum of joint and generalized spectral radius of diagonal entries (in fact the diagonal entries of block matrices).

Theorem 2.6. *Let $\Sigma = \{T_1, \dots, T_n\}$ be a finite and product bounded set of upper triangular matrices with entries in Banach algebra A, then*

$$r_*(\Sigma) = \max_{1\le j\le l} r_*\left(\Sigma^{(j)}\right), \tag{2.4}$$

and

$$\hat{r}(\Sigma) = \max_{1\le j\le l} \hat{r}\left(\Sigma^{(j)}\right). \tag{2.5}$$

Proof. For the first equation, let $T \in \Sigma^k$ with block matrices $T^{(1)}, \dots, T^{(l)}$. By Proposition 2.1, $r(T) = \max_{1\le i\le m} r(T(ii))$. Now any $T^{(j)}$ matrix is upper triangular and we can use Proposition 2.1 again, so $r(T) = \max_{1\le j\le l} r\left(T^{(j)}\right)$, then

$$\begin{aligned} r_k(\Sigma) &= \sup\left\{r(T) : T \in \Sigma^k\right\} \\ &= \sup\left\{\max_{1\le j\le l} r\left(T^{(j)}\right) : T^{(j)} \in \left(\Sigma^{(j)}\right)^k\right\} \end{aligned}$$

$$= \max_{1\le j\le l} \sup\left\{ r\left(T^{(j)}\right) : T^{(j)} \in \left(\Sigma^{(j)}\right)^k \right\}$$
$$= \max_{1\le j\le l} r_k\left(\Sigma^{(j)}\right),$$

and so

$$\begin{aligned} r_*(\Sigma) &= \limsup \left(r_k(\Sigma)\right)^{\frac{1}{k}} \\ &= \limsup \left(\max_{1\le j\le l} r_k\left(\Sigma^{(j)}\right)\right)^{\frac{1}{k}} \\ &= \max_{1\le j\le l}\left(\limsup \left(r_k\left(\Sigma^{(j)}\right)\right)^{\frac{1}{k}}\right) \\ &= \max_{1\le j\le l} r_*\left(\Sigma^{(j)}\right). \end{aligned}$$

For equation (2.5), let $T \in \Sigma^k$ and k be fixed. Then one can write $T = D+N$, such that D is a diagonal and N is a nilpotent matrix and

$$\|T\| = \sup\nolimits_{1\le i\le m}\left(\sum\nolimits_{j=1}^m \|T(ij)\|\right);$$
$$D = \begin{pmatrix} T^{(1)} & \dots & 0 \\ & \ddots & \\ 0 & & T^{(l)} \end{pmatrix}.$$

This norm has the following properties:

(i) If any element of T is substituted with zero, its norm decreases, so $\|D\| \le \|T\|$;

(ii) For D, diagonal part of T, $\|D\| = \max_{1\le j\le l}\left\|T^{(j)}\right\|$.

Hence

$$\begin{aligned} \hat{r}_k(\Sigma) &= \sup\left\{\|T\| : T \in \Sigma^k\right\} \\ &\ge \sup\left\{\|D\| : T = D + N \in \Sigma^k\right\} \\ &= \sup\left\{ \max_{1\le j\le l}\left\|T^{(j)}\right\| : D = \begin{pmatrix} T^{(1)} & \dots 0 \\ & \ddots \\ 0 & T^{(l)} \end{pmatrix} \right\} \\ &= \max_{1\le j\le l}\sup\left\{ \left\|T^{(j)}\right\| : T^{(j)} \in \left(\Sigma^{(j)}\right)^k \right\} \\ &= \max_{1\le j\le l} \hat{r}_k\left(\Sigma^{(j)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \hat{r}(\Sigma) &= \limsup\left(\hat{r}_k(\Sigma)\right)^{\frac{1}{k}} \\ &\ge \limsup\left(\max_{1\le j\le l}\hat{r}_k\left(\Sigma^{(j)}\right)\right)^{\frac{1}{k}} \\ &= \max_{1\le j\le l}\limsup\left(\hat{r}_k\left(\Sigma^{(j)}\right)\right)^{\frac{1}{k}} \end{aligned}$$

