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Vector Optimization Problems and Generalized Vector Variational-Like Inequalities

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Abstract. In this paper, some properties of pseudoinvex functions, defined by means of limiting subdifferential, are discussed. Furthermore, the Minty vector variational-like inequality, the Stampacchia vector variational-like inequality and the weak formulations of these two inequalities defined by means of limiting subdifferential are studied. Moreover, some relationships between the vector variational-like inequalities and vector optimization problems are established.

1. Introduction

Two types of vector variational inequalities (VVI), Stampacchia-type VVI and Minty-type VVI, have been studied in the literatures. Giannessi [8] extended the classical Stampacchia variational inequality for vector-valued functions, called Stampacchia vector variational inequality (SVVI), which has been applied to various problems. In particular, it has been used as a tool to solve vector optimization problems (VOP) [1, 2, 5, 7, 21]. Giannessi [7] provided a necessary and sufficient conditions for a point to be a solution of VOP in terms of Minty vector variational inequality (MVVI) for differentiable and convex functions. Yang et al. [20] extended the results of Giannessi [7] for differentiable but pseudoconvex functions. Recently, Yang and Yang [21] have extended the results of Giannessi [7] and Yang et al. [20] for differentiable but pseudoinvex functions. Yang and Yang [21] also proved that SVVI and MVVI are equivalent under continuity assumption. The vector variational-like

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inequality (VVLI) is a generalization of a VVI in which the term $y-x$ is replaced by a bifunction $\eta(y,x)$. Generalized Minty vector variational-like inequality problems (GMVVLI) have been studied by Al-Homidan and Ansari [1]. They also generalized Stampacchia vector variational-like inequality problems (GSVVLI) defined by means of Clarke's subdifferential for nonsmooth VOP under nonsmooth invexity along with studying the relationships between GMVVLI and SVVLI. Furthermore, they considered the weak formulations of GMVVLI and GSVVLI and provided some relations between the solutions of these problems and a weak efficient solution of VOP. Chen and Huang [5] generalized some of the main results of Al-Homidan and Ansari [1] and Yang and Yang [19]. They introduced MVVLI, SVVLI and the weak formulations of these two inequalities defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. They also established some relations between the vector variational-like inequalities and VOP. The results of [1, 5] for generalized pseudoinvex functions involving limiting subdifferentials under nonsmooth invexity have been extended in [2]. They presented some properties of pseudoinvex functions defined by means of limiting subdifferentials along with providing a necessary and sufficient conditions for a solution to be efficient one of a VOP in terms of GMVVLI. They also presented some relationships between the solutions of GMVVLI, GSVVLI and VOP under pseudoinvexity condition.

In this paper, we rectify some results in [2], by using the correct assumptions. The paper is organized as follows. In Section 2, some basic definitions and preliminary results are given. Some relations between various kinds of invexities are established in Section 3. In Section **??**, some relationships between the solutions of GMVVLI, GSVVLI, efficient solution of VOP and weak formulations of these problems are established.

2. Preliminaries

Let X be a Banach space being an Asplund space; that is, a Banach space that its separable subspaces have separable duals and the topological dual of X denoted by X^*. The duality pairing between X and X^*, the line segment joining x and y and $[x,y]\setminus\{x,y\}$ are denoted by $\langle .,.\rangle$, $[x,y]$ and (x,y), respectively.

Let $F: X \rightrightarrows X^*$ be a set-valued mapping/multifunction. Then the sequential Painlevé-Kuratowski upper/outer limit of F at $\bar{x}$ is defined by

$$\limsup_{x\to\bar{x}} F(x) \coloneqq \left\{ x^* \in X^* | \; \exists\; x_k \to \bar{x},\; x_k^* \overset{w^*}{\to} x^* \text{s.t} x_k^* \in F(x_k),\; k \in \mathbb{N} \right\},$$

where w^* denotes the weak* topology on X^*.

