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About One Sweep Algorithm for Solving Linear-Quadratic Optimization Problem with Unseparated Two-Point Boundary Conditions

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Abstract. In the paper a linear-quadratic optimization problem (LCTOR) with unseparated two-point boundary conditions is considered. To solve this problem is proposed a new sweep algorithm which increases doubles the dimension of the original system. In contrast to the well-known methods, here it refuses to solve linear matrix and nonlinear Riccati equations, since the solution of such multi-point optimization problems encounters serious difficulties in passing through nodal points. The results are illustrated with a specific numerical example.

1. Introduction

As in known, optimization problems with unseparated two-point and multi-point boundary conditions [4, 5, 7, 12] play a great role in solving many practical problems as constructing optimal program trajectories and control systems [3, 11, 16, 17, 26], for motion of gas and gas-fluid mixture in annular space and hoist [9, 20], when producing gas by gas-lift method. As even for solving nonlinear optimization problems [4, 11, 16] the quasilinearization method is used with the help of solving a linearly square optimization problem (LSOP), so for the latter attracts attention of researchers [2, 21, 22, 25].

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There exist various methods for solving LSOP with unseparated boundary conditions, where missing boundary data are found by solving corresponding Euler-Lagrange differential equation and then optimal solutions that are called a method that raises the dimension of the original system [5, 7] are recovered. In the case when the length of the interval determining the system is long, the Moshinsky method [1, 3, 19] that twice increases the length of the interval and twice raises the dimension of the original system, is used. Here it is difficult to ensure good stipulation of final systems for finding both initial data and at any point inside the interval. Therefore, in [3–5, 7, 23] the sweep method is developed for solving LSOP with unseparated boundary conditions and this requires to solve some nonlinear systems of differential equations not raising the dimension of the original system. In [22], the results of [4, 5, 7] are given in the case of multi-point boundary conditions, in [2, 21] a counter-example with non-optimality of the obtained results is given. Apparently, in [2, 21, 22] the problems are associated with passage of matrix equations from nodal points to Riccati equations, that requires complex researches.

In the present paper, based on the method raising the dimension [3, 24] of the original system, we give a sweep method for solving LSOP with unseparated boundary conditions, in the three-point case such annoyances [21] do not happen, i.e., the example cited in [21] is easily solved here. Proceeding from the results of [2] here, we study a discrete case which can be extended to discontinuous case.

2. Problem Statement and the Method Raising the Dimension of the Original System

At first, we consider linear square optimization problem with two-point unseparated boundary conditions, i.e., let the motion of the object be described by the following discrete linear controllable system [13, 16]

$$x\,(i+1)=\psi(i)x(i)+\Gamma(i)u(i),\quad i=1,2,\ldots,\ell, \tag{2.1}$$

with unseparated boundary conditions

$$\Phi_1 x(0)-\Phi_2 x(\ell)=q, \tag{2.2}$$

where $x(i)-n$- dimensional phase vector, $u(i)$ is $m-$ dimensional control action, $\psi(i)$, $\Gamma(i)$ and the constants Φ_1, Φ_2 are $n\times n$, $n\times m$ and $k\times n$-dimensional matrices, respectively, the known constant vector q is of dimension $k\times 1$, respectively.

It is required to find such vectors $x(i)$, $u(i)$ that the square functional

$$J=\frac{1}{2}\sum_{i=0}^{l-1}\left(x'(i)R(i)x(i)+u'(i)C(i)u(i)\right) \tag{2.3}$$

at constraint (2.1), (2.2) obtains minimal value, where $R(i) = R'(i) \geq 0$, $C(i) = C'(i) > 0$ are $n \times n$; $m \times m$-dimensional symmetric matrices, the prime denotes the transposition operation.

Using necessary optimality conditions of problem (2.1)-(2.3) in the form of Euler-Lagrange, from [13, 16] it is easy to show that the solution of problems (2.1)-(2.3) is reduced to the solution of the following system of $2n$-th order finite-difference equations [15]

$$\left.\begin{array}{l} x(i+1) = \psi(i)x(i) - M(i)\lambda(i+1) \\ \lambda(i) = R(i)x(i) + \psi'(i)\lambda(i+1) \end{array}\right\} \tag{2.4}$$

with the following boundary conditions

$$\begin{array}{l} \Phi_1' v + \lambda(0) = 0 \\ -\Phi_{21}' v + \lambda(\ell) = 0 \end{array} \tag{2.5}$$

and (2.2), $M(i) = \Gamma(i)C^{-1}(i)\Gamma'(i)$.

