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Best Proximity Point Results for Almost Contraction and Application to Nonlinear Differential Equation

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Abstract. Berinde [V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum **9** (2004), 43-53] introduced almost contraction mappings and proved Banach contraction principle for such mappings. The aim of this paper is to introduce the notion of multivalued almost Θ-contraction mappings and to prove some best proximity point results for this new class of mappings. As applications, best proximity point and fixed point results for weak single valued Θ-contraction mappings are obtained. Moreover, we give an example to support the results presented herein. An application to a nonlinear differential equation is also provided.

1. Introduction and Preliminaries

The following concept was introduced by Berinde as 'weak contraction' in [9]. But in [10], Berinde renamed 'weak contraction' as 'almost contraction' which is appropriate.

Definition 1.1. Let (X, d) be a metric space. A mapping $F : X \to X$ is called almost contraction or (δ, L)-contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that for any $x, y \in X$, we have

$$d(Fx, Fy) \leq \delta d(x, y) + Ld(y, Fx). \tag{1.1}$$

Von Neumann [32] considered fixed points of multivalued mappings in the study of game theory. Indeed, the fixed point results for multivalued

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mappings play a significant role in the study of control theory and in solving many problems of economics and game theory.

Nadler [27] used the concept of the Hausdorff metric to obtain fixed points of multivalued contraction mappings and obtained the Banach contraction principle as a special case.
Here, we recall that a Hausdorff metric H induced by a metric d on a set X is given by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \tag{1.2}$$

for every $A, B \in CB(X)$, where $CB(X)$ is the collection of the closed and bounded subsets of X.

M. Berinde and V. Berinde [11] introduced the notion of multivalued almost contraction as follows:

Let (X,d) be a metric space. A mapping $F : X \to CB(X)$ is called multivalued almost contraction if there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that for any $x, y \in X$, we have

$$H(Fx,Fy) \leq \delta d(x,y) + LD(y,Fx). \tag{1.3}$$

Berinde [11] proved Nadler's fixed point theorem in ([27]):

Theorem 1.2. *Let (X,d) be a complete metric space and $F : X \to CB(X)$ be multivalued almost contraction. Then F has a fixed point.*

Jleli *et al.* [23] defined Θ-contraction mapping as follows:

A mapping $F : X \to X$ is called Θ-contraction if for any $x, y \in X$

$$\Theta(d(Fx,Fy)) \leq [\Theta(d(x,y))]^k \tag{1.4}$$

where, $k \in (0,1)$ and $\Theta : (0,\infty) \to (1,\infty)$ is a mapping which satisfies the following conditions.

- (Θ_1) Θ is nondecreasing;
- (Θ_2) for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty} (\alpha_n) = 0$;
- (Θ_3) there exist $0 < k < 1$ and $l \in (0,\infty)$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

Denote

$$\Omega = \{\Theta : (0,\infty) \to (1,\infty) : \Theta \text{ satisfies } \Theta_1 - \Theta_3\}. \tag{1.5}$$

Theorem 1.3 ([23]). *Let (X,d) be a complete metric space and $F : X \to X$ be a Θ-contraction, then F has a unique fixed point.*

Hancer *et al.* [21] introduced the notion of multi-valued Θ-contraction mapping as follows:

Let (X, d) be a metric space and $F : X \to CB(X)$ be a multivalued mapping. Suppose that there exist $\Theta \in \Omega$ and $0 < k < 1$ such that

$$\Theta(H(Fx, Fy)) \leq [\Theta(d(x, y))]^k, \tag{1.6}$$

for any $x, y \in X$ provided that $H(Fx, Fy) > 0$, where $CB(X)$ is a collection of all nonempty closed and bounded subsets of X.

Theorem 1.4. *Let (X, d) be a complete metric space and $F : X \to K(X)$ be a multi-valued Θ-contraction. Then F has a fixed point.*

Let A and B be two nonempty subsets of a metric space (X, d) and $F : A \to CB(B)$. A point $x^* \in A$ is called a best proximity point of F if

$$\begin{aligned} D(x^*, Fx^*) &= \inf\{d(x^*, y) : y \in Fx^*\} \\ &= dist(A, B), \end{aligned}$$

where

$$dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

If $A \cap B \neq \phi$, then x^* is a fixed point of F. If $A \cap B = \phi$, then $D(x, Fx) > 0$ for all $x \in A$ and F has no fixed point.

Consider the following optimization problem:

$$\min\{D(x, Fx) : x \in A\}. \tag{1.7}$$

It is then important to study necessary conditions so that the above minimization problem has at least one solution.

Since

$$dist(A, B) \leq D(x, Fx) \tag{1.8}$$

for all $x \in A$, hence the optimal solution to the problem

$$\min\{D(x, Fx) : x \in A\} \tag{1.9}$$

for which the value $dist(A, B)$ is attained is indeed a best proximity point of multivalued mapping F.