The epigraph of a function $f : X \to \mathbb{R}$ is defined by

$$\text{epi}f = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}.$$

Leting $f : X \to Y$ be a mapping between Banach spaces, recall that f is Fréchet differentiable at $\bar{x}$ if there is a linear continuous operator $\nabla f(\bar{x}) : X \to Y$ such that

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

Let $\varepsilon \geq 0$, then the set of ε-normals to a set Ω at x is defined by

$$\hat{N}_\varepsilon(x; \Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon \right\}. \tag{2.1}$$

For $\varepsilon = 0$, we write $\hat{N}(x; \Omega) := \hat{N}_0(x; \Omega)$ in (2.1) and call it the prenormal cone or the Fréchet normal cone to Ω at x. If $x \notin \Omega$, we set $\hat{N}_\varepsilon(x; \Omega) := \emptyset$ for all $\varepsilon \geq 0$.

Recall also that the notation $u \xrightarrow{\Omega} x$ for $\Omega \subset X$ means $u \to x$ with $u \in \Omega$. The (basic, limiting, Mordukhovich) normal cone $N(\bar{x}, \Omega)$ is obtained from $\hat{N}_\varepsilon(x; \Omega)$ by taking the sequential Painlevé-Kuratowski upper/outer limit in the weak* topology on X^* as

$$N(\bar{x}; \Omega) := \limsup_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \hat{N}_\varepsilon(x; \Omega),$$

and we let $N(\bar{x}; \Omega) := \emptyset$ for $\bar{x} \notin \Omega$.

Let $f : X \to \bar{\mathbb{R}}$ be a function and $\bar{x} \in X$ so that $|f(\bar{x})| < \infty$. Then the set

$$\partial_L f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N((\bar{x}, f(\bar{x})), \text{epi}f)\},$$

is called the limiting subdifferential of f at $\bar{x}$ and its elements are called limiting subdifferentials of f at this point. We put $\partial_L f(\bar{x}) = \emptyset$ if $|f(\bar{x})| = \infty$.

Theorem 2.1 ([13]). *Let X be an Asplund space and $\varphi : X \to \mathbb{R}$ be locally Lipschitz around $\bar{x}$. Then $\partial_L \varphi(\bar{x}) \neq \emptyset$.*

Theorem 2.2 ([13]). *(mean value inequality for Lipschitzian functions). Let X be an Asplund space and φ be locally Lipschitz on an open set containing $[x, y]$. Then there exists $c \in [x, y)$ such that for all $x^* \in \partial_L \varphi(c)$, we have*

$$\varphi(y) - \varphi(x) \leq \langle x^*, y - x\rangle.$$

Theorem 2.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $\bar{x}$, then $\partial_L f(\bar{x})$ is closed. In fact, if $x_n \to \bar{x}$, $x_n^* \in \partial_L f(x_n)$ and $x_n^* \to x^*$, then $x^* \in \partial_L f(\bar{x})$.*

Proof. Since $x_n^* \in \partial_L f(x_n)$ and $\partial_L f(\bar{x}) = \limsup_{x \xrightarrow{\varphi} \bar{x}} \hat{\partial} f(x)$ ([13, Theorem 1.89]), for each n, we can find sequences $x_{n,k} \to x_n$ and $x_{n,k}^* \in \hat{\partial} f(x_{n,k})$ with $x_{n,k}^* \xrightarrow{w^*} x_n^*$ as $k \to \infty$. We pick the sequences $x_{k,k}$, which converges to $\bar{x}$ and $x_{k,k}^* \in \hat{\partial} f(x_{k,k})$ with $x_{k,k}^* \xrightarrow{w^*} x^*$ (this is possible because the weak-star and norm topologies are agree in finite dimensions). This means that $x^* \in \partial_L f(\bar{x})$. ☐

Definition 2.4. (monotonicity) A mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \text{ whenever } v_1 \in T(u_1),\ v_2 \in T(u_2),$$

and strictly monotone if this inequality is strict for $u_1 \neq u_2$.

3. Generalized Pseudoinvex Functions

Let $\eta : K \times K \to X$ be a vector-valued mapping and $f : K \to \mathbb{R}$ be a function where $K \subseteq X$ is a nonempty set.

Definition 3.1 ([3])**.** The mapping $\eta : K \times K \to X$ is said to be skew if

$$\eta(y, x) + \eta(x, y) = 0,\ x, y \in K.$$

Definition 3.2 ([3, 10])**.** Let x be an arbitrary point of K. The set K is said to be invex at x with respect to η if for all $y \in K$ and $t \in [0, 1]$ we have

$$x + t\eta(y, x) \in K.$$

If K is invex at every point $x \in K$ with respect to η, then K is said to be invex with respect to η.

