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Bornological Completion of Locally Convex Cones

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Abstract. In this paper, firstly, we obtain some new results about bornological convergence in locally convex cones (which was studied in [1]) and then we introduce the concept of bornological completion for locally convex cones. Also, we prove that the completion of a bornological locally convex cone is bornological. We illustrate the main result by an example.

1. Introduction

A nonempty set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers is called a *cone* whenever the addition is associative and commutative, there is a neutral element $0 \in \mathcal{P}$ and for the scalar multiplication the usual associative and distributive properties hold. Therefore, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a+\beta a$, $\alpha(a+b) = \alpha a+\alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

The theory of locally convex cones as developed in [6] and [8], uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the latter. For recent researches see [4, 5].

A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:

(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

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Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there exists $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0, a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \text{ resp. } (a)U = \{b \in P : (a, b) \in U\}, \; U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

Let $\mathcal{P}$ be a cone. A subset B of $\mathcal{P}^2$ is called *uniformly convex* whenever it has the properties (U_1) and (U_3). For $F \subseteq \mathcal{P}^2$, $uch(F)$ is the smallest uniformly convex subset of $\mathcal{P}^2$ which contains F. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called a *uc-cone* whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$ (see [2]).

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + (+\infty) = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. We set $\mathcal{V} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \left\{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\right\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the linear operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We denote the set of all linear functional on $\mathcal{P}$ by $L(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$ we define *polar* F° as follows

$$F^\circ = \{\mu \in L(\mathcal{P}) : \mu(a) \leq \mu(b) + 1, \; \forall (a, b) \in F\}.$$

Clearly $(\{(0, 0)\})^\circ = L(\mathcal{P})$. A linear functional μ on $(\mathcal{P}, \mathfrak{U})$ is (uniformly) continuous if $\mu \in U^\circ$ for some $U \in \mathfrak{U}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

We shall say that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ has the strict separation property if the following holds:

(SP) For all $a, b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $(a, b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ ([6], II, 2.12).

Also, we shall say that the subset V of $\mathcal{P}^2$ has the property (CP) if the following holds:

(CP) if $(a, b) \notin V$, then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in V$.

Suppose that $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone. We shall say that $F \subseteq \mathcal{P}^2$ is u-bounded (uniformly bounded) if it is absorbed by each $U \in \mathfrak{U}$. A subset A of $\mathcal{P}$ is called bounded above (below) whenever $A \times \{0\}$(res. $\{0\} \times A$) is u-bounded (see [2]).

In locally convex cone $(\mathcal{P}, \mathfrak{U})$ the closure of $a \in \mathcal{P}$ was defined to be the set

$$\overline{a} = \bigcap_{U \in \mathfrak{U}} U(a),$$

(see [6], chapter I). The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called separated if $\overline{a} = \overline{b}$ implies $a = b$ for $a, b \in \mathcal{P}$. It was proved in [6] that a locally convex cone is separated if and only if its symmetric topology is Hausdorff.

In [6], a dual pair is defined as follows.

A dual pair $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping

$$(a, x) \to < a, x >: \mathcal{P} \times \mathcal{Q} \to \overline{\mathbb{R}}.$$

If $(\mathcal{P}, \mathcal{Q})$ is a dual pair, then every $x \in \mathcal{Q}$ is a linear mapping on $\mathcal{P}$. We denote the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $x \in \mathcal{Q}$ continuous by $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$. In fact $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q}))$ is the projective limit of $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ by $x \in \mathcal{Q}$ as linear mappings on $\mathcal{P}$ (projective limits of locally convex cones were defined in [7]).

Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair. We shall say that a subset B of $\mathcal{P}$ is $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below whenever it is bounded below in locally convex cone $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q}))$. Let $\mathfrak{B}$ be a collection of $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below subsets of $\mathcal{P}$ such that

(a) $\alpha B \in \mathfrak{B}$ for all $B \in \mathfrak{B}$ and $\alpha > 0$,
(b) For all $X, Y \in \mathfrak{B}$ there is $Z \in \mathfrak{B}$ such that $X \cup Y \subset Z$.
(c) $\mathcal{P}$ is spanned by $\bigcup_{B \in \mathfrak{B}} B$.

