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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 17
Number: 2
Pages: 139-159

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2018.94775.506

Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 2 (2020), 139-159
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.94775.506

Inequalities of Ando's Type for n-convex Functions

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Abstract. By utilizing different scalar equalities obtained via Hermite's interpolating polynomial, we will obtain lower and upper bounds for the difference in Ando's inequality and in the Edmundson-Lah-Ribarič inequality for solidarities that hold for a class of n-convex functions. As an application, main results are applied to some operator means and relative operator entropy.

1. Introduction

The theory for connections and means of pairs of positive operators has been developed by Kubo and Ando in [9]. A connection σ, as a binary operation on the set of positive definite operators, is characterized by the relation

$$A\sigma B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}, \tag{1.1}$$

where f is a positive operator monotone function on $(0,\infty)$ called the representing function for σ. The axiomatic properties of connections are as follows:

(i) $A \leq C$, $B \leq D$ implies $A\sigma B \leq C\sigma D$,
(ii) $C\,(A\sigma B)\,C \leq (CAC)\sigma(CBC)$,
(iii) $A_k \downarrow A$ and $B_k \downarrow B$ imply $A_k\sigma B_k \downarrow A\sigma B$.
A mean is a connection with normalization condition
(iv) (4) $I\sigma I = I$.

The basic examples of connections and their representing functions are:

2010 *Mathematics Subject Classification.* 47A63, 47A64, 26A51.
Key words and phrases. Solidarities, Ando's inequality, Edmundson-Lah-Ribarič inequality, n-convex functions, Operator means.
Received: 30 September 2018, Accepted: 28 November 2018.
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- The weighted arithmetic mean
$$A\nabla_\alpha B = (1-\alpha)A + \alpha B, \quad 0 \le \alpha \le 1,$$
with representing function $t \mapsto (1-\alpha) + \alpha t$.
- The weighted harmonic mean
$$A\,!_\alpha B = \left[(1-\alpha)A^{-1} + \alpha B^{-1}\right]^{-1}, \quad 0 \le \alpha \le 1,$$
with representing function $t \mapsto \frac{t}{(1-\alpha)t+\alpha}$.
- The weighted geometric mean
$$A\,\#_\alpha B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}, \quad 0 \le \alpha \le 1,$$
with representing function $t \mapsto t^\alpha$.
- The weighted power mean
$$A\,\#_{p,\alpha} B = A^{1/2}\left[(1-\alpha)I + \alpha\left(A^{-1/2}BA^{-1/2}\right)^p\right]^{1/p} A^{1/2},$$
such that $0 \le \alpha \le 1$ and $-1 \le p \le 1$, with representing function $t \mapsto [(1-\alpha) + \alpha t^p]^{1/p}$.

A binary operation σ on the set of positive definite operators is called solidarity if the representing function f in (1.1) is just operator monotone on $(0,\infty)$. The theory of solidarities has been developed in [6]. The relative operator entropy $S(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$ is an example of solidarity.

The following properties are proved in [6].

Theorem 1.1. *If σ is a solidarity, then*

(i) $(A+B)\sigma(C+D) \ge A\sigma C + B\sigma D$ *(subaditivity).*

(ii) $(\lambda A_1 + (1-\lambda)A_2)\,\sigma\,(\lambda B_1 + (1-\lambda)B_2) \ge \lambda\,(A_1\sigma B_1) + (1-\lambda)\,(A_2\sigma B_2)$, $0 \le \lambda \le 1$ *(joint concavity).*

A simple consequence is:

Corollary 1.2. *Let $p_i \ge 0$, $A_i, B_i > 0$, $i = 1,\dots,n$. Then*
$$\sum_{i=1}^n p_i A_i \sigma B_i \le \left(\sum_{i=1}^n p_i A_i\right)\sigma\left(\sum_{i=1}^n p_i B_i\right), \tag{1.2}$$
for any solidarity σ.

The following theorem is proved in [11], and it gives us the Edmundson-Lah-Ribarič inequality for the special type of solidarities that includes connections.