Notice that by taking $\eta(y, x) = y - x$, each convex set is invex but the converse is not true in general.

We sometimes need some more assumptions on η in the sequal.

Condition 3.3 ([20])**.** Let $K \subseteq X$ be an invex set with respect to η and $f : K \to \mathbb{R}$ be a function. Then, for all $x, y \in K$ we have

$$f(x + \eta(y, x)) \leq f(y).$$

Condition 3.4 ([12])**.** Let $K \subseteq X$ be an invex set with respect to $\eta : K \times K \to X$. Then, for all $x, y \in K$ and all $t \in [0, 1]$,

(a) $\eta(x, x + t\eta(y, x)) = -t\eta(y, x)$,

(b) $\eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x)$.

Remark 3.5. It is shown in [14] that if $\eta : K \times K \to X$ satisfies Condition 3.4, then

$$\eta(x + t\eta(y, x), x) = t\eta(y, x),\ \text{for all } t \in [0, 1] \text{ and all } x, y \in K.$$

We assume that $K \subseteq X$ is an invex set with respect to η in this section unless otherwise specified.

Definition 3.6 ([3, 10])**.** A function $f : K \to \mathbb{R}$ is said to be

(i) pre-quasiinvex with respect to η on K if for all $x, y \in K$ and all $t \in [0,1]$ we have

$$f(y) \leq f(x) \quad \Rightarrow \quad f(x + t\eta(y,x)) \leq f(x).$$

(ii) quasiinvex with respect to η on K if for all $x, y \in K$ and all $\zeta \in \partial_L f(x)$ we have

$$f(y) \leq f(x) \quad \Rightarrow \quad \langle \zeta, \eta(y,x) \rangle \leq 0.$$

(iii) generalized pseudoinvex with respect to η on K [2] if for all $x, y \in K$ and for all $\zeta \in \partial_L f(x)$ we have

$$\langle \zeta, \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad f(x) \leq f(y).$$

(iv) generalized strict pseudoinvex with respect to η [2] if for all $x, y \in K$ and for all $\zeta \in \partial_L f(x)$ we have

$$\langle \zeta, \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad f(x) < f(y).$$

Ansari and Rezaei in [2, Theorem 2.5] showed that if f has a local maximum at $\bar{x}$, then $0 \in \partial_L f(\bar{x})$ which is incorrect. The simplest example can be given as $f(x) = -|x|$ at $\bar{x} = 0$, where

$$\partial_L f(0) = \{-1, 1\},$$

and $\bar{x} = 0$ is a maximum point of f but $0 \notin \partial_L f(0)$. What follows is a correction of this result.

Proposition 3.7. *Let $K \subseteq X = \mathbb{R}^n$ be a nonempty open set and $\eta : K \times K \to \mathbb{R}^n$ satisfy Condition 3.3. If $f : K \to \mathbb{R}$ is locally Lipschitz and generalized pseudoinvex with respect to η, then f is pre-quasiinvex with respect to the same η.*

Proof. In contrast, suppose that f is not pre-quasiinvex with respect to η. Then, there exist $x, y \in K$ and $\hat{t} \in [0,1]$, such that $f(y) \leq f(x)$ while $f(x + \hat{t}\eta(y,x)) > f(x)$. For $\hat{x} = x + \hat{t}\eta(y,x)$, we get

$$f(\hat{x}) > f(x) \geq f(y). \tag{3.1}$$

Now let $\varphi(t) = f(x + t\eta(y,x))$, $t \in [0,1]$. Since f is continuous, then φ attains its maximum over $[0,1]$. Since $\varphi(0) = f(x) < f(\hat{x})$ from (3.1), then $t = 0$ is not a maximizer and thus $f(x + \eta(y,x)) = \varphi(1) \leq f(y) < f(\hat{x})$ by Condition 3.3 and (3.1), which implies that $t = 1$ is not a maximizer, as well. Hence, there exists a $\bar{t} \in (0,1)$ such that $f(\bar{x}) = \max_{t \in [0,1]} f(x + t\eta(y,x))$, where $\bar{x} = x + \bar{t}\eta(y,x)$.