For $B \in \mathfrak{B}$ we set

$$U_B = \left\{(x, y) \in \mathcal{Q}^2 :< b, x > \leq < b, y > +1, \text{ for all } b \in B\right\},$$

and $\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P}) = \{U_B : B \in \mathfrak{B}\}$. It is proved in [6], page 37, that $\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P})$ is a convex quasiuniform structure on $\mathcal{Q}$ and $(\mathcal{Q},\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P}))$ is a locally convex cone.

2. Bornological Completion

Bornological locally convex cones studied in [2]. Suppose that $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones and $T : \mathcal{P} \to \mathcal{Q}$ is a linear operator. We shall say T is *u-bounded* if $(T \times T)(F)$ is u-bounded in $\mathcal{Q}^2$ for every u-bounded subset F of $\mathcal{P}^2$. We shall say $(\mathcal{P},\mathfrak{U})$ is a *bornological cone* if every u-bounded linear operator from $(\mathcal{P},\mathfrak{U})$ into any locally convex cone is continuous (see [2]).

Let $\mathcal{P}$ be a cone and U be a uniformly convex subset of $\mathcal{P}$. We set $\mathcal{P}_U = \{a \in \mathcal{P} : \exists\lambda > 0, \quad (0,a) \in \lambda U\}$ and $\mathfrak{U}_U = \{\alpha(U \cap \mathcal{P}_B^2) : \alpha > 0\}$. Then $(\mathcal{P}_U,\mathfrak{U}_U)$ is a locally convex cone (a *uc*-cone). In [2], it is shown that there is the finest convex quasiuniform structure $\mathfrak{U}_\tau$ on locally convex cone $(\mathcal{P},\mathfrak{U})$ such that $\mathcal{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$. The locally convex cone $(\mathcal{P},\mathfrak{U}_\tau)$ is the inductive limit of the *uc*-cones $(\mathcal{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$, where $\mathfrak{B}$ is the collection of all uniformly convex u-bounded subsets of $\mathcal{P}^2$ (the inductive limit of locally convex cones were defined in [7]). If $(\mathcal{P},\mathfrak{U})$ is bornological, then $\mathfrak{U}$ and $\mathfrak{U}_\tau$ are equivalent.

A net $(x_i)_{i\in\mathcal{I}}$ in $(\mathcal{P},\mathfrak{U})$ is called upper (lower) Cauchy if for every $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\alpha,x_\beta) \in U$ (respectively, $(x_\beta,x_\alpha) \in U$) for all $\alpha,\beta \in \mathcal{I}$ with $\beta \geq \alpha \geq \gamma_U$. Also $(x_i)_{i\in\mathcal{I}}$ is called symmetric Cauchy if for each $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta,x_\alpha) \in U$ for all $\alpha,\beta \in \mathcal{I}$ with $\beta,\alpha \geq \gamma_U$.