$$= \max_{1\le j\le l} \hat{r}\left(\Sigma^{(j)}\right).$$

Conversely, for any $\delta > \max_{1\le j\le l} \hat{r}\left(\Sigma^{(j)}\right)$, by Lemma 2.3, $\hat{r}(\Sigma/\delta) \le 1$ and by Proposition 2.1,

$$\hat{r}(\Sigma/\delta) = \inf_{\nu\in\mathfrak{N}} \sup_{T\in\Sigma/\delta} \nu(T).$$

($\mathfrak{N}$ is the family of all multiplicative equivalent norms on matrix algebra) Then $\inf_{\nu\in\mathfrak{N}} \sup_{T\in\Sigma/\delta} \nu(T) \le 1$ and there is a norm $\nu_0 = \nu_\delta$, such that $\sup_{T\in\Sigma/\delta} \nu_0(T) \le 1$ and

$$\sup_{T\in\Sigma} \nu_0(T) \le \delta. \tag{2.6}$$

Now, for any $T \in \Sigma^k$ there are $T_1, \ldots, T_k$, $N_1, \ldots, N_k$ and $D_1, \ldots, D_k$ such that

$$\begin{aligned} T = T_1 \ldots T_k &= (D_1 + N_1) \ldots (D_k + N_k) \\ &= D_1 \ldots D_k + \ldots + N_1 \ldots N_k. \end{aligned}$$

There is n such that the product in n terms of N_i's is zero (N_i's are nilpotent), then for suitable k and constant C we have

$$\begin{aligned} \sup_{T\in\Sigma^k} \nu_0(T) &\le \sum_{i=1}^{n-1} \binom{k}{i} \left(\sup_{N\in\Sigma_N^k} \nu_0(N)\right)^i \left(\sup_{D\in\Sigma_D^k} \nu_0(D)\right)^{k-i} \\ &\le Ck^{n-1} \left(\sup_{D\in\Sigma_D^k} \nu_0(D)\right)^k. \end{aligned}$$

Since $\|\cdot\|$ and ν are equivalence, there is a constant C_1 such that $\|T\| \le C_1\nu_0(T)$, but $\|D\| \le \|T\|$, then $\nu_0(D) \le \nu_0(T) < \delta$. By the above equations, we have

$$\sup\nolimits_{T\in\Sigma^k} \nu_0(T) \le Ck^{n-1}\delta^k,$$

and so

$$\begin{aligned} \sup_{T\in\Sigma^k} \|T\| &\le \sup_{T\in\Sigma^k} C_1\nu_0(T) \\ &= C_1 \sup_{T\in\Sigma^k} \nu_0(T) \\ &\le C_1 Ck^{n-1}\delta^k. \end{aligned}$$

Then

$$\begin{aligned} \hat{r}(\Sigma) &= \limsup \left(\hat{r}_k(\Sigma)\right)^{\frac{1}{k}} \\ &\le \limsup \left(C_1 C\right)^{\frac{1}{k}} \left(k^{n-1}\right)^{\frac{1}{k}} \delta = \delta. \end{aligned}$$

Therefore $\hat{r}(\Sigma) = \max_{1\le j\le l} \hat{r}\left(\Sigma^{(j)}\right)$. □

In this paper, we compare jsr with gsr and try to simplify the calculation of them. We search that "Is the jsr equals gsr?"
The following theorem proved by L. Gurvits in [5]:

Theorem 2.7. *For any two positive numbers $0 < \alpha < \beta$, there exist two isomorphisms A_1, A_2 in Hilbert space l_2, such that the generalized spectral radius $r_* (\{A_1, A_2\}) = \alpha$ while the joint spectral radius $\hat{r} (\{A_1, A_2\}) = \beta$.*