Denoting $\Phi = [\Phi_1, -\Phi_2]$ and using the results of [14], the matrix Φ' is represented in the form

$$\Phi' = P^{-1}\begin{bmatrix} E \\ 0 \end{bmatrix} Q^{-1},$$

where P and Q are some square, E is unit matrix of dimensions $2n \times 2n$, $n \times n$ and $k \times k$, respectively.

Let the matrix P be decomposed into blocks

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

where the matrices P_1, P_2 and P_3, P_4 are of dimension $k \times n$ and $(2n-k)\times n$, respectively. Then, it is shown in [3] that the solution of problems (2.1), (2.2) is reduced to the solution of equation (2.4) with the following $2n$ boundary conditions

$$\begin{bmatrix} \Phi_1 & 0 \\ 0 & -P_3 \end{bmatrix}\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} -\Phi_2 & 0 \\ 0 & P_3 \end{bmatrix}\begin{bmatrix} x(\ell) \\ \lambda(\ell) \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix} \tag{2.6}$$

and the control law $u(i)$ is determined in the form

$$u(i) = -C^{-1}(i)\Gamma'(i)\lambda(i+1)\,. \tag{2.7}$$

As was shown in [3], the missing boundary data are determined from the following system of linear algebraic equations

$$\begin{bmatrix} \psi(0,\ell) & 0 & -E & -M(0,\ell) \\ R(0,\ell) & -E & 0 & \psi'(0,\ell) \\ \Phi_1 & 0 & -\Phi_2 & 0 \\ 0 & -P_3 & 0 & P_4 \end{bmatrix}\begin{bmatrix} x(0) \\ \lambda(0) \\ x(\ell) \\ \lambda(\ell) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q \\ 0 \end{bmatrix}, \tag{2.8}$$

where $\psi(0,\ell)$, $M(0,\ell)$, and $R(0,\ell)$ satisfy the following recurrent relations

$$\begin{aligned}
\psi(i,j) &= \psi(i+j-1)Q(i,j-1)\psi(i,j-1),\ \psi(i,1) \\
&= \psi(i), \\
M(i,j) &= M(i+j-1)+\psi(i+j-1)Q(i,j-1) \\
&\quad \times M(i,j-1)\psi'(i+j-1), \quad M(i,1)=M(i), \\
R(i,j) &= R(i,j-1)+\psi'(i,j-1)R(i+j-1) \\
&\quad \times Q(i,j-1)\psi(i,j-1), \quad R(i,1)=R(i) \\
Q(i,j) &= (E+M(i,j)R(i+j))^{-1}
\end{aligned} \tag{2.9}$$

for $i=0,\ j=\ell$.

Having solving the system of linear algebraic equations (SLAE) (2.8), we find the initial and final values $x(0)$, $\lambda(0)$, $x(\ell)$, and $\lambda(\ell)$. In what follows, as was shown in [3], the current values $x(i)$ and $\lambda(i)$ are determined from the following SLAE

$$\begin{bmatrix} E & M(0,i) \\ 0 & \psi'(0,i) \end{bmatrix} \begin{bmatrix} x(i) \\ \lambda(i) \end{bmatrix} = \begin{bmatrix} \psi(0,i) & 0 \\ -R(0,i) & E \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} \tag{2.10}$$

or

$$\begin{bmatrix} \psi(i,\ell-i) & 0 \\ R(i,\ell-i) & -E \end{bmatrix} \begin{bmatrix} x(i) \\ \lambda(i) \end{bmatrix} = \begin{bmatrix} E & M(i,\ell-i) \\ 0 & -\psi'(i,\ell-i) \end{bmatrix} \begin{bmatrix} x(\ell) \\ \lambda(\ell) \end{bmatrix} \tag{2.11}$$

depending on degeneracy of the matrices $\psi(i)$. Thus, finding $x(0)$, $\lambda(0)$, and $x(\ell)$ from SLAE (2.8), we restore the current values $x(i)$ and $\lambda(i)$ from (2.10) and (2.11) depending on what points i degenerates $\psi(i)$. Then we restore optimal control $u(i)$ by formula (2.7). Note that $\psi(0,\ell)$, $M(0,\ell)$, and $R(0,\ell)$ are determined from (2.9) for $i=0,\ j=\ell$.