For more results in this direction, we refer to [1, 2, 4, 5, 7, 8, 14, 15, 18, 19, 22, 26, 33, 34] and references mentioned therein.

Let A and B be two nonempty subsets of X. Denote

$$\begin{aligned} A_0 &= \{a \in A : d(a, b) = dist(A, B) \text{ for some } b \in B\}, \\ B_0 &= \{b \in B : d(a, b) = dist(A, B) \text{ for some } a \in A\}. \end{aligned}$$

Definition 1.5. [31] Let (X, d) be a metric space and $A_0 \neq \phi$, we say that the pair (A, B) has the P-property if

$$\left.\begin{aligned} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{aligned}\right\} \text{ implies that } d(x_1, x_2) = d(y_1, y_2), \tag{1.10}$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 1.6 ([35])**.** Let (X,d) be a metric space and $A_0 \neq \phi$, we say that the pair (A,B) has the weak P-property if

$$\left.\begin{array}{c} d(x_1,y_1)=dist(A,B) \\ d(x_2,y_2)=dist(A,B) \end{array}\right\} \text{ implies that } d(x_1,x_2)\leq d(y_1,y_2), \tag{1.11}$$

where $x_1,x_2 \in A$ and $y_1,y_2 \in B$.

Definition 1.7 ([13])**.** Let (X,d) be a metric space, A,B be two subsets of X and $\alpha : A \times A \to [0,\infty)$. A mapping $F : A \to 2^B \backslash \{\phi\}$ is called α-proximal admissible if

$$\left.\begin{array}{c} \alpha(x_1,x_2)\geq 1, \\ d(u_1,y_1)=dist(A,B), \\ d(u_2,y_2)=dist(A,B) \end{array}\right\} \text{ implies that } \alpha(u_1,u_2)\geq 1 \tag{1.12}$$

for all $x_1,x_2,u_1,u_2 \in A,\ y_1 \in Fx_1$ and $y_2 \in Fx_2$.

Definition 1.8 ([13])**.** Let $F : X \to CB(Y)$ be a multi-valued mapping, where (X,d_1), (Y,d_2) are two metric spaces. A mapping F is said to be continuous at $x \in X$ if $H(Fx,Fx_n)\to 0$ whenever $d_1(x,x_n)\to 0$ as $n \to \infty$.

The aim of this paper is to obtain some best proximity point results for multivalued almost Θ-contraction mappings. We also present some best proximity point and fixed point results for single valued mappings. Moreover, an example to prove the validity and application to nonlinear differential equation for the usability of our results is presented. Our results extend, unify and generalize the comparable results in the literature.

2. Best Proximity Points of Multivalued Almost Θ-contraction

We begin with the following definition:

Definition 2.1. Let A,B be two nonempty subsets of a metric space (X,d) and $\alpha : A\times A \to [0,\infty)$. Let $\Theta \in \Omega$ be a continuous function. A multivalued mapping $F : A \to 2^B\backslash\{\emptyset\}$ is called almost Θ-contraction if for any $x,y \in A$, $H(Fx,Fy)>0$, we have

$$\alpha(x,y)\Theta[H(Fx,Fy)] \leq [\Theta(d(x,y)+\lambda(D(y,Fx)-dist(A,B)))]^k \tag{2.1}$$

where $k \in (0,1)$ and $\lambda \geq 0$.

Theorem 2.2. *Let (X,d) be a complete metric space and A, B nonempty closed subsets of X such that $A_0 \neq \emptyset$. Suppose that $F : A \to K(B)$ is a continuous mapping such that*

- (i) *$Fx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(ii) *F is α-proximal admissible mapping;*
(iii) *there exist $x_0, x_1 \in A_0$ and $y_0 \in Fx_0 \subseteq B_0$ such that $d(x_1, y_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$;*
(iv) *F is multivalued almost Θ-contraction.*

Then F has a best proximity point in A.

Proof. Let x_0, x_1 be two given points in A_0 and $y_0 \in Fx_0 \subseteq B_0$ such that $d(x_1, y_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$. If $y_0 \in Fx_1$, then $dist(A, B) \leq D(x_1, Fx_1) \leq d(x_1, y_0) = dist(A, B)$ implies that $D(x_1, Fx_1) = dist(A, B)$ and x_1 is a best proximity point of F. If $y_0 \notin Fx_1$ then,

$$0 < D(y_0, Fx_1) \leq H(Fx_0, Fx_1).$$

Since $Fx_1 \in K(B)$, we can choose $y_1 \in Fx_1$ such that

$$1 < \Theta[d(y_0, y_1)] \leq \Theta[H(Fx_0, Fx_1)].$$

As F is multivalued almost Θ-contraction mapping, we have

$$\begin{aligned} 1 < \Theta[d(y_0, y_1)] &\leq \alpha(x_0, x_1)\Theta[H(Fx_0, Fx_1)] \\ &\leq [\Theta(d(x_0, x_1) + \lambda(D(x_1, Fx_0) - dist(A, B)))]^k \\ &= [\Theta(d(x_0, x_1))]^k. \end{aligned} \tag{2.2}$$