Corollary 2.8. *Let $\Sigma = \{T_1, \dots, T_n\}$ be a finite set of upper triangular matrices with entries in Banach algebra A, then*

$$r_* (\Sigma) \neq \hat{r} (\Sigma) . \tag{2.7}$$

Proof. Set $A = B (l_2)$ and $m = 1$, so by Theorem 2.7 for $U_1 (B (l_2))$, $r_* (\Sigma) \neq \hat{r} (\Sigma)$. □

3. Spectral radius and norm of a set

Definition 3.1. Let Σ be a bounded family of $m \times m$ upper triangular matrices with entries in Banach algebra A and $\|\cdot\|$ be a norm on $M_m (A)$. Define

$$\|\Sigma\| := \hat{r}_1 (\Sigma) = \sup_{i \in I} \|T_i\|.$$

Definition 3.2. Let Σ be a bounded family of $m \times m$ upper triangular matrices with entries in Banach algebra A is said to be nondefective if the multiplicative semigroup $M^+ = \bigcup_{k=1}^{\infty} \Sigma^k$ is a bounded set of matrices. Equivalently, if there exists $\alpha \in \mathbb{R}$ such that for all k,

$$\sup \left\{ \|T\| : T \in \Sigma^k \right\} \leq \alpha \left(\hat{r} (\Sigma) \right)^k.$$

Remark 3.3. We prove that two terms in Definition 3.2 are equivalent: Let the first definition be true and the second one be not true, so there is some k such that for all $\alpha \in \mathbb{R}$

$$\sup \left\{ \|T\| : T \in \Sigma^k \right\} > \alpha \left(\hat{r} (\Sigma) \right)^k.$$

Which contradicts the boundedness of M^+. Now assume there is $\alpha \in \mathbb{R}$ such that for any k

$$\sup \left\{ \|T\| : T \in \Sigma^k \right\} \leq \alpha \left(\hat{r} (\Sigma) \right)^k.$$

By Lemma 2.3, $\hat{r} (\Sigma) \leq 1$, then

$$\hat{r} (\Sigma) \geq \left(\hat{r} (\Sigma) \right)^2 \geq \cdots \geq \left(\hat{r} (\Sigma) \right)^k \geq \cdots .$$

Now set $M = \alpha \hat{r} (\Sigma)$, then for any k there exists M which

$$\sup \left\{ \|T\| : T \in \Sigma^k \right\} \leq M,$$

but for any $T \in \bigcup_{k=1}^{\infty} \Sigma^k$ there exist k such that $T \in \Sigma^k$, so $\|T\| \leq M$ and this means $M^+ = \bigcup_{k=1}^{\infty} \Sigma^k$ is bounded.

We get simple examples similar numerical matrices, and calculate jsr and gsr for them. The first case shows a Σ that is defective:

Example 3.4. Let X be a Banach algebra and $A = B (X)$, the algebra of all operators on X, then $B \left(X^2 \right) \cong M_2 (B (X))$ and hence $U_2 (B (X))$ is subalgebra of $B \left(X^2 \right)$. Set

$$\Sigma = \left\{ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & i \\ 0 & i \end{pmatrix} \right\},$$

which $i : X \to X$ is the identity map. By Theorem 2.6, $\hat{r}(\Sigma) = \max_{1\leq j\leq 2} \hat{r}\left(\Sigma^j\right)$. We have $\Sigma^1 = \{i, i\} = \{i\}$ and $\Sigma^2 = \{i, i\} = \{i\}$, so

$$\begin{aligned}\hat{r}\left(\Sigma^1\right) &= \hat{r}\left(\Sigma^2\right)\\ &= \hat{r}(\{i\})\\ &= \limsup_{k\to\infty} \left(\hat{r}_k(\{i\})\right)^{\frac{1}{k}},\end{aligned}$$