It should be noted that for large dimension of problems (2.1)-(2.3) such a method can encounter difficulties, i.e., for finding initial data $x(0)$ (vector of dimension n) it is required to solve $4n$-dimensional SLAE (2.8). For determining $x(i)$ for the general case ($\psi(i)$ does not exist) it is also required to solve SLAE (2.10), (2.11).

Therefore, in [18, 21] a sweep method is offered for solving problem (2.1)-(2.3), that despite (2.8), (2.10), and (2.11) considerably decreases dimension of similar equations for determining $x(0)$ and $x(i)$. However, when solving optimization problems of multi-point boundary value problems, these methods face difficulties. In the following point of the function, we give a new sweep algorithm that can be successfully extended [21] to the multi-point general case as well [22].

3. New Sweep Method

We use the idea of the sweep method [3–7, 23] instead of equations (2.8) for determining $x(0)$, $\lambda(0)$, $x(\ell)$, and $\lambda(\ell)$ using the first two equations, i.e. substituting $\lambda(\ell)$ from the second equations of (2.5) in the second equation of (2.8), and have

$$R(0,\ell)x(0) + \left(\Phi_1^{'} - \psi'(0,\ell)\Phi_2^{'}\right) v = 0\,. \tag{3.1}$$

Now, having determined $x(\ell)$ from the first equation of (2.8) in the form

$$x(\ell) = \psi(0,\ell)x(0) - M(0,\ell)\lambda(\ell) \tag{3.2}$$

we substitute this formula (3.2) and (2.2), after some transformations have

$$\Phi_1 x(0) - \Phi_2\psi(0,\ell)x(0) + \Phi_2 M(0,\ell)\lambda(\ell) = q\,. \tag{3.3}$$

Then, taking into account from the last relation $\lambda(\ell)$ from (2.5), in (3.3) we get

$$\left(\Phi_1 - \Phi_2\psi(0,\ell)\right)x(0) + \Phi_2 M(0,\ell)\Phi_2^{'} v = q\,, \tag{3.4}$$

where having joined (3.1), (3.4) for determining $x(0)$, v we have the following system of linear algebraic equations (SLAE)

$$\begin{bmatrix} R(0,\ell) & \Phi_1^{'} - \psi'(0,\ell)\Phi_2^{'} \\ \Phi_1 - \Phi_2\psi(0,\ell) & -\,\Phi_2 M(0,\ell)\Phi_2^{'} \end{bmatrix} \begin{bmatrix} x(0) \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ q \end{bmatrix}. \tag{3.5}$$

Note that the main matrix of SLAE (3.5) is symmetric, and this allows to solve this problem more precisely [8].

We stop on calculation of $x(i)$ and $u(i)$ without using $\lambda(i)$. For this, we suppose that $\psi^{-1}(i)$ from (2.4) exists. Then from the second equation of (2.4), we represent $\lambda(i+1)$ in the form

$$\lambda(i+1) = -\psi'^{-1}(i)R(i)x(i) + \psi'^{-1}(i)\lambda(i)\,. \tag{3.6}$$

From (3.6) and the first equation of (2.5), for $i = 0$, we have

$$\lambda(1) = -\psi'^{-1}(0)R(0)x(0) - \psi'^{-1}(0)\Phi_1^{'} v\,,$$

and for $i = 1$

$$\begin{aligned}\lambda(2) &= -\psi'^{-1}(1)R(1)x(1) + \psi'^{-1}(1)\lambda(1) \\ &= -\psi'^{-1}(1)R(1)x(1) - \psi'^{-1}(1)\psi'^{-1}(0)R(0)\lambda(0) - \psi'^{-1}(1)\psi'^{-1}(0)\Phi_1^{'} v\end{aligned}$$

and so on. By mathematical induction, we can sow that

(3.7)
$$\lambda(i+1) = -\sum_{k=0}^{i}\left(\prod_{j=k}^{i}\psi'^{-1}(i+k-1)\right)R_k x_k - \prod_{k=0}^{i}\left(\psi'^{-1}(i-1)\right)\Phi_1' v\,.$$

Taking into account formulas (3.7) in (3.6), we find $x(i+1)$ by $x(i)$ and v in the following form