Theorem 1.3. *Let A_i, B_i be positive definite operators such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$, $p_i \ge 0$, and let Φ_i be positive linear* *maps, $i = 1, \ldots, n$. Suppose that σ is a solidarity generated by an operator monotone and operator concave function f. Then*

$$\sum_{i=1}^{n} p_i \Phi_i \left(A_i \sigma B_i\right) \geq \frac{M \sum_{i=1}^{n} p_i \Phi_i\left(A_i\right) - \sum_{i=1}^{n} p_i \Phi_i\left(B_i\right)}{M-m} f(m) + \frac{\sum_{i=1}^{n} p_i \Phi_i\left(B_i\right) - m \sum_{i=1}^{n} p_i \Phi_i\left(A_i\right)}{M-m} f(M). \tag{1.3}$$

More general property holds for connections (see [2]), and it is known as Ando's inequality.

Theorem 1.4. *If Φ is a positive linear map, then for any connection σ and for each $A, B > 0$,*

$$\Phi\left(A \sigma B\right) \leq \Phi(A) \sigma \Phi(B). \tag{1.4}$$

Remark 1.5. It is easy to deduce from Theorem 1.4 that inequality

$$\sum_{j=1}^{n} p_j \Phi_j (A_j \sigma B_j) \leq \left(\sum_{j=1}^{n} p_j \Phi_j (A_j) \right) \sigma \left(\sum_{j=1}^{n} p_j \Phi_j (B_j) \right), \tag{1.5}$$

also holds, where σ is a connection, A_j, B_j are positive definite operators, Φ_j are positive linear maps, $p_j \geq 0$, $j = 1, \ldots, n$.

We also need to mention Davis-Choi's inequality, which states that for an operator convex function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ an interval, Φ a normalized positive linear map, it holds

$$f\left(\Phi(A)\right) \leq \Phi\left(f(A)\right), \tag{1.6}$$

where A is a selfadjoint operator with $\mathrm{Sp}(A) \subseteq I$ (see [5], [3]).

2. Preliminaries

For some recent results on the converses of the Ando and Davis-Choi inequalities the reader is referred to [11], [12] and [15]. The methods used in those papers and related results can be found in the monographs [7] and [8]. Unlike the mentioned results, which require operator convexity of the involved functions, the main objective of this paper is to obtain inequalities of the Edmundson-Lah-Ribarič type that hold for n-convex functions, which will also give us inequalities of Ando's type for n-convex functions.

Definition of n-convex functions is characterized by n-th order divided differences. The n-th order divided difference of a function $f\colon [a, b] \to \mathbb{R}$ at mutually distinct points $t_0, t_1, ..., t_n \in [a, b]$ is defined recursively by

$$f[t_i] = f(t_i), \quad i = 0, \ldots, n,$$

$$f[t_0,\ldots,t_n]=\frac{f[t_1,\ldots,t_n]-f[t_0,\ldots,t_{n-1}]}{t_n-t_0}.$$

The value $f[t_0,...,t_n]$ is independent of the order of the points $t_0,...,t_n$.

Definition of divided differences can be extended to include the cases in which some or all the points coincide (see e.g. [1], [14]):

$$f[\underbrace{a,\ldots,a}_{n\text{ times}}]=\frac{1}{(n-1)!}f^{(n-1)}(a),\quad n\in\mathbb{N}.$$

A function $f\colon[a,b]\to\mathbb{R}$ is said to be n-convex ($n\geq 0$) if and only if for all choices of $(n+1)$ distinct points $t_0,t_1,\ldots,t_n\in[a,b]$, we have $f[t_0,\ldots,t_n]\geq 0$. For different notions of convexity and corresponding properties see for example [16].