Since $y_1 \in Fx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, y_1) = dist(A, B)$ and $\alpha(x_1, x_2) \geq 1$. By weak P-property of the pair (A, B) we obtain that $d(x_2, x_1) \leq d(y_0, y_1)$. If $y_1 \in Fx_2$, then x_2 is a best proximity point of F. If $y_1 \notin Fx_2$, then

$$D(y_1, Fx_2) \leq H(Fx_1, Fx_2).$$

We now choose $y_2 \in Fx_2$ such that

$$\begin{aligned} 1 < \Theta[d(y_1, y_2)] &\leq \Theta[H(Fx_1, Fx_2)] \\ &\leq \alpha(x_1, x_2)\Theta[H(Fx_1, Fx_2)] \\ &\leq [\Theta(d(x_1, x_2) + \lambda(D(x_2, Fx_1) - dist(A, B)))]^k \\ &= [\Theta(d(x_1, x_2))]^k. \end{aligned} \tag{2.3}$$

Continuing this process, we can obtain two sequences $\{x_n\}$ and $\{y_n\}$ in $A_0 \subseteq A$ and $B_0 \subseteq B$, respectively, such that $y_n \in Fx_n$ and it satisfies

$$d(x_{n+1}, y_n) = dist(A, B) \text{ with } \alpha(x_n, x_{n+1}) \geq 1,$$

where $n = 0, 1, 2, \ldots$. Also,

(2.4)

$$\begin{aligned} 1 < \Theta[d(y_n, y_{n+1})] &\leq \Theta[H(Fx_n, Fx_{n+1})] \\ &\leq \alpha(x_n, x_{n+1})\Theta[H(Fx_n, Fx_{n+1})] \\ &\leq [\Theta(d(x_n, x_{n+1}) + \lambda(D(x_n, Fx_{n+1}) - dist(A, B)))]^k \end{aligned}$$

$$= [\Theta(d(x_n, x_{n+1}))]^k$$

implies that

$$1 < \Theta[d(y_n, y_{n+1})] \leq [\Theta(d(x_n, x_{n+1}))]^k. \tag{2.5}$$

Since

$$d(x_{n+1}, y_n) = dist(A, B) \tag{2.6}$$

and

$$d(x_n, y_{n-1}) = dist(A, B) \tag{2.7}$$

for all $n \geq 1$, it follows by the weak P-property of the pair (A, B) that

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \tag{2.8}$$

for all $n \in \mathbb{N}$. Now by repeated application of (2.5), (2.8) and the monotone property of Θ, we have

(2.9)

$$\begin{aligned} 1 < \Theta[d(x_n, x_{n+1})] &\leq \Theta(d(y_{n-1}, y_n)) \leq \Theta(H(Fx_{n-1}, Fx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\Theta(H(Fx_{n-1}, Fx_n)) \\ &\leq [\Theta(d(x_{n-1}, x_n) + \lambda(D(x_n, Fx_{n-1}) - dist(A, B)))]^k \\ &= (\Theta(d(x_{n-1}, x_n)))^k \leq (\Theta(d(y_{n-2}, y_{n-1})))^k \\ &\leq (\Theta(H(Fx_{n-2}, Fx_{n-1})))^k \\ &\leq (\alpha(x_{n-2}, x_{n-1})(\Theta(H(Fx_{n-2}, Fx_{n-1})))^k \\ &\leq [\Theta(d(x_{n-2}, x_{n-1}) + \lambda(D(x_{n-1}, Fx_{n-2}) - dist(A, B)))]^{k^2} \\ &= (\Theta(d(x_{n-2}, x_{n-1})))^{k^2} \\ &\ \ \vdots \\ &\leq (\Theta(d(x_0, x_1)))^{k^n} \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This shows that $\lim_{n\to\infty} \Theta(d(x_n, x_{n+1})) = 1$ and hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. From (Θ_3), there exist $0 < r < 1$ and $0 < l \leq \infty$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l. \tag{2.10}$$

Assume that $l < \infty$ and $\beta = l/2$. From the definition of the limit there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\Theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - l \right| \leq B, \quad \text{for all } n \geq n_0$$

which implies that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq l - \beta = \beta \quad \text{for all } n \geq n_0.$$