The concept of bornological convergence studied in [1]. We review some definitions from [1]. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. We shall say that the net $(x_i)_{i\in\mathcal{I}}$ is *upper (lower or symmetric) bornological Cauchy* whenever there is a u-bounded uniformly convex subset U of $\mathcal{P}^2$ such that $(x_i)_{i\in\mathcal{I}}$ is upper (lower or symmetric) Cauchy in the *uc*-cone $(\mathcal{P}_U,\mathfrak{U}_U)$. Similarly, we shall say that the net $(x_i)_{i\in\mathcal{I}}$ is *upper (lower or symmetric) bornological convergent* whenever there is a u-bounded uniformly convex subset U of $\mathcal{P}^2$ such that $(x_i)_{i\in\mathcal{I}}$ is upper (lower or symmetric) convergent in the *uc*-cone $(\mathcal{P}_U,\mathfrak{U}_U)$. We shall say that the locally convex cone $(\mathcal{P},\mathfrak{U})$ is *upper (lower or symmetric) bornological complete* whenever every upper (lower or symmetric) bornological Cauchy sequence is upper (lower or symmetric) bornological convergent. In [1], we proved that every upper (lower or symmetric) complete locally convex cone with (SP) is upper (lower or symmetric) bornological complete. Note that one can apply sequences instead of nets in the studding of bornological convergence (see [1]). In the following, we work with sequences instead of nets.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone and $A \subseteq \mathcal{P}$. We denote the closures of $A \subseteq \mathcal{P}$, with respect to the upper, lower and symmetric topologies by $\overline{A}^u$, $\overline{A}^l$ and $\overline{A}^s$, respectively. For a sequence $(x_n)_{n\in\mathbb{N}}$ in $(\mathcal{P}, \mathfrak{U})$ and $x \in \mathcal{P}$ we write $x_n \xrightarrow{ub} x$, $x_n \xrightarrow{lb} x$ or $x_n \xrightarrow{sb} x$ whenever $(x_n)_{n\in\mathbb{N}}$ is upper, lower or symmetric bornological convergent to x.

Definition 2.1. Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. For $A \subseteq \mathcal{P}$, we set

$$\overline{A}^{ub} = \left\{ a \in \mathcal{P} : \exists (a_n)_{n\in\mathbb{N}} \subseteq A \;\; s.\,t.\; a_n \xrightarrow{ub} a \right\},$$
$$\overline{A}^{lb} = \left\{ a \in \mathcal{P} : \exists (a_n)_{n\in\mathbb{N}} \subseteq A \;\; s.\,t.\; a_n \xrightarrow{lb} a \right\},$$
$$\overline{A}^{sb} = \left\{ a \in \mathcal{P} : \exists (a_n)_{n\in\mathbb{N}} \subseteq A \;\; s.\,t.\; a_n \xrightarrow{sb} a \right\},$$

and we call them upper, lower and symmetric bornological closure of A, respectively.

Definition 2.2. Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone and $A \subseteq \mathcal{P}$. The set A is called upper, lower or symmetric bornological closed whenever $\overline{A}^{ub} = A$, $\overline{A}^{lb} = A$ or $\overline{A}^{sb} = A$, respectively.

If M is a subcone of $\mathcal{P}$, then the upper, lower and symmetric bornological closure of M is also a subcone of $\mathcal{P}$, since the algebraic operations of $\mathcal{P}$ is continuous with respect to bornological convergence (see [1], Lemma 2.8).

Proposition 2.3. *Let $(\mathcal{P}, \mathfrak{U})$ be an upper (a lower or a symmetric) bornological complete locally convex cone and M be an upper (a lower or a symmetric) bornological closed subcone of $\mathcal{P}$. Then M is an upper (a lower or a symmetric) bornological complete subcone of $\mathcal{P}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be an upper (a lower or a symmetric) bornological Cauchy sequence in M. Then $(a_n)_{n\in\mathbb{N}}$ is upper (lower or symmetric) bornological Cauchy in $(\mathcal{P}, \mathfrak{U})$. Since $(\mathcal{P}, \mathfrak{U})$ is upper (lower or symmetric) bornological complete, there is $a \in \mathcal{P}$ such that $(a_n)_{n\in\mathbb{N}}$ is upper (lower or symmetric) bornological convergent to a. Now, we have $a \in M$, since M is upper (lower or symmetric) bornological closed. □

Proposition 2.4. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. For $A, B \subseteq \mathcal{P}$ and $\alpha > 0$ the following statements hold:*

(a) $\overline{A}^{ub} \subseteq \overline{A}^u$, $\overline{A}^{lb} \subseteq \overline{A}^l$ *and* $\overline{A}^{sb} \subseteq \overline{A}^s$,

(b) $\overline{A}^{sb} \subseteq \overline{A}^{ub} \bigcap \overline{A}^{lb}$,

(c) $\overline{\alpha A}^{ub} = \alpha \overline{A}^{ub}$, $\overline{\alpha A}^{lb} = \alpha \overline{A}^{lb}$ *and* $\overline{\alpha A}^{sb} = \alpha \overline{A}^{sb}$, (*d*) $\overline{A}^{ub}+\overline{B}^{ub}\subseteq\overline{A+B}^{ub}$, $\overline{A}^{lb}+\overline{B}^{lb}\subseteq\overline{A+B}^{lb}$ *and* $\overline{A}^{sb}+\overline{B}^{sb}\subseteq\overline{A+B}^{sb}$.