On the other hand, since K is an open set and f is locally Lipschitz, there exists $0 < \bar{\delta} < 1$ such that $B_{\bar{\delta}}(\bar{x}) \subset K$ and f is Lipschitz on an open set containing $[\bar{x} - \delta\eta(y, \bar{x}), \bar{x}] \subset B_{\bar{\delta}}(\bar{x}) \subset K$ for all $\delta \in (0, \bar{\delta}]$. Now by Theorem 2.2, there exists $c_\delta \in [\bar{x} - \delta\eta(y, \bar{x}), \bar{x})$ such that

$$f(\bar{x}) - f(\bar{x} - \delta\eta(y, \bar{x})) \leq \delta\langle x^*_\delta, \eta(y, \bar{x})\rangle,$$

for some $x^*_\delta \in \partial_L f(c_\delta)$. It follows from [13, Proposition 1.85] that the sequence $\{x^*_\delta\}$ is bounded. Hence, by Proposition 2.3, $\{x^*_\delta\}$ has a subsequence converging to some $x^* \in \partial_L f(\bar{x})$ when $\delta \downarrow 0$, because $\lim_{\delta\downarrow 0} c_\delta = \bar{x}$. This leads to

$$0 \leq \lim_{\delta\downarrow 0} \frac{f(\bar{x}) - f(\bar{x} - \delta\eta(y, \bar{x}))}{\delta} \leq \langle x^*, \eta(y, \bar{x})\rangle.$$

Now, by the generalized pseudo-invexity assumption, we obtain $f(y) \geq f(\bar{x})$. So, from (3.1), we have $f(\hat{x}) > f(\bar{x})$. Since $\bar{t}$ is a maximizer of φ on $[0, 1]$, this is a contradiction. □

Now, we replace locally Lipschitz and generalized pseudoinvexity assumptions on f by monotonicity assumption in Proposition 3.7 for $\eta : K \times K \to \mathbb{R}$ in the following proposition.

Proposition 3.8. *Let $\eta : K \times K \to \mathbb{R}$ satisfy Condition 3.3 and $f : K \to \mathbb{R}$ be a monotone function. Then, f is pre-quasiinvex with respect to the same η.*

Proof. Assume that f is not pre-quasiinvex with respect to η. Then, there exist $x, y \in K$ and $t \in [0, 1]$, such that $f(y) \leq f(x)$ and

$$f(x + t\eta(y, x)) > f(x). \tag{3.2}$$

From (3.2) and Condition 3.3, we have

$$f(x + t\eta(y, x)) > f(x) \geq f(y) \geq f(x + \eta(y, x)).$$

Since f is monotone on K, we have

$$\langle f(x + t\eta(y, x)) - f(x), t\eta(y, x)\rangle \geq 0,$$

and thus $\eta(y, x) \geq 0$. Also, we have

$$\langle f(x + t\eta(y, x)) - f(x + \eta(y, x)), (t - 1)\eta(y, x)\rangle \geq 0,$$

resulting to $\eta(y, x) \leq 0$. Hence, $\eta(y, x) = 0$, which contradicts (3.2) and thus f is pre-quasiinvex. □

Definition 3.9 ([15])**.** A set-valued mapping $F : K \to 2^{X^*}$ is said to be

(i) invariant monotone with respect to η on K if $\forall x, y \in K$, $\forall \zeta \in F(x)$, $\forall \xi \in F(y)$, we have

$$\langle \zeta, \eta(y, x)\rangle + \langle \xi, \eta(x, y)\rangle \leq 0.$$

(ii) invariant pseudomonotone with respect to η if $\forall x, y \in K,\ \forall \zeta \in F(x),\ \forall \xi \in F(y)$, we have

$$\langle \zeta, \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad \langle \xi, \eta(x,y) \rangle \leq 0.$$

(iii) invariant strictly-pseudomonotone with respect to η if $\forall x, y \in K,\ \forall \zeta \in F(x),\ \forall \xi \in F(y)$, we have

$$\langle \zeta, \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad \langle \xi, \eta(x,y) \rangle < 0.$$

Theorem 3.10 ([18, Theorems 3.4 and 3.5])**.** *Let $\eta : K \times K \to \mathbb{R}^n$ satisfy Condition 3.4. If $f : K \to \mathbb{R}$ is locally Lipschitz and generalized pseudoinvex with respect to η, then it is both strictly pre-quasiinvex and pre-quasiinvex with respect to the same η.*