Hence

$$n[d(x_n, x_{n+1})]^r \leq n\alpha[\Theta(d(x_n, x_{n+1})) - 1] \quad \text{for all } n \geq n_0,$$

where $\alpha = 1/\beta$. Assume that $l = \infty$. Let $\beta > 0$ be a given real number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq \beta \quad \text{for all } n \geq n_0$$

implies that

$$n[d(x_n, x_{n+1})]^r \leq n\alpha[\Theta(d(x_n, x_{n+1})) - 1] \quad \text{for all } \ n \geq n_0,$$

where $\alpha = 1/\beta$. Hence, in all cases there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(x_n, x_{n+1})]^r \leq n\alpha[\Theta(d(x_n, x_{n+1})) - 1] \quad \text{for all } n \geq n_0.$$

From (2.9), we have

$$n[d(x_n, x_{n+1})]^r \leq n\alpha[\Theta(d(x_0, x_1)) - 1] \quad \text{for all } n \geq n_0.$$

On taking the limit as $n \to \infty$ on both sides of the above inequality, we have

$$\lim_{n \to \infty} n[d(x_n, x_{n+1})]^r = 0. \tag{2.11}$$

It follows from (2.11) that there exists $n_1 \in \mathbb{N}$ such that

$$n[d(x_n, x_{n+1})]^r \leq 1 \quad \text{for all } n > n_1.$$

This implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} \quad \text{for all } \ n > n_1.$$

Now, for $m > n > n_1$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Since $0 < r < 1$, $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A and B, respectively. Next, we assume that there exist elements $u \in A$ and $v \in B$ such that

$$x_n \to u \text{ and } y_n \to v \; as \; n \to \infty.$$

Taking limit as $n \to \infty$ in (2.6), we obtain that

$$d(u,v) = dist(A,B). \tag{2.12}$$

Now, we claim that $v \in Fu$. Since $y_n \in Fx_n$, we have

$$D(y_n, Fu) \leq H(Fx_n, Fu).$$

Taking limit as $n \to \infty$ on both sides of above inequality, we have

$$\begin{aligned} D(v, Fu) &= \lim_{n\to\infty} D(y_n, Fu) \\ &\leq \lim_{n\to\infty} H(Fx_n, Fu) \\ &= 0. \end{aligned}$$

As $Fu \in K(B)$, $D(v, Fu) = 0$ implies that $v \in Fu$. By (2.12), we have

$$\begin{aligned} D(u, Fu) &\leq d(u,v) \\ &= dist(A,B) \\ &\leq D(u, Fu), \end{aligned}$$

which implies that $D(u, Fu) = dist(A,B)$ and hence u is a best proximity point of F in A. □

Remark 2.3. In the next theorem, we replace the continuity assumption on F with the following condition:

If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k. If the above condition is satisfied then we say that the set A satisfies α-subsequential property.

Theorem 2.4. *Let (X,d) be a complete metric space and (A,B) a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $F : A \to K(B)$ be a multivalued mapping such that conditions (i)-(iv) of Theorem 2.2 are satisfied. Then F has a best proximity point in A provided that A satisfies α-subsequential property.*

Proof. Following arguments similar to those in the proof of Theorem 2.2, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in A and B, respectively, such that

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{2.13}$$

$$x_n \to u \in A \text{ and } y_n \to v \in B \text{ as } n \to \infty, \tag{2.14}$$

and

$$d(u,v) = dist(A,B). \tag{2.15}$$

By given assumption, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Since $y_{n(k)} \in Fx_{n(k)}$ for all $k \geq 1$, applying condition (iv) of Theorem 2.2, we obtain that

(2.16)

$$\begin{aligned} 1 < \Theta(D(y_{n(k)}, Fu)) &\leq \Theta(H(Fx_{n(k)}, Fu)) \\ &\leq \alpha(x_{n(k)}, u)\Theta(H(Fx_{n(k)}, Fu)) \\ &= (\Theta(d(x_{n(k)}, u)) + \lambda(D(u, Fx_{n_k}) - dist(A,B)))^k. \end{aligned}$$

On taking limit as $k \to \infty$ in (2.16) and using the continuity of Θ, we have $\Theta(D(v, Fu)) = 1$. Therefore, by (Θ_2) we obtain that $D(v, Fu) = 0$. As shown in the proof of Theorem 2.2, we have $D(u, Fu) = dist(A,B)$ and hence u is a best proximity point of F in A. □

Remark 2.5. To obtain the uniqueness of the best proximity point of multivalued almost Θ-contraction mappings, we propose the following $\mathcal{H}$ condition:

$\mathcal{H}$: for any best proximity points x_1, x_2 of mapping F, we have

$$\alpha(x_1, x_2) \geq 1.$$

Theorem 2.6. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$ and $F : A \to K(B)$ be a continuous multivalued mapping satisfying the conditions of Theorem 2.2 (respectively in Theorem 2.4). Then the mapping F has a unique best proximity point provided that it satisfies the condition $\mathcal{H}$.*