(3.8)
$$x(i+1) = \psi(i)x(i) - M(i)\left\{\sum_{k=0}^{i}\left(\prod_{j=k}^{i}\psi'^{-1}(i+k-j)\right)R(k)x(k) - \left(\prod_{k=0}^{i}\psi_{i-k}^{-1}\right)\Phi_1' v\right\},\quad (i=0,1,\ldots,\ell-1)$$

$u(i)$ will be

$$u(i) = C^{-1}(i)\Gamma'(i)\left\{\sum_{k=0}^{i}\left(\prod_{j=k}^{i}\psi'^{-1}(i+k-j)\right)R(k)x(k) + \left(\prod_{k=0}^{i}\psi'(i-k)\right)\Phi_1' v\right\},\quad i=0,1,\ldots,\ell-1\,. \tag{3.9}$$

For the given matrices from (2.1)-(2.3) - $\psi(i), \Gamma(i), \Phi_1, \Phi_2, C(i), R(i), q$ from SLAE (3.5) we find $x(0)$, v. Then we determine the control $u(i)$ from relations (3.9), the trajectory $x(i)$ from (3.8).

4. Algorithm and Example

Thus, we have the following computing

Algorithm.

- We formulate the given matrices $\psi(i)$, $\Gamma(i)$, Φ_1, Φ_2, $C(i)$, and $R(i)$ and q vector from problem (2.1)-(2.3).
- From relations (2.9) under the matrix conditions $\psi(0,1)=\psi(0)$, $M(0,1)=M(0)$, $R(0,1)=R(0)$ we calculate $\psi(0,\ell)$, $R(0,\ell)$ and $M(0,\ell)$.
- We formulate the main matrix and vector in the right hand side of SLAE (3.5).
- We solve SLAE (3.5).
- The control $u(i)$ is calculated from (3.9), the trajectory $x(i)$ from (3.8).

It should be noted that algorithm (2.1) is a “sweep” algorithm. In fact this algorithm does not use the adjoint vector $\lambda(i)$.

Example 4.1. We consider the case when in problem (2.1)-(2.3) $n = m = k = 1,\ \ell = 4$, and the constants $\psi(i),\ \Gamma(i),\ \phi_1,\ \phi_2,\ q,\ R(i)$ and $C(i)$ are determined as follows

$$\left.\begin{array}{l} \psi(0) = \psi(1) = \psi(2) = \psi(3) = 1, \\ \Gamma(0) = \Gamma(1) = \Gamma(2) = \Gamma(3) = 1, \\ \Phi_1 = 1,\ \Phi_2 = 1,\ q = 1, \\ R(0) = R(1) = R(2) = R(3) = 1, \\ C(0) = C(1) = C(2) = C(3) = 1\,. \end{array}\right\} \tag{4.1}$$

Then problem (2.1)-(2.3) takes the following form

$$\begin{aligned} & x(i+1) = x(i) + u(i), \quad i = 0,1,2,3, \\ & x(0) - x(4) = 1 \\ & J = \frac{1}{2}\sum_{i=0}^{3}\left[x^2(i) + u^2(i)\right] \to \min\,. \end{aligned} \tag{4.2}$$

Using the recurrent relation (2.9), after some calculations we have

$$\psi(0,4) = \frac{1}{13}, \qquad M(0,4) = \frac{21}{13}, \qquad R(0,4) = \frac{21}{13}.$$

Substituting these values in (3.5), we get the system of linear algebraic equations

$$\left\{\begin{array}{l} \frac{21}{13}x(0) + \frac{12}{13}v = 0 \\ \\ \frac{12}{13}x(0) - \frac{21}{13}v = 1 \end{array}\right.$$

whose solution equals $x(0) = \frac{4}{15},\ \ v = -\frac{7}{15}\,.$

Then, using formulas (3.8) and (3.9), we find the solution of problem (4.2) in the following form

$$x(0) = \tfrac{4}{15}, \quad x(1) = \tfrac{1}{15}, \quad x(2) = -\tfrac{1}{15}, \quad x(3) = -\tfrac{4}{15}, \quad x(4) = -\tfrac{11}{15}$$

$$u(0) = -\tfrac{1}{5}, \quad u(1) = -\tfrac{2}{15}, \quad u(2) = -\tfrac{1}{5}, \quad u(3) = -\tfrac{7}{15}.$$

And the functional obtains its minimal value $J = \frac{7}{30}$.

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