Following representations of the left side in the scalar Edmundson-Lah-Ribarič inequality are obtained in [10] by using Hermite's interpolating polynomials in terms of divided differences. In these results, the usual notation is used:

$$\alpha_f=\frac{f(b)-f(a)}{b-a},\qquad \beta_f=\frac{bf(a)-af(b)}{b-a}.$$

Lemma 2.1. *Let a,b be real numbers such that $a<b$. For a function $f\in\mathcal{C}^n([a,b])$, $n\geq 3$ the following identities hold:*

(2.1)
$$f(t)-\alpha_f t-\beta_f=\sum_{k=2}^{n-1}f[a;\underbrace{b,\ldots,b}_{k\ times}](t-a)(t-b)^{k-1}+R_1(t)$$

$$f(t)-\alpha_f t-\beta_f=f[a,a;b](t-a)(t-b)$$

$$+\sum_{k=2}^{n-2}f[a,a;\underbrace{b,\ldots,b}_{k\ times}](t-a)^2(t-b)^{k-1}+R_2(t), \tag{2.2}$$

where

$$R_m(t)=(t-a)^m(t-b)^{n-m}f[t;\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,b,\ldots,b}_{(n-m)\ times}]. \tag{2.3}$$

In addition, if $n>m\geq 3$, then we have

$$f(t)-\alpha_f t-\beta_f=(t-a)\left(f[a,a]-f[a,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$$

$$+\sum_{k=1}^{n-m}f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}](t-a)^m(t-b)^{k-1}+R_m(t). \tag{2.4}$$

Lemma 2.2. *Let a, b be real numbers such that $a < b$. For a function $f \in \mathcal{C}^n([a,b])$, $n \geq 3$, the following identities hold:*

(2.5)

$$f(t) - \alpha_f t - \beta_f = \sum_{k=2}^{n-1} f[b; \underbrace{a, \ldots, a}_{k \text{ times}}](t-b)(t-a)^{k-1} + R_1^*(t)$$

$$f(t) - \alpha_f t - \beta_f = f[b, b; a](t-b)(t-a) + \sum_{k=2}^{n-2} f[b, b; \underbrace{a, \ldots, a}_{k \text{ times}}](t-b)^2(t-a)^{k-1} + R_2^*(t), \tag{2.6}$$

where

$$R_m^*(t) = f[t; \underbrace{b, \ldots, b}_{m \text{ times}}; \underbrace{a, a, \ldots, a}_{(n-m) \text{ times}}](t-b)^m (t-a)^{n-m}. \tag{2.7}$$

If $n > m \geq 3$, then additionally we have

$$f(t) - \alpha_f t - \beta_f = (b-t)\left(f[a,b] - f[b,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!}(t-b)^k + \sum_{k=1}^{n-m} f[\underbrace{b, \ldots, b}_{m \text{ times}}; \underbrace{a, \ldots, a}_{k \text{ times}}](t-b)^m (t-a)^{k-1} + R_m^*(t). \tag{2.8}$$

3. Inequalities of the Edmundson-Lah-Ribarič type

In this section, the difference generated by the Edmundson-Lah-Ribarič inequality is estimated from below and from above by Hermite's interpolating polynomials in terms of divided differences.

For the rest of the paper, let Φ_i be normalized positive linear maps and let A_i, B_i, $i = 1, \ldots, r$, be positive definite operators such that $aA_i \leq B_i \leq bA_i$ for some $0 < a < b < \infty$ and $p_i \geq 0$ such that $\sum_{i=1}^r p_i = 1$. Let σ be a solidarity generated by an operator monotone function $f \in \mathcal{C}^n([a,b])$. In order to simplify the obtained relations, we introduce the following notations:

$$\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) = \sum_{i=1}^r p_i \Phi_i(A_i);$$

$$\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}, \boldsymbol{B}; g(t)) = \sum_{i=1}^r p_i \Phi_i\left(A_i^{\frac{1}{2}} g\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}\right) A_i^{\frac{1}{2}}\right).$$

Our first result is a generalization of the Edmundosn-Lah-Ribarič inequality for solidarities that holds for the class of n-convex functions.

Theorem 3.1. *If the function f is n-convex and if $n > m \geq 3$ are of different parity, then we have*

$$\begin{aligned}(3.1)\quad &\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\ &\leq (f[a,a] - f[a,b]) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) \\ &\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right) \\ &\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right).\end{aligned}$$

Inequality (3.1) also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign in (3.1) is reversed.