and

$$\begin{aligned}\hat{r}_k(\{i\}) &= \sup\left\{\left\|\underbrace{i\ldots i}_{k\ times}\right\|\right\}\\ &= \sup\{\|i\|\} = 1,\end{aligned}$$

and then $\hat{r}\left(\Sigma^1\right) = \hat{r}\left(\Sigma^2\right) = 1$ and $\hat{r}(\Sigma) = 1$. For $r_*(\Sigma)$ with similar calculation

$$\Sigma^1 = \Sigma^2 = \{i, i\} = \{i\},$$

and so

$$r_*\left(\Sigma^1\right) = r_*\left(\Sigma^2\right) = r_*(\{i\}) = \limsup_{k\to\infty} \left(r_k(\{i\})\right)^{\frac{1}{k}},$$

and

$$\begin{aligned}r_k(\{i\}) &= \sup\left\{r\left(\underbrace{i\ldots i}_{k\ times}\right)\right\}\\ &= \sup\{r(i)\}\\ &= \lim_{k\to\infty} \left\|i^k\right\|^{\frac{1}{k}}\\ &= \lim_{k\to\infty} \|i\|^{\frac{1}{k}} = 1,\end{aligned}$$

and then $r_*\left(\Sigma^1\right) = r_*\left(\Sigma^2\right) = 1$ and $r_*(\Sigma) = 1$.

Otherwise

$$\bigcup_{k=1}^{\infty} \Sigma^k = \bigcup_{k=1}^{\infty} \{T_1 \ldots T_k : T_i \in \Sigma\},$$

and

$$\begin{aligned}\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}^2 &= \begin{pmatrix} i & 0\\ 0 & i\end{pmatrix},\\ \begin{pmatrix} i & i\\ 0 & i\end{pmatrix}^2 &= \begin{pmatrix} i & 2i\\ 0 & i\end{pmatrix},\\ &\vdots\\ \begin{pmatrix} i & i\\ 0 & i\end{pmatrix}^k &= \begin{pmatrix} i & ki\\ 0 & i\end{pmatrix},\\ \begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix} i & i\\ 0 & i\end{pmatrix} &= \begin{pmatrix} i & i\\ 0 & i\end{pmatrix},\\ \begin{pmatrix} i & i\\ 0 & i\end{pmatrix}\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix} &= \begin{pmatrix} i & i\\ 0 & i\end{pmatrix}.\end{aligned}$$

Then

$$\bigcup_{k=1}^{\infty}\Sigma^k=\left\{\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix},\begin{pmatrix} i & i\\ 0 & i\end{pmatrix},\begin{pmatrix} i & ki\\ 0 & i\end{pmatrix}: k\in\mathbb{N}\right\}$$

which is unbounded and therefor Σ is defective. For any norm, $\|\Sigma\|\neq 1$. If $\|\Sigma\|_*=1$, then

$$\left\|\begin{pmatrix} i & ki\\ 0 & i\end{pmatrix}_*\right\|\leq 1,$$

for any $k\in\mathbb{N}$. That is impossible, because all norms on $M_2(A)$ are equivalence and there are some norms should have not this property.

Now the following example shows a set Σ which is nondefective.

Example 3.5. As above, let X be a Banach algebra and $A=B(X)$, the algebra of all operators on X, then $B\left(X^2\right)\cong M_2(B(X))$ and hence $U_2(B(X))$ is a subalgebra of $B\left(X^2\right)$. Set

$$\Sigma=\left\{\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix},\begin{pmatrix} -i & 0\\ 0 & i\end{pmatrix}\right\},$$

which $i:X\to X$ is the identity map. By forward calculation we get

$$\Sigma^k=\left\{\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix},\begin{pmatrix} -i & 0\\ 0 & i\end{pmatrix}\right\}=\Sigma$$