Proof. Let x_1, x_2 be two best proximity points of F such that $x_1 \neq x_2$, then by the given hypothesis $\mathcal{H}$, we have $\alpha(x_1, x_2) \geq 1$ and $D(x_1, Fx_1) = dist(A,B) = D(x_2, Fx_2)$. Since Fx_1 and Fx_2 are compact sets, there exist elements $y_0 \in Fx_1$ and $y_1 \in Fx_2$ such that

$$d(x_1, y_0) = dist(A,B), \qquad d(x_2, y_1) = dist(A,B).$$

By the weak P-property of the pair (A,B), we have

$$d(x_1, x_2) \leq d(y_0, y_1).$$

Since F is multivalued almost Θ-contraction mapping, we obtain that

$$\begin{aligned} \Theta(d(x_1, x_2)) \leq \Theta(d(y_0, y_1)) &\leq \Theta(H(Fx_1, Fx_2)) \\ &\leq \alpha(x_1, x_2)\Theta(H(Fx_1, Fx_2)) \\ &\leq [\Theta(d(x_1, x_2) + \lambda(D(x_2, Fx_1) - dist(A,B)))]^k \\ &= (\Theta(d(x_1, x_2)))^k \end{aligned}$$

$$< \Theta(d(x_1, x_2)),$$

a contradiction. Hence $d(x_1, x_2) = 0$ and $x_1 = x_2$. □

If the pair (A, B) satisfies weak P-property, then it satisfies the P-property, we have the following corollaries:

Corollary 2.7. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Suppose that a continuous mapping $F : A \to K(B)$ satisfies the following properties:*

(i) *$Fx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the P-property;*
(ii) *F is multivalued α-proximal admissible mapping;*
(iii) *there exist $x_0, x_1 \in A_0$ and $y_0 \in Fx_0 \subseteq B_0$ such that $d(x_1, y_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$;*
(iv) *F is multivalued almost Θ-contraction.*

Then F has a best proximity point in A.

Corollary 2.8. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that $A_0 \neq \emptyset$. Let $F : A \to K(B)$ be a multi-valued mapping such that conditions (i)-(iv) of Corollary 2.7 are satisfied. Then F has a best proximity point in A provided that A has $\alpha-$subsequential property.*

Now we give an example to support Theorem 2.2.

Example 2.9. Let $X = \mathbb{R}^2$ be a usual metric space. Let

$$A = \{(-2, 2), (2, 2), (0, 4)\} \tag{2.17}$$

and

(2.18)
$$B = \{]-8, \gamma[: \gamma \in [-8, 0]\} \cup \{]8, \gamma[: \gamma \in [-8, 0]\} \cup \{]\beta, -8[: \beta \in]-8, 8[\}.$$

Then $dist(A, B) = 8$, $A_0 = \{(-2, 2), (2, 2)\}$ and $B_0 = \{(-8, 0), (8, 0)\}$.

Define the mapping $F : A \to K(B)$ by

$$Fx = \begin{cases} \{(-8, 0)\} & \text{if } x = (-2, 2) \\ \{(8, 0)\} & \text{if } x = (2, 2) \\ \{]\beta, -8[: \beta \in]-8, 8[\} & \text{if } x = (0, 4). \end{cases}$$

and $\alpha : A \times A \to [0, \infty)$ by

$$\alpha((x, y), (u, v)) = \frac{11}{10}. \tag{2.19}$$

Clearly, $F(A_0) \subseteq B_0$. For $(-2, 2), (2, 2) \in A$ and $(-8, 0), (8, 0) \in B$, we have

$$\begin{cases} d((-2, 2), (-8, 0)) = dist(A, B) = 8, \\ d((2, 2), (8, 0)) = dist(A, B) = 8. \end{cases}$$

Note that

$$d((-2,2),(2,2)) < d((-8,0),(8,0)). \tag{2.20}$$

that is, the pair (A, B) has weak P-property. Also, F is α-proximal admissible mapping. Now we show that F is multivalued almost Θ-contraction where $\Theta :]0,\infty[\to]1,\infty[$ is given by $\Theta(t) = 5^t$.