Proof. Since $f \in \mathcal{C}^n([a,b])$, it is continuous and its n-th order divided difference $f_n(t) = f[t;\underbrace{a,\ldots,a}_{m\text{ times}};\ \underbrace{b,b,\ldots,b}_{(n-m)\text{ times}}]$ is also continuous, so consequently the function $R_m(t)$ defined in (2.3) is also continuous.

From $aA_i \leq B_i \leq bA_i$ easily follows $a\mathbf{1} \leq A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} \leq b\mathbf{1}$, $i = 1,\ldots,n$. Using the functional calculus and (2.4), we get

$$\begin{aligned}&f\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}\right) - \alpha_f A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - \beta_f\mathbf{1} \\ &= (f[a,a]-f[a,b])\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - a\mathbf{1}\right) + \sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - a\mathbf{1}\right)^k \\ &\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}]\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - a\mathbf{1}\right)^m\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - b\mathbf{1}\right)^{k-1} \\ &\quad + R_m\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}\right).\end{aligned}$$

After multiplying the obtained relation twice by $A_i^{\frac{1}{2}}$, acting by Φ_i, then finally multiplying by p_i and summing, it follows:

(3.2)

$$\begin{aligned}&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\ &= (f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) + \sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right) \\ &\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t)).\end{aligned}$$

We want to remove the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))$ from the equality above, so we need to check its positivity (negativity). Due to the monotonicity property, it is actually enough to study positivity and negativity of the function:

$$R_m(t)=(t-a)^m\,(t-b)^{n-m}\,f[t;\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,b,\ldots,b}_{(n-m)\text{ times}}].$$

Since $a\le t\le b$, we have $(t-a)^m\ge 0$ for any choice of m. For the same reason we have $(t-b)\le 0$. Trivially it follows that $(t-b)^{n-m}\le 0$ when n and m are of different parity, and $(t-b)^{n-m}\ge 0$ when n and m are of equal parity. When the function f is n-convex, then its nth order divided differences are nonnegative, and when it is n-concave, then those divided differences are less or equal to zero.

Now, we see that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))\le 0$ when the function f is n-convex and n and m are of different parity or when f is n-concave and n and m are of equal parity, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))\ge 0$ in the remaining cases, so inequality (3.1) easily follows from (3.2). □

Remark 3.2. Sum of positive definite operators is again positive definite, and for positive definite operators A and B, operator $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is positive definite, so in the proof of Theorem 3.1, in the discussion about the positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))$ it was enough to discuss the positivity and negativity of function $R_m(t)$ because for a continuous and positive function R_m and a selfadjoint operator A, the operator $R_m(A)$ is positive definite.

Following result provides us with a similar generalization of the Edmundson-Lah-Ribarič inequality, and it is obtained from Lemma 2.2.

Theorem 3.3. *If the function f is n-convex and if $n>m$, where $m\ge 3$ is an odd number, then*

$$\begin{aligned}(3.3)\quad &\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\ &\le (f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\\ &\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)\\ &\quad+\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,\ldots,a}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right).\end{aligned}$$

Inequality (3.3) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when *the function f is n-concave and m is odd, the inequality sign in (3.3) is reversed.*

Proof. As in the proof of Theorem 3.1, since all the involved functions are continuous, we can replace t with operator $A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}$ in (2.8), multiply the obtained relation twice by $A_i^{\frac{1}{2}}$, act by Φ_i, then finally multiply it by p_i and sum it and get

$$\begin{aligned}(3.4)\quad \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) &= (f[a,b] - f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \\ &+ \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right) \\ &+ \sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,\dots,a}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right) \\ &+ \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_m^*(t)\right).\end{aligned}$$

As before, in order to remove the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t))$, we need to know when it is positive, and when it is negative. Due to the monotonicity property, it is enough to check positivity and negativity of the function:

$$R_m^*(t) = (t-b)^m (t-a)^{n-m} f[t;\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,a,\dots,a}_{(n-m)\text{ times}}].$$

Since $t \in [a,b]$, we have $(t-a)^{n-m} \geq 0$ for every t and any choice of m. For the same reason we have $(t-b) \leq 0$. Trivially it follows that $(t-b)^m \leq 0$ when m is odd, and $(t-b)^m \geq 0$ when m is even. If the function f is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function f is n-concave, then its n-th order divided differences are less or equal to zero.