Then $\bigcup_{k=1}^{\infty}\Sigma^k$ is bounded and Σ is nondefective. Similar to Example 3.4 $\hat{r}(\Sigma)=r_*(\Sigma)=1$. Consider

$$\|T\|_*=\sup_{1\leq i\leq 2}\sum_{i=1}^{2}\|T_{ij}\|,$$

so

$$\|\Sigma\|_*=\sup_{T\in\Sigma}\|T\|_*=\sup\{1,1\}=1.$$

Then $\|\Sigma\|_*=1=\hat{r}(\Sigma)=r_*(\Sigma)$.

In the following theorem and corollary we prove that what happens in the two above examples is true. In general nondefectivity is sufficient and necessary to have a norm such that $\hat{r}(\Sigma)=\|\Sigma\|$.

Theorem 3.6. *Let Σ be a product bounded family of $m\times m$ upper triangular matrices with entries in a Banach algebra, then we have:*

- (i) *Σ is nondefective if and only if there exists an operator norm ν on $M_m(A)$ such that $\nu(\Sigma)\leq 1$.*
- (ii) *There is a norm ν' such that $\hat{r}(\Sigma)=\nu'(\Sigma)$ if and only if there exists an operator norm ν on $M_m(A)$ such that $\nu(\Sigma)\leq 1$.*

Proof. (i) By assumption Σ is product bounded (then it is nondefective), so $M=\bigcup_{k=0}^{\infty}\Sigma^k$ is a bounded set of matrices. We set

$$\nu(x)=\sup\{\|Tx\|:T\in M\},$$

for any $0\neq x\in A^m$. Let ν' be the induced norm by ν on $M_m(A)$ i.e. for any $T\in M_m(A)$,

$$\nu'(T)=\sup_{\nu(x)\neq 0}\frac{\nu(Tx)}{\nu(x)}.$$

For any $T \in M$, $\nu(x) \geq \|Tx\|$, then $\frac{1}{\nu(x)} \leq \frac{1}{\|Tx\|}$ and by boundedness of M, there is some k such that

$$\begin{aligned} \nu(Tx) &= \sup\{\|STx\| : S \in M\} \\ &\leq \sup\{\|S\|\,\|Tx\| : S \in M\} \\ &\leq k\,\|Tx\|. \end{aligned} \tag{3.1}$$

(ii) Let ν be a norm on $M_m(A)$ such that $\nu(\Sigma) \leq 1$. By equation (1.5) and Definition 3.1 $\hat{r}(\Sigma) = \inf_{\|\ \| \in \mathfrak{N}} \|\Sigma\| \leq \nu(\Sigma)$. Now if $\nu(\Sigma) = \hat{r}(\Sigma)$, ν' is equal to ν, otherwise $\hat{r}(\Sigma) < \nu(\Sigma)$. Set $\delta = \nu(\Sigma)$, by using assumption and part $[i]$ Σ is nondefective then for constant δ,

$$M^{+}\left(\tfrac{\Sigma}{\delta}\right) = \textstyle\bigcup_{k=1}^{\infty}\left(\tfrac{\Sigma}{\delta}\right)^{k}$$

is bounded, part $[i]$ again there exist ν^* such that $\nu^*\left(\frac{\Sigma}{\delta}\right) \leq 1$ and so $\nu^*(\Sigma) \leq \delta$ hence $\hat{r}(\Sigma) \leq \nu^*(\Sigma) \leq \nu(\Sigma)$, similarly if ν^* is equal to ν' this prove is completed, otherwise by repeating this process obtain ν' because by continuing $\delta \to 0$ and consequently $\hat{r}(\Sigma) = \nu^*(\Sigma)$.
Conversely suppose ν' be such that $\hat{r}(\Sigma) = \nu'(\Sigma)$ and by lemma 2.3 $\hat{r}(\Sigma) \leq 1$.