Note that

$$\alpha((-2,2),(2,2))\Theta[H(F(-2,2),F(2,2))] = \frac{11}{10}(5^{16}), \tag{2.21}$$

and

$$[\Theta(d((-2,2),(2,2)) + \lambda(D((2,2),(-8,0)) - 8))]^k. \tag{2.22}$$

If we take $k \in]0.829, 1[$ and $\lambda = 4$ in (2.22), we have

$$\frac{11}{10}(5^{16}) < \left(\ 5^{20}\right)^k. \tag{2.23}$$

Similarly,

$$\alpha(x,y)\Theta[H(Fx,Fy)] \leq [\Theta(d(x,y)+\lambda(D(y,Fx)-dist(A,B)))]^k \tag{2.24}$$

holds for the remaining pairs. Hence all the conditions of Theorem 2.2 are satisfied. Moreover, $(-2, 2)$, $(2, 2)$ are best proximity points of F in A.

Remark 2.10. Note that mapping F in the above example does not hold for the case of Nadler [27] as well as for Hancer *et al.* [21]. For if, take $x = (-2,2), y = (2,2) \in A$, we have

$$\Theta(H(Fx,Fy)) = 5^{16} > 5^4 > (5^4)^k = [\Theta(d(x,y))]^k,$$

for $k \in (0, 1)$. Also

$$H(Fx,Fy) = 16 > 4 = d(x,y) > \alpha d(x,y)$$

for $\alpha \in (0, 1)$.

Remark 2.11. In the above example 2.9, the pair (A, B) does not satisfy the P-property and hence the Corollary 2.7 is not applicable in this case.

3. Application to Single Valued Mappings

In this section, we obtain some best proximity point results for singlevalued mappings as applications of our obtained results in section 2.

Definition 3.1. [23] Let (X, d) be a metric space and A, B two subsets of X, a nonself mapping $F : A \to B$ is called α-proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ d(u_1, Fx_1) = dist(A, B), \text{ implies } \alpha(u_1, u_2) \geq 1 \\ d(u_2, Fx_2) = dist(A, B), \end{cases} \tag{3.1}$$

for all $x_1, x_2, u_1, u_2 \in A$ where $\alpha : A \times A \to [0, \infty)$.

Definition 3.2. Let $\alpha : A \times A \to [0, \infty)$ and $\Theta : (0, \infty) \to (1, \infty)$ be a nondecreasing and continuous function. A mapping $F : A \to B$ is called almost Θ-contraction if for any $x, y \in A$, we have

$$\alpha(x, y)\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y) + \lambda(d(y, Fx) - dist(A, B)))]^k, \tag{3.2}$$

where $k \in (0, 1)$ and $\lambda \geq 0$.

Theorem 3.3. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. If $F : A \to B$ is a continuous mapping such that*

(i) *$F(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*
(ii) *F is α-proximal admissible mapping;*
(iii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Fx_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$;*
(iv) *F is almost Θ-contraction,*

then F has a best proximity point in A.

Proof. As for every $x \in X, \{x\}$ is compact in X. Define a multivalued mapping $T : A \to K(B)$ by $Tx = \{Fx\}$ for $x \in A$. The continuity of F implies that T is continuous. Now $F(A_0) \subseteq B_0$ implies that $Tx = \{Fx\} \subseteq B_0$ for each $x \in A_0$. If $x_1, x_2, v_1, v_2 \in A$, $y_1 \in Tx_1 = \{Fx_1\}$ and $y_2 \in Tx_2 = \{Fx_2\}$ are such that

$$\alpha(x_1, x_2) \geq 1, \quad d(v_1, y_1) = dist(A, B), \quad d(v_2, y_2) = dist(A, B). \tag{3.3}$$

That is,

$$\alpha(x_1, x_2) \geq 1, \quad d(v_1, Fx_1) = dist(A, B), \quad d(v_2, Fx_2) = dist(A, B). \tag{3.4}$$

Then we have $\alpha(v_1, v_2) \geq 1$ as F is α-proximal admissible mapping. Hence T is α-proximal admissible mapping.

Suppose there exist $x_0, x_1 \in A_0$ such that $d(x_1, Fx_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$. Let $y_0 \in Tx_0 = \{Fx_0\} \subseteq B_0$. Then $d(x_1, Fx_0) = dist(A, B)$ gives that $d(x_1, y_0) = dist(A, B)$. By condition (iii), there exist $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1, y_0) = dist(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

Since F is almost Θ-contraction, we have

(3.5)
$$\alpha(x,y)\Theta(H(Tx,Ty)) = \alpha(x,y)\Theta[d(Fx,Fy)] \leq [\Theta(d(x,y)+\lambda(D(y,Fx)-dist(A,B)))]^k,$$

for any $x, y \in A$, which implies that T is multivalued almost Θ-contraction. Thus, all the conditions of Theorem 2.2 are satisfied and hence T has a best proximity point x^* in A. Thus we have $D(x^*, Tx^*) = dist(A,B)$ and hence $d(x^*, Fx^*) = dist(A,B)$, that is x^* is a best proximity point of F in A. □

Theorem 3.4. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. If $F : A \to B$ is a mapping such that conditions (i)-(iv) of Theorem 3.3 are satisfied, then F has a best proximity point in A provided that A satisfies α-subsequential property.*

Proof. Let $T : A \to K(B)$ be as given in proof of Theorem 3.3. Following arguments similar to those in the proof of Theorem 3.3, we obtain that

(i) $Tx \subseteq B_0$ for each $x_0 \in A_0$;
(ii) T is multi-valued α-proximal admissible mapping;
(iii) there exist $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1, y_0) = dist(A,B)$ and $\alpha(x_0, x_1) \geq 1$;
(iv) T is multivalued almost Θ-contraction.