Now, it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t)) \leq 0$ when the function f is n-convex and m is odd or f is n-concave and m is even. In the remaining cases the inequality sign is reversed, (3.3) easily follows from (3.4). □

As a direct consequence of Theorem 3.1 and Theorem 3.3, we get lower and upper bounds for the difference in the Edmundson-Lah-Ribarič inequality that hold for the class of n-convex functions.

Corollary 3.4. *If the function f is n-convex, where n is an odd number, and if $m \geq 3$ is odd, then*

$$(3.5)\quad (f[a,a] - f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)$$

$$
\begin{aligned}
&+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right)\\
&+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m\,(t-b)^{k-1}\right)\\
\le\ &\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)\\
\le\ &\left(f[a,b]-f[b,b]\right)\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)\right)\\
&+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)\\
&+\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ times};\underbrace{a,\ldots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right).
\end{aligned}
$$

Inequality (3.5) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (3.5) are reversed.

The following result also provides us with a lower and upper bound for the difference in the Edmundson-Lah-Ribarič inequality, and it is obtained from Lemma 2.1.

Theorem 3.5. *If the function f is n-convex and if $n \geq 3$ is odd, then*

$$
\begin{aligned}
(3.6)\qquad &\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right)\\
&\le\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)\\
&\le f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)\right)\\
&\quad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right),
\end{aligned}
$$

where $x \in H$ is a unit vector. Inequalities (3.6) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (3.6) are reversed.

Proof. Again, using the functional calculus, we can replace t with $A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}$ in (2.1) and (2.2), multiply obtained relations twice by $A_i^{\frac{1}{2}}$, act by Φ_i, then finally multiply them by p_i and sum them. In that way we get (3.7)

$$\begin{aligned}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right),\\ =\sum_{k=2}^{n-1}f[a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right)+\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1(t)\right)\end{aligned}$$

and

$$\begin{aligned}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)&-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)\\ &=f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)\right)\\ &+\sum_{k=2}^{n-2}f[a,a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right)\\ &+\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2(t)\right).\end{aligned}\tag{3.8}$$

We have already discussed positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_m(t)\right)$ in the proof of Theorem 3.1. For $m=1$ it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1(t)\right)\geq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n is even, and when the function f is n-concave and n is odd, or when f is n-convex and n even the inequality sign is reversed, so the relation (3.7) gives us

$$\begin{aligned}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)&-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)\\ &\geq\sum_{k=2}^{n-1}f[a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right),\end{aligned}$$

for $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1(t)\right)\geq 0$, and in case $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1(t)\right)\leq 0$ the inequality sign is reversed.

In the same way, for $m=2$ it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2(t)\right)\leq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n is even, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2(t)\right)\geq 0$ in the remaining cases.

The relation (3.8) for $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2(t)\right)\leq 0$ gives us

$$\begin{aligned}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)&-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right)\\ &\leq f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)\right)\\ &+\sum_{k=2}^{n-2}f[a,a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right),\end{aligned}$$

and when $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2(t)\right)\geq 0$ the inequality sign is reversed.

When we combine the two inequalities obtained above, we get exactly (3.6). ☐

By utilizing Lemma 2.2, we can get similar lower and upper bounds for the difference in the Edmundson-Lah-Ribarič operator inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 3.6. *If the function f is n-convex, $n \geq 3$, then*

$$\begin{aligned} &f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)\right) \\ &\qquad+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right) \\ &\quad\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right) \\ &\quad\leq \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right). \end{aligned} \tag{3.9}$$

If the function f is n-concave, the inequality signs in (3.9) are reversed.