Theorem 3.11 ([16, Theorem 3.2])**.** *Let $f : K \to \mathbb{R}$ be locally Lipschitz pre-quasiinvex function and η be continuous with respect to the second argument satisfying Condition 3.4, then f is quasiinvex.*

Proposition 3.12. *Let $K \subseteq X = \mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous in terms of the second argument. Furthermore, let Condition 3.4 hold and $f : K \to \mathbb{R}$ be locally Lipschitz. If f is generalized pseudoinvex with respect to η, then $\partial_L f$ is invariant pseudomonotone with respect to the same η.*

Proof. Assume the contrary, which is: there exist $x, y \in K$, $\zeta \in \partial_L f(x)$ and $\xi \in \partial_L f(y)$ such that

$$\langle \zeta, \eta(y,x) \rangle \geq 0, \qquad \langle \xi, \eta(x,y) \rangle > 0. \tag{3.3}$$

Since f is generalized pseudoinvex with respect to η, from the first inequality in (3.3), we will have $f(y) \geq f(x)$. On the other hand, f is pre-quasiinvex with respect to η, by Theorem 3.10. From Theorem 3.11 and the second inequality in (3.3), we get $f(x) > f(y)$, which is a contradiction. Therefore, $\partial_L f(x)$ is invariant pseudomonotone with respect to η. □

Remark 3.13. Proposition 3.12 has been established in [2, Proposition 3.9]. But there is a gap in the proof, because the authors applied

$$\langle \xi, \eta(x,y) \rangle > 0 \quad \Rightarrow \quad f(x) \geq f(y),$$

which does not contradict $f(y) \geq f(x)$.

The following result is about generalized strict pseudoinvex functions. The proof is straightforward.

Proposition 3.14. *If $f : K \to \mathbb{R}$ is locally Lipschitz and generalized strict pseudoinvex with respect to $\eta : K \times K \to X$, then $\partial_L f$ is invariant strictly-pseudomonotone with respect to the same η.*

4. Vector Optimization

In this section, we study the relationships between the solutions of the following (weak) Minty and (weak) Stampacchia (VVLI) and (weak) efficient solutions of vector optimization problems. For the rest of this article, unless otherwise specified, we assume that K is a nonempty subset of an Asplund space X and $\eta : K \times K \to X$ is a vector-valued mapping. The interior of K is denoted by int$K$.

Definition 4.1 ([1, 5]). Let K be an invex set with respect to η and $f = (f_1, \dots, f_\ell) : X \to \mathbb{R}^\ell$ be a vector-valued function.

(i) Generalized Minty vector variational-like inequality problem (GMVVLIP) is finding a vector $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial_L f_i(y)$, $i \in \{1, \dots, \ell\}$, we have

$$\langle \xi, \eta(y, \bar{x}) \rangle_\ell = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \dots, \langle \xi_\ell, \eta(y, \bar{x}) \rangle) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

(ii) Weak generalized Minty vector variational-like inequality problem (WGMVVLIP) is finding a vector $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial_L f_i(y)$, $i \in \{1, ...\ell\}$, we have

$$\langle \xi, \eta(y, \bar{x}) \rangle_\ell = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \dots, \langle \xi_\ell, \eta(y, \bar{x}) \rangle) \notin -\text{int } \mathbb{R}^\ell_+.$$

(iii) Generalized Stampacchia vector variational-like inequality problem (GSVVLIP) is finding a vector $\bar{x} \in K$ such that for all $y \in K$, there exists $\zeta_i \in \partial_L f_i(\bar{x})$, $i \in \{1, \dots, \ell\}$, such that

$$\langle \zeta, \eta(y, \bar{x}) \rangle_\ell = (\langle \zeta_1, \eta(y, \bar{x}) \rangle, \dots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

(iv) Weak generalized Stampacchia vector variational-like inequality problem (WGSVVLIP) is finding a vector $\bar{x} \in K$ such that for all $y \in K$, there exists $\zeta_i \in \partial_L f_i(\bar{x})$, $i \in \{1, \dots, \ell\}$, such that

$$\langle \zeta, \eta(y, \bar{x}) \rangle_\ell = (\langle \zeta_1, \eta(y, \bar{x}) \rangle, \dots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle) \notin -\text{int } \mathbb{R}^\ell_+.$$