Proof. For (a), let $a\in\overline{A}^{ub}$. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$, which is upper bornological convergent to a. Since every upper bornological convergent is upper convergent (see [1]), we conclude that $(a_n)_{n\in\mathbb{N}}$ is convergent to a. This implies that $a\in\overline{A}^{u}$. The other statements are proved in a similar way.

For (b), let $a\in\overline{A}^{sb}$. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$, which is symmetric bornological convergent to a. This means that there is a uniformly convex u-bounded subset U of $\mathcal{P}^2$ such that $(a_n)_{n\in\mathbb{N}}$ is symmetric convergent to a in the uc-cone $(\mathcal{P}_U,\mathfrak{U}_U)$. Since the symmetric topology of $(\mathcal{P}_U,\mathfrak{U}_U)$ is finer than the upper and lower topologies, we conclude that $(a_n)_{n\in\mathbb{N}}$ is upper and lower convergent to a in $(\mathcal{P}_U,\mathfrak{U}_U)$. Therefore $(a_n)_{n\in\mathbb{N}}$ is upper and lower bornological convergent to a. Then $a\in\overline{A}^{ub}\bigcap\overline{A}^{lb}$.

For (c), let $a\in\overline{\alpha A}^{ub}$. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $(\alpha a_n)_{n\in\mathbb{N}}$ is upper bornological convergent to a. This shows that $(a_n)_{n\in\mathbb{N}}$ is upper bornological convergent to $\dfrac{1}{\alpha}a$. Therefore, $\dfrac{1}{\alpha}a\in\overline{A}^{ub}$. In the other words $a\in\alpha\overline{A}^{ub}$. The converse inclusion and the other statements are proved similarly.

Since the algebraic operations of a cone are continuous with respect to bornological convergence, all statements of (d) are true. □

Corollary 2.5. *In a locally convex cone every upper (lower or symmetric) closed subset is upper (lower or symmetric) bornological closed.*

We know that the symmetric bornological convergence implies the symmetric convergence, but the converse is not true in general (see [2]). In the following proposition we find a condition under which the symmetric convergence concludes the symmetric bornological convergence.

Proposition 2.6. *Let $(\mathcal{P},\mathfrak{U})$ be a separated locally convex cone such that for each symmetric compact subset K of $\mathcal{P}$ there is a u-bounded subset B of $\mathcal{P}^2$ such that K is symmetric compact in $(\mathcal{P}_B,\mathfrak{U}_B)$. Then the symmetric convergence implies the symmetric bornological convergence.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to the symmetric topology of $(\mathcal{P},\mathfrak{U})$. Then the set $A=\{x_n: n\in\mathbb{N}\}\cup\{x\}$ is symmetric compact in $(\mathcal{P},\mathfrak{U})$ and by the assumption it is compact in the symmetric topology of $(\mathcal{P}_B,\mathfrak{U}_B)$ for some u-bounded subset B of $\mathcal{P}^2$. Since the inclusion mapping $I:\mathcal{P}_B\mapsto\mathcal{P}$ is continuous with respect to the symmetric topologies of $\mathcal{P}_B$ and $\mathcal{P}$ then the symmetric topologies of $\mathcal{P}_B$ and $\mathcal{P}$ coincide on A (in fact each continuous one to one mapping from a compact space into a Hausdorff space is a homeomorphism). Then we conclude that $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to the symmetric topology of $(\mathcal{P}_B, \mathfrak{U}_B)$. This shows that $(x_n)_{n\in\mathbb{N}}$ is symmetric bornological convergent to x. □