Proof. This proof follows the lines of the proof of Theorem 3.5. We start with replacing t with operator $A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}$ in (2.5) and (2.6) respectively, and after multiplying obtained relations twice by $A_i^{\frac{1}{2}}$, acting by Φ_i, then finally multiplying them by p_i and summing them we get

$$\begin{aligned} &\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right) \\ &=\sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right)+\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1^*(t)\right), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\sigma\boldsymbol{B}\right)&-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{B}\right)-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}\right) \\ &=f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)\right) \\ &\quad+\sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right) \\ &\quad+\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_2^*(t)\right). \end{aligned} \tag{3.11}$$

Now, we return to the discussion about positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_m^*(t)\right)$ from the proof of Theorem 3.3. For $m=1$ we have $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};R_1^*(t)\right) \geq 0$ when the function f is n-concave, and

$\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}, \boldsymbol{B}; R_1^*(t)) \leq 0$ when the function f is n-convex, so the relation (3.10) for a n-convex function f gives us

$$\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \leq \sum_{k=2}^{n-1} f[b; \underbrace{a, \ldots, a}_{k \text{ times}}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}, \boldsymbol{B}; (t-b)(t-a)^{k-1}\right),$$

and if the function f is n-concave, the inequality sign is reversed.

Similarly, for $m = 2$ we have $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}, \boldsymbol{B}; R_2^*(t)) \geq 0$ when the function f is n-convex, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}, \boldsymbol{B}; R_2^*(t)) \leq 0$ when the function f is n-concave. In this case the identity (3.11) for a n-convex function f gives us

$$\begin{aligned} \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) &- \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\ &\geq f[b, b; a] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}, \boldsymbol{B}; (t-b)(t-a)) \\ &\quad + \sum_{k=2}^{n-2} f[b, b; \underbrace{a, \ldots, a}_{k \text{ times}}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}, \boldsymbol{B}; (t-b)^2(t-a)^{k-1}\right), \end{aligned}$$

and if the function f is n-concave, the inequality sign is reversed.

When we combine the two results from above, we get exactly (3.9). □

4. Jensen-Type Inequalities

The aim of this section is to utilize the results from the previous section, as well as Lemma 2.1 and Lemma 2.2, in order to obtain some Jensen-type inequalities that hold for the class of n-convex functions. In that way, we will obtain lower and upper bounds for the difference generated by the Jensen inequality for solidarities.

Again, let Φ_i be normalized positive linear maps and let A_i, B_i, $i = 1, \ldots, r$, be positive definite operators such that $aA_i \leq B_i \leq bA_i$ for some $0 < a < b < \infty$ and $p_i \geq 0$ such that $\sum_{i=1}^r p_i = 1$. Let σ be a solidarity generated by an operator monotone function $f \in \mathcal{C}^n([a, b])$.

Our first result is a consequence of Corollary 3.4 and Lemma 2.1 and 2.2.

Theorem 4.1. *Let n be an odd number. If the function f is n-convex, $n > m$, and if $m \geq 3$ is odd, then*

(4.1)

$$(f[a,a] - f[b,b]) \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) + (b(f[b,b] - f[a,b]) - a(f[a,a] - f[a,b])) \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})$$
$$+ \sum_{k=2}^{m-1} \left[\frac{f^{(k)}(a)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A}, \boldsymbol{B}; (t-a)^k\right) - \frac{f^{(k)}(b)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \right.$$

$$
\begin{aligned}
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - b\mathbf{1} \right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \Bigg] \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times}; \underbrace{b,\ldots,b}_{k\ times}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}} \left(\boldsymbol{A}, \boldsymbol{B}; (t-a)^{m} (t-b)^{k-1} \right) \\
&\quad - \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ times}; \underbrace{a,\ldots,a}_{k\ times}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - b\mathbf{1} \right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - a\mathbf{1} \right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \\
&\le \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\, \sigma \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \\
&\le \left(f[b,b] - f[a,a] \right) \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \\
&\quad + \left(b(f[a,b] - f[b,b]) - a(f[a,b] - f[a,a]) \right) \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\quad + \sum_{k=2}^{m-1} \Bigg[\frac{f^{(k)}(b)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}} \left(\boldsymbol{A}, \boldsymbol{B}; (t-b)^{k} \right) - \frac{f^{(k)}(a)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - a\mathbf{1} \right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \Bigg] \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ times}; \underbrace{a,\ldots,a}_{k\ times}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}} \left(\boldsymbol{A}, \boldsymbol{B}; (t-b)^{m} (t-a)^{k-1} \right). \\
&\quad - \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times}; \underbrace{b,\ldots,b}_{k\ times}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - a\mathbf{1} \right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{-\frac{1}{2}} - b\mathbf{1} \right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \right)^{\frac{1}{2}}.
\end{aligned}
$$