We consider the following vector-minimization problem (VOP):

$$\min_{x \in K} \left(f(x) = \left(f_1(x), f_2(x), \dots, f_\ell(x) \right) \right).$$

Definition 4.2. [2] A point $\bar{x} \in K$ is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of (VOP) if

$$f(y) - f(\bar{x}) = \left(f_1(y) - f_1(\bar{x}), \dots, f_\ell(y) - f_\ell(\bar{x}) \right) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \forall y \in K,$$

$$\left(\textit{respectively}, f(y) - f(\bar{x}) = \left(f_1(y) - f_1(\bar{x}), \dots, f_\ell(y) - f_\ell(\bar{x}) \right) \notin -\text{int } \mathbb{R}^\ell_+, \quad \forall y \in K \right).$$

where $\mathbb{R}^\ell_+$ is the nonnegative orthant of $\mathbb{R}^\ell$. Hereafter $\mathbf{0}$ stands for the zero vector.

It is clear that every efficient solution is a weak efficient solution. See [4, 6, 9] and the references therein for more details on vector optimization theory. See also [4, 11, 17] and the references therein for Pareto optimality and its applications.

Proposition 4.3. *Let $K \subseteq X = \mathbb{R}^n$ be a nonempty open invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally Lipschitz. If $\bar{x} \in K$, is a weak efficient solution of (VOP), then it is a solution of (WGSVVLIP).*

Proof. Assume that $\bar{x} \in K$ is a weak efficient solution of (VOP) but not a solution of (WGSVVLIP). So, there exists $y \in K$ such that for all $\xi_i \in \partial_L f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, we have

$$(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle) \in -\text{int } \mathbb{R}^\ell_+,$$

that is

$$\langle \xi_i, \eta(y, \bar{x})\rangle < 0, \quad i = 1, 2, \ldots, \ell. \tag{4.1}$$

Since K is an open set and f_i, $i = 1, 2, \ldots, \ell$, is locally Lipschitz, then there exists $0 < \bar{\delta} < 1$ such that $B_{\bar{\delta}}(\bar{x}) \subset K$ and f_i, $i = 1, 2, \ldots, \ell$, is Lipschitz on an open set containing $[\bar{x}, \bar{x} + \delta\eta(y, \bar{x})] \subset B_{\bar{\delta}}(\bar{x})$ for each $\delta \in (0, \bar{\delta}]$. Now there exists $c_{\delta_i} \in [\bar{x}, \bar{x} + \delta\eta(y, \bar{x}))$, by Theorem 2.2, so that we have

$$f_i(\bar{x} + \delta\eta(y, \bar{x})) - f_i(\bar{x}) \leq \delta\langle x^*_{\delta_i}, \eta(y, \bar{x})\rangle, \quad i = 1, 2, \ldots, \ell,$$

for some $x^*_{\delta_i} \in \partial f_i(c_{\delta_i})$. It follows from [13, Proposition 1.85], the sequence $\{x^*_{\delta_i}\}$ is bounded, $i = 1, 2, \ldots, \ell$. Hence, when $\delta \downarrow 0$, by Proposition 2.3, $\{x^*_{\delta_i}\}$ has a subsequence that converges to some $x^*_i \in \partial_L f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, because $\lim_{\delta\downarrow 0} c_{\delta_i} = \bar{x}$. From (4.1) we have

$$\lim_{\delta\downarrow 0} \frac{f_i(\bar{x} + \delta\eta(y, \bar{x})) - f_i(\bar{x})}{\delta} \leq \langle x^*_i, \eta(y, \bar{x})\rangle < 0.$$

Thus for each $i = 1, 2, \ldots, \ell$, there exists λ_i such that for each $0 < \delta < \lambda_i$, we have $f_i(\bar{x} + \delta\eta(y, \bar{x})) - f_i(\bar{x}) < 0$. If $\bar{\lambda} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_\ell, 1\}$, then for each $0 < \delta < \bar{\lambda}$ we have

$$f_i(\bar{x} + \delta\eta(y, \bar{x})) - f_i(\bar{x}) < 0, \quad i = 1, 2, \ldots, \ell.$$