Thus, all the conditions of Theorem 2.4 are satisfied and hence T has a best proximity point x^* in A, that is,

$$D(x^*, Tx^*) = dist(A,B).$$

Consequently, $d(x^*, Fx^*) = dist(A,B)$ and x^* is a best proximity point of F in A. □

Corollary 3.5. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. If $F : A \to B$ is a continuous mapping such that*

(i) *$F(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;*
(ii) *F is α-proximal admissible mapping;*
(iii) *there exists $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A,B)$ and $\alpha(x_0, x_1) \geq 1$;*
(iv) *F is almost Θ-contraction,*

then F has a best proximity point in A.

Proof. Replace the condition of weak P-property with P-property in Theorem 3.3. □

Corollary 3.6. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. If $F : A \to B$ is a mapping such that conditions (i)-(iv) of Corollary* 3.5 *are satisfied. Then F has a best proximity point in A provided that A satisfies $\alpha-$ subsequential property.*

Proof. Replace the condition of weak P-property with P-property in Theorem 3.4. □

4. Fixed point results for single and multi-valued mappings

In this section, fixed points of singlevalued and multivalued almost Θ-contraction mappings are obtained.

Taking $A = B = X$ in Theorem 2.2 (Theorem 2.4), we obtain corresponding fixed point results for multivalued almost Θ-contraction mappings.

Theorem 4.1. *Let (X, d) be a complete metric space. If $F : X \to K(X)$ is a continuous mapping satisfying*

(i) *F is α-proximal admissible mapping;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$;*
(iii) *F is multivalued almost Θ-contraction,*

then F has a fixed point in X.

Theorem 4.2. *Let (X, d) be a complete metric space. Let $F : X \to K(X)$ be a multi-valued mapping such that conditions (i)-(iii) of Theorem 4.1 are satisfied. Then F has a fixed point in X provided that X satisfies $\alpha-$subsequential property.*

Taking $A = B = X$ in Theorem 3.3 (in Theorem 3.4), we obtain the corresponding fixed point results of almost Θ-contraction mappings.

Theorem 4.3. *Let (X, d) be a complete metric space. Let $F : X \to X$ be a continuous mapping satisfying*

(i) *F is α-proximal admissible mapping;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$;*
(iii) *F is almost Θ-contraction.*

Then F has a fixed point in X.

Theorem 4.4. *Let (X, d) be a complete metric space. Let $F : X \to X$ be a multi-valued mapping such that conditions (i)-(iii) of Theorem 4.1 are satisfied. Then F has a fixed point in X provided that X has a $\alpha-$subsequential property.*

Remark 4.5. In Theorem 4.1 (respectively in 4.3)

(i) If we take $\alpha(x,y) = 1$, we obtain the main results of Durmaz [17] and Altun [3].
(ii) Taking $\lambda = 0$ and $\alpha(x,y) = 1$, we obtain the main result of Hancer *et al.* [21] and Jelli [23], respectively.
(iii) Taking $\alpha(x,y) = 1$ and $\Theta(t) = e^t$, we obtain the main result of Berinde [11] and [9].
(iv) Taking $\alpha(x,y) = 1$, $\lambda = 0$ and $\Theta(t) = e^t$, we obtain the main result of Nadler [27] and Banach [6].

5. Application to Nonlinear Differential Equations

Let $C([0,1])$ be the set of all continuous functions defined on $[0,1]$ and $d : C([0,1]) \times C([0,1]) \to \mathbb{R}$ be the metric defined by

$$d(x,y) = ||x-y||_\infty = \max_{t\in[0,1]} |x(t) - y(t)|. \tag{5.1}$$

It is known that $(C([0,1]), d)$ is a complete metric space.

Let us consider the two-point boundary value problem of the second-order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t,x(t)) & t \in [0,1]; \\ x(0) = x(1) = 0 \end{cases} \tag{5.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

The Green function associated with (5.2) is defined by

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \le t \le s \le 1, \\ s(1-t) & \text{if } 0 \le s \le t \le 1. \end{cases} \tag{5.3}$$

Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function.