Inequalities (4.1) also hold when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (4.1) are reversed.

Proof. From $aA_j \le B_j \le bA_j$ it follows that

$$a\mathbf{1} \le \left(\sum_{j=1}^{r} p_j \Phi_j(A_j)\right)^{-\frac{1}{2}} \left(\sum_{j=1}^{r} p_j \Phi_j(B_j)\right) \left(\sum_{j=1}^{r} p_j \Phi_j(A_j)\right)^{-\frac{1}{2}} \le b\mathbf{1},$$

so using functional calculus and (2.4), and then multiplying twice by $\left(\sum_{i=1}^{r} p_i \Phi_i(A_1)\right)^{\frac{1}{2}}$ we get the following relation:

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= (f[a,a] - f[a,b]) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} R_m\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.
\end{aligned}$$

We want to remove the last term from the equality above, so we need to check its positivity (negativity). Again, due to the monotonicity property and Remark 3.2, it is actually enough to study positivity and negativity of the function $R_m(t)$, and we already have that discussion in the proof of Theorem 3.1, so for n-convex function f and n and $m \ge 3$ of different parity, or n-concave function f and n and $m \ge 3$ of the same parity, from the previous equality it follows

(4.2)

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\le (f[a,a] - f[a,b]) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},
\end{aligned}$$

and for n-convex function f and n and $m \ge 3$ of the same parity, or n-concave function f and n and $m \ge 3$ of different parity, the inequality sign is reversed.

In the same way, we can replace t with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}$ in (2.8) and then multiplying twice by $\left(\sum_{i=1}^{r} p_i \Phi_i(A_1)\right)^{\frac{1}{2}}$ we get

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\, \sigma \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= (f[a,b] - f[b,b]) \left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}}; \underbrace{a,\ldots,a}_{k \text{ times}}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} R_m^{*} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.
\end{aligned}
$$

To remove the last term from the previous equality, we need to study its positivity and negativity. For the same reasons as before, it is enough to check positivity and negativity of the function R_m^{*}, and we have that discussion in the proof of Theorem 3.3. The equality above now turns into

(4.3)

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\, \sigma \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\leq (f[a,b] - f[b,b]) \left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}}; \underbrace{a,\ldots,a}_{k \text{ times}}] \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - b\mathbf{1}\right)^{m} \\
&\quad \times \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},
\end{aligned}
$$

for n-convex function f and an odd number $m \geq 3$ or n-concave function f and an even number $m \geq 3$. If f is n-convex and m is even, or if f is n-concave and m is odd, the inequality is reversed.

By combining inequalities (4.2) and (4.3) we get that

(4.4)

$$
\begin{aligned}
&(f[a,a] - f[a,b]) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}} \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1}\right)^{k} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{m}\\
&\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
\le\ &\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
\le\ &(f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\\
&+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&+\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,\ldots,a}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{m}\\
&\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},
\end{aligned}
$$

holds if n is odd and f is n-convex and m is odd, or f is n-concave and m is even. If f is n-convex and m is even, or f is n-concave and m is odd, then the inequality signs are reversed.

When we multiply series of inequalities (4.4) by -1 and add to (3.5), we get exactly (4.1), and the proof is complete. □

Next result also provides us with an estimate from below and from above for the difference generated by Ando's inequality, and it is obtained from Theorem 3.5 and Lemma 2.1.