Therefore, by choosing $\hat{t} \in (0, \bar{\lambda})$ we have $\hat{x} = \bar{x} + \hat{t}\eta(y, \bar{x}) \in K$ and $f_i(\hat{x}) < f_i(\bar{x})$ for any $i \in \{1, 2, \ldots, \ell\}$ which implies that $\bar{x}$ is not a weak efficient solution of (VOP). □

Proposition 4.4. *Let $K \subseteq X$ be invex with respect to $\eta : K \times K \to X$, such that η is skew. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally Lipschitz and generalized strict pseudoinvex with respect to η. If $\bar{x} \in K$ is a solution of (WGSVVLIP), then it is a solution of (GMVVLIP).*

Proof. Let $\bar{x} \in X$ be a solution of (WGSVVLIP) but not a solution of (GMVVLIP). Then, there exist $y \in K$ and $\xi_i \in \partial_L f_i(y)$, $i = 1, 2, \ldots, \ell$, such that

$$(\langle \xi_1, \eta(\bar{x}, y)\rangle, \ldots, \langle \xi_\ell, \eta(\bar{x}, y)\rangle) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\},$$

implying

$$\langle \xi_i, \eta(\bar{x}, y)\rangle \geq 0, \quad i = 1, 2, \ldots, \ell,$$

holds strictly for some $k \in \{1, 2, \ldots, \ell\}$. Since f_i, $1 \leq i \leq \ell$, is generalized strict pseudoinvex with respect to η, each $\partial_L f_i$ is invariant strictly-pseudomonotone with respect to η, from Theorem 3.14 and then we will have

$$\langle \zeta_i, \eta(y, \bar{x})\rangle < 0, \quad \forall \zeta_i \in \partial_L f_i(\bar{x}), \quad i = 1, 2, \ldots, \ell.$$

Therefore, for all $\zeta_i \in \partial_L f_i(\bar{x})$, we have

$$(\langle \zeta_i, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle) \in -\text{int } \mathbb{R}^\ell_+,$$

Thus, $\bar{x} \in K$ is not a solution of (WGSMVVLIP), which contradicts our assumption. □

Proposition 4.5. *Let $K \subseteq X = \mathbb{R}^n$ be invex with respect to $\eta : K \times K \to \mathbb{R}^n$. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally Lipschitz. If $\bar{x} \in K$ is a solution of (WGMVVLIP), then it is a solution of (WGSMVVLIP).*

Proof. Suppose that $\bar{x} \in K$ is a solution of (WGMVVLIP). For any $y \in K$ and any sequence $\{\alpha_m\} \searrow 0$ with $\alpha_m \in (0, 1]$, we have

$$y_m := \bar{x} + \alpha_m \eta(y, \bar{x}) \in K,$$

since K is invex. Since $\bar{x}$ is a solution of (WGMVVLIP), then for all $\xi_i^m \in \partial_L f_i(y_m)$, $i = 1, 2, \ldots, \ell$, we have

$$(\langle \xi_1^m, \eta(y_m, \bar{x})\rangle, \ldots, \langle \xi_\ell^m, \eta(y_m, \bar{x})\rangle) \notin -\text{int } \mathbb{R}^\ell_+.$$

The sequence $\{\xi_i^m\}$ is bounded due to [13, Proposition 1.85]. Hence, when $m \to \infty$, by Proposition 2.3, $\{\xi_i^m\}$ has a subsequence that converges to some $\xi_i \in \partial_L f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, because $y_m \to \bar{x}$ as $m \to \infty$. Therefore, for any $y \in K$, there exists $\xi_i \in \partial_L f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, such that

$$(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle) \notin -\text{int } \mathbb{R}^\ell_+,$$

which shows that $\bar{x}$ is a solution of (WGSMVVLIP). □

From Propositions 4.4 and 4.5, we have the following result.