□

Corollary 3.7. *Let Σ be a bounded family of $m \times m$ upper triangular matrices with entries in a Banach algebra. Then Σ is nondefective if and only if Σ has a norm ν such that $\hat{r}(\Sigma) = \nu'(\Sigma)$.*

Remark 3.8. Let Σ be a product bounded set of $m \times m$ upper triangular matrices with entries in Banach algebra A, then Σ is nondefective. It is obvious by definition of product bounded and nondefective set. By Theorem 3.6 the product bounded set Σ has a norm ν such that $\hat{r}(\Sigma) = \nu'(\Sigma)$.

4. Joint and generalized spectral subradius

As presented in [11] and [5] for a set of upper triangular matrices with entries in a Banach algebra, Σ instead of taking sup and limsup, in definition of joint and generalized spectral radius, we take inf and liminf; this guides to define the joint and generalized spectral subradius as following:

(i) The limit

$$\hat{\rho}(\Sigma) := \liminf_{k\to\infty} [\hat{\rho}_k(\Sigma)]^{1/k}, \tag{4.1}$$

where

$$\hat{\rho}_k(\Sigma) := \inf\left\{\|T\| : T \in \Sigma^k\right\},$$

is called the *joint spectral subradius* of the matrices Σ.

(ii) The limit

$$\rho_*(\Sigma) := \liminf_{k\to\infty} [\rho_k(\Sigma)]^{1/k}, \tag{4.2}$$

where

$$\rho_k(\Sigma) := \inf\left\{r(T) : T \in \Sigma^k\right\},$$

is called the *generalized spectral radius* of the matrices Σ.

We compare there two values in this section. In a similar way J. Theys proved for a set of complex matrices in [11], we prove this equality for the set Σ.

Lemma 4.1. *For any set of upper triangular matrices with entries of a Banach algebra,* $\rho_* (\Sigma) = \hat{\rho} (\Sigma)$.

Proof. By $r(T) \leq \|T\|$, for any $T \in \Sigma^k$;

$$\inf_{T\in\Sigma^k} r(T) \leq \inf_{T\in\Sigma^k} \|T\|.$$

So

$$(\rho_k (\Sigma))^{\frac{1}{k}} \leq (\hat{\rho}_k (\Sigma))^{\frac{1}{k}}.$$

Then

$$\rho_* (\Sigma) \leq \hat{\rho} (\Sigma).$$

For converse, by definition of inf for any n_0 and $\epsilon > 0$, there exist $T_{l_0}, \ldots, T_{l_{n_0}}$ such that

$$r\left(T_{l_0} \cdots T_{l_{n_0}}\right) \leq \rho_{n_0} (\Sigma) + \epsilon.$$

Then for any n_0,

$$\begin{aligned}\liminf_{k\to\infty} [\hat{\rho}_{n_0 k} (\Sigma)]^{1/n_0 k} &\leq \left(\liminf_{k\to\infty} \left[\left\|\left(T_{l_0} \cdots T_{l_{n_0}}\right)^k\right\|^{1/k}\right]\right)^{\frac{1}{n_0}} \\ &= r\left(T_{l_0} \cdots T_{l_{n_0}}\right)^{\frac{1}{n_0}} \\ &\leq (\rho_{n_0} (\Sigma) + \epsilon)^{\frac{1}{n_0}}.\end{aligned}$$

Thus by tending $\epsilon \to 0$

$$[\hat{\rho}_k (\Sigma)]^{1/k} \leq \liminf_{k\to\infty} [\hat{\rho}_{n_0 k} (\Sigma)]^{1/n_0 k} \leq (\rho_{n_0} (\Sigma))^{\frac{1}{n_0}},$$

and so

$$\hat{\rho} (\Sigma) \leq \liminf_{n_0\to\infty} (\rho_{n_0} (\Sigma))^{\frac{1}{n_0}} = \rho_* (\Sigma).$$

□

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