Theorem 4.2. *If the function f is n-convex and if $n \geq 3$ is odd, then*

(4.5)
$$
\begin{aligned}
&\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right)\\
&\quad-f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\quad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad-\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{2}\\
&\quad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\le\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\\
&\le f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b))\\
&\quad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^{2}(t-b)^{k-1}\right)
\end{aligned}
$$

$$-\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.$$

Inequalities (4.5) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (4.5) are reversed.

Proof. By following a similar procedure as in the proof of the previous theorem, we start by replacing t with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}$ in relations (2.1) and (2.2) from Lemma 2.1, and then multiplying them twice with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$. We get

$$\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})$$
$$=\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}R_1\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},$$

and

$$\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})$$
$$=f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^2$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}R_2\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},$$

respectively. After discussing the positivity an negativity of the last terms in the equalities from above and removing them in the same way as in the proof Theorem 4.1, we get a series of inequalities

(4.6)
$$\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)$$

$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})$$
$$\leq f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\sum_{k=2}^{n-2}f[a,a;\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{2}$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}},$$

that holds when n is odd and f is n-convex, or when n is even and f is n-concave. If n is odd and f is n-concave, or if n is even and f is n-convex, then the inequality signs in (4.6) are reversed.

Inequalities (4.5) are obtained after multiplying (4.6) by -1 and adding it to (3.6). □

In the analogue way as described in the proof of the previous theorem, but this time utilizing Lemma 2.2 and Theorem 3.6, we can get similar lower and upper bounds for the difference generated by Ando's inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 4.3. *Let $A \in \mathcal{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathcal{C}^n([a,b])$. If the function f is n-convex, $n \geq 3$, then*

(4.7)

$$f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a))$$
$$-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)$$
$$\times\sum_{k=2}^{n-1}f[b;\underbrace{a,\dots,a}_{k\ times}]\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\sum_{k=2}^{n-2}f[b,b;\underbrace{a,\dots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right)$$
$$\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})$$
$$\leq f[b,b;a]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(b\mathbf{1}-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)(\langle Ax,x\rangle-a\mathbf{1})$$
$$+\sum_{k=2}^{n-1}f[b;\underbrace{a,\dots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right)$$
$$-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{2}$$

$$\times \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}] \left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}} - a\mathbf{1} \right)^{k-1} \left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.$$

If the function f is n-concave, the inequality signs in (4.7) are reversed.

5. Applications

As applications of the results from Section 3 and Section 4, we give reverses of this type for basic examples of some operator means and relative operator entropy. For some different, recent results on operator means and relative operator entropy, the reader is reffered to [4] and [13].

Let A_i, B_i, $i = 1, \ldots, r$, be positive definite operators such that $aA_i \leq B_i \leq bA_i$ for some $0 < a < b < \infty$ and $p_i \geq 0$ such that $\sum_{i=1}^{r} p_i = 1$.

Some examples of connections and solidarities and their representing functions to which our results are applicable are as follows.

- The weighted harmonic mean
$$A\,!_\alpha B = \left[(1-\alpha)A^{-1} + \alpha B^{-1}\right]^{-1}, \quad 0 \leq \alpha \leq 1,$$
has representing function $f(t) = \frac{t}{(1-\alpha)t+\alpha}$. We can calculate that for $n \in \mathbb{N}$
$$f^{(n)}(t) = \alpha(-1)^{n-1} n!(1-\alpha)^{n-1}\left((1-\alpha)t+\alpha\right)^{-n-1}.$$
Since $\alpha \in [0,1]$, it is easy to see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.
- The weighted geometric mean
$$A\,\#_\alpha B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\alpha} A^{1/2}, \quad 0 \leq \alpha \leq 1,$$
has representing function $f(t) = t^\alpha$. After an easy calculation we get that
$$f^{(n)}(t) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)t^{\alpha-n}.$$
Because $\alpha \in [0,1]$, we see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.
- The relative operator entropy
$$S(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$
has representing function $f(t) = \log t$. After an easy calculation we get that
$$f^{(n)}(t) = (-1)^{n-1}(n-1)!t^{-n}.$$

We immediately see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.

Acknowledgment. The publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008).

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