Theorem 4.6. *Let $K \subseteq X = \mathbb{R}^n$ be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally Lipschitz and generalized strict pseudoinvex with respect to η. Then $\bar{x} \in K$ is a solution of (WGSVVLIP) if and only if it is a solution of (WGMVVLIP).*

Proposition 4.7. *Let $K \subseteq X$ be invex with respect to $\eta : K \times K \to X$ such that η is skew. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, ..., \ell$, be locally Lipschitz and generalized strict pseudoinvex with respect to η. If $\bar{x} \in K$ is a weak efficient solution of (VOP), then it is a solution of (GMVVLIP).*

Proof. If $\bar{x}$ is a weak efficient solution of (VOP) but not a solution of (GMVVLIP), then there exist $y \in K$ and $\xi_i \in \partial_L f_i(y)$, $i = 1, \ldots, \ell$ such that

$$\langle \xi_i, \eta(\bar{x}, y) \rangle \geq 0, \quad \forall i = 1, 2, \ldots, \ell,$$

and $\langle \xi_k, \eta(\bar{x}, y) \rangle > 0$ for some $k \in \{1, 2, \ldots, \ell\}$. Generalized strict pseudoinvexity of f_i with respect to η implies that

$$f_i(\bar{x}) > f_i(y), \quad \forall i = 1, 2, \ldots, \ell.$$

Therefore, we have

$$(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})) \in -\text{int } \mathbb{R}^\ell_+,$$

which contradicts that $\bar{x} \in K$ is a weak efficient solution of (VOP). □

Proposition 4.8. *Let $K \subseteq X = \mathbb{R}^n$ be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ such that η is skew and continuous in terms of the second argument satisfying Condition 3.4. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally Lipschitz and generalized pseudoinvex with respect to η. Then $\bar{x} \in K$ is a weak efficient solution of (VOP) if and only if it is a solution of (WGMVVLIP).*

Proof. Assume that $\bar{x}$ is a weak efficient solution of (VOP) and not a solution of (WGMVVLIP). Then, there exist $y \in K$ and $\xi_i \in \partial_L f_i(y)$, $i = 1, \ldots, \ell$, such that

$$\langle \xi_i, \eta(\bar{x}, y) \rangle > 0, \quad \forall i = 1, 2, \ldots, \ell.$$

Since each f_i is generalized pseudoinvex with respect to η, applying Theorems 3.10 and 3.11, we get each f_i is quasiinvex with respect to η. Thus

$$\langle \xi_i, \eta(\bar{x}, y) \rangle > 0 \Rightarrow f_i(\bar{x}) > f_i(y), \quad i = 1, 2, \ldots, \ell.$$

Hence,

$$(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})) \in -\text{int } \mathbb{R}^\ell_+,$$

which contradicts the assumption that $\bar{x}$ is a weak efficient solution of (VOP).
Conversely, the weak efficiency of (VOP) can be resulted from Proposition 4.5 and [2, Proposition 4.6]. □

Proposition 4.9. *Let $K \subseteq X = \mathbb{R}^n$ be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ such that η is continuous in terms of the second argument and satisfies Condition 3.4. Let $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, \ell$, be locally*

Lipschitz and generalized pseudoinvex with respect to η. If $\bar{x} \in K$ is a solution of (GSVVLIP), then it is an efficient solution of (VOP).

Proof. Let $\bar{x}$ be a solution of (GSVVLIP) and not an efficient solution of (VOP). Then, there exists $y \in K$ such that

$$f(y) - f(\bar{x}) = \big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}\,.$$

That is,

$$f_i(y) \leq f_i(\bar{x}), \quad i = 1, 2, \ldots, \ell, \tag{4.2}$$

and $f_k(y) < f_k(\bar{x})$ for some $k \in \{1, 2, \ldots, \ell\}$. From the generalized pseudoinvexity of f_k with respect to η we have

$$\langle \xi_k, \eta(y, \bar{x})\rangle < 0, \ \forall\ \xi_k \in \partial_L f_k(\bar{x}).$$

On the other hand, for any $i \in \{1, 2, \ldots, \ell\}$ such that $f_i(y) = f_i(\bar{x})$ in (4.2), from Theorems 3.10 and 3.11, we have

$$\langle \xi_i, \eta(y, \bar{x})\rangle \leq 0, \quad \forall \xi_i \in \partial_L f_i(\bar{x}).$$

Therefore,

$$\langle \xi, \eta(y, \bar{x})\rangle = (\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}\,,$$

which contradicts the assumption that $\bar{x}$ is a solution of (GSVVLIP). □

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