Assume that the following conditions hold:

(i) $|f(t,a) - f(t,b)| \le \max_{a,b\in\mathbb{R}} |a-b|$ for all $t \in [0,1]$ and $a, b \in \mathbb{R}$ with $\phi(a,b) \ge 0$;
(ii) there exists $x_0 \in C[0,1]$ such that $\phi(x_0(t), Fx_0(t)) \ge 0$ *for all* $t \in [0,1]$ where $F : C[0,1] \to C[0,1]$;
(iii) *for each* $t \in [0,1]$ *and* $x, y \in C[0,1]$, $\phi(x(t), y(t)) \ge 0$ *implies* $\phi(Fx(t), Fy(t)) \ge 0$*;*
(iv) *for each* $t \in [0,1]$*, if* $\{x_n\}$ *is a sequence in* $C[0,1]$ *such that* $x_n \to x$ *in* $C[0,1]$ *and* $\phi(x_n(t), x_{n+1}(t)) \ge 0$ *for all* $n \in \mathbb{N}$*, then* $\phi(x_n(t), x(t)) \ge 0$ *for all* $n \in \mathbb{N}$.

We now prove the existence of a solution of the second order differential equation (5.2).

Theorem 5.1. *Under the assumptions (i)-(iv), (5.2) has a solution in* $C^2([0,1])$.

Proof. It is well known that $x \in C^2([0,1])$ is a solution of (5.2) is equivalent to $x \in C([0,1])$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds, \quad t \in [0,1]. \tag{5.4}$$

Let $F : C[0,1] \to C[0,1]$ be a mapping defined by

$$Fx(t) = \int_0^1 G(t,s) f(s, x(s)) ds. \tag{5.5}$$

Suppose that $x, y \in C([0,1])$ such that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$. By applying (i), we obtain that

$$\begin{aligned} |Fu(x) - Fv(x)| &= \int_0^1 G(t,s) f(s, x(s)) ds - \int_0^1 G(t,s) f(s, y(s)) ds \\ &\leq \int_0^1 G(t,s)[f(s,x(s)) - f(s,y(s))] ds \\ &\leq \int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s))| ds \\ &\leq \int_0^1 G(t,s) \cdot (\max |x(s) - y(s)|) ds \\ &\leq ||x-y||_\infty \cdot \sup_{t \in [0,1]} \left(\int_0^1 G(t,s) ds \right). \end{aligned}$$

Since $\int\limits_0^1 G(t,s) ds = -(t^2/2) + (t/2)$, for all $t \in [0,1]$, we have

$$\sup_{t \in [0,1]} \left(\int_0^1 G(t,s) ds \right) = \frac{1}{8}.$$

It follows that

$$||Fx - Fy||_\infty \leq \frac{1}{8}(||x-y||_\infty. \tag{5.6}$$

Taking exponential on the both sides, we have

$$\begin{aligned} e^{||Fx-Fy||_\infty} &\leq e^{\frac{1}{8}(||x-y||_\infty)} \\ &= [e^{(||x-y||_\infty)}]^{\frac{1}{8}}, \end{aligned} \tag{5.7}$$

for all $x, y \in C[0,1]$. Now consider a function $\Theta : (0,\infty) \to (1,\infty)$ by $\Theta(t) = e^t$. Define

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \phi(x(t), y(t)) \geq 0, t \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Then from (5.7) with $k = \frac{1}{8}$, we obtain that

$$\alpha(x, y)\Theta(||Fx - Fy||_\infty) \le [\Theta(d(x, y))]^k \le [\Theta(d(x, y) + \lambda d(y, Fx))]^k.$$

Therefore, the mapping F is almost Θ-contraction.

From (ii) there exists $x_0 \in C[0, 1]$ such that $\alpha(x_0, Fx_0) \ge 1$. Next, for any $x, y \in C[0, 1]$ with $\alpha(x, y) \ge 1$, we have

$$\begin{aligned} & \phi(x(t), y(t)) \ge 0 \quad \text{for all } t \in [0, 1] \\ \Rightarrow \quad & \phi(Fx(t), Fy(t)) \ge 0 \quad \text{for all } t \in [0, 1] \\ \Rightarrow \quad & \alpha(Fx, Fy) \ge 1, \end{aligned}$$

and hence F is α-proximal admissible. It follows from Theorem 4.3 that F has a fixed point x in $C([0, 1])$ which in turns is the solution of (5.2). □

6. Conclusion

This paper is concerned with the existence and uniqueness of the best proximity point results for Berinde type contractive conditions via auxiliary function $\Theta \in \Omega$ in the framework of complete metric spaces. Also, some fixed point results as a special cases of our best proximity point results in the relevant contractive conditions are studied. Moreover, the corresponding fixed point results are obtained. An example is discussed to show the significance of the investigation of this paper. An application to a nonlinear differential equation is presented to illustrate the usability of the new theory.

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