Corollary 2.7. *Let $(\mathcal{P}, \mathfrak{U})$ be a separated locally convex cone such that for each symmetric compact subset K of $\mathcal{P}$ there is a u-bounded subset B of $\mathcal{P}^2$ such that K is symmetric compact in $(\mathcal{P}_B, \mathfrak{U}_B)$. Then for each subset $A \subseteq \mathcal{P}$, we have $\overline{A}^s = \overline{A}^{sb}$ by Propositions 2.4 and 2.6.*

Example 2.8. Consider the locally convex cone $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$, and let c_0 be the cone of all sequences in $\overline{\mathbb{R}}$ which are symmetric convergent to 0. We set $\mathcal{P} = \prod_{c_0} \overline{\mathbb{R}}$. We consider on $\mathcal{P}$ the projective limit convex quasiuniform structure and denote it by $\mathfrak{U}$. Define $x_n \in \mathcal{P}$ by $(x_n)_\mu := \mu(n) = \mu_n$, where $n \in \mathbb{N}$ and $\mu \in c_0$. Then $(x_n)_{n\in\mathbb{N}}$ is symmetric convergent to 0 in $(\mathcal{P}, \mathfrak{U})$, because this is true for each component. The sequence $(x_n)_{n\in\mathbb{N}}$ is not symmetric bornological convergent to 0, otherwise, by Proposition 2.4 from [1], there exist a u-bounded uniformly convex subset B of $\mathcal{P}^2$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive reals converging to infinity such that $(\lambda_n x_n, 0) \in B$ and $(0, \lambda_n x_n) \in B$ for all $n \in \mathbb{N}$. We project the first relation on the component $\mu = \left(\frac{1}{\sqrt{\lambda_n}}\right)_{n\in\mathbb{N}}$. Then we have $(\lambda_n \cdot \frac{1}{\sqrt{\lambda_n}}, 0) \in (P_\mu \times P_\mu)(B)$, where P_μ is the component mapping. Then $(\sqrt{\lambda_n}, 0) \in (P_\mu \times P_\mu)(B)$ for all $n \in \mathbb{N}$. This is a contradiction, since $(P_\mu \times P_\mu)(B)$ is u-bounded in $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$. Then (x_n) is not symmetric bornological convergent to 0.

In fact in the above example, the subset $K = \{x_n : n \in \mathbb{N}\} \cup \{0\}$ of $\mathcal{P}$ is symmetric compact in $(\mathcal{P}, \mathfrak{U})$, but it is not symmetric compact in any $(\mathcal{P}_B, \mathfrak{U}_B)$. This shows that the conditions of Proposition 2.6 is necessary.

The concept of completion for locally convex cones has been established in [3]. For a locally convex cone $(\mathcal{P}, \mathfrak{U})$ with (SP), the completion $\hat{\mathcal{P}}$ of $\mathcal{P}$, is the subcone $\bigcap_{U\in\mathfrak{U}}(\mathcal{P} + (\{0\} \times U^\circ)^\circ)$ of $L(\mathcal{P}^*)$ endowed with the convex convex quasiuniform structure $\hat{\mathfrak{U}} = \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$, where $\mathfrak{B} = \{U^\circ : U \in \mathfrak{U}\}$. For details see [3].

Lemma 2.9. *The completion of a uc-cone with (SP) is a uc-cone.*

Proof. Let $(\mathcal{P}, \mathfrak{U})$ be a uc-cone with (SP) and $\mathfrak{U} = \{\alpha U : \alpha > 0\}$. As mentioned above, the completion $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ of $(\mathcal{P}, \mathfrak{U})$ is the subcone $\bigcap_{U\in\mathfrak{U}}(\mathcal{P} + (\{0\} \times U^\circ)^\circ)$ of $L(\mathcal{P}^*)$ endowed with the convex convex quasiuniform structure $\mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$, where $\mathfrak{B} = \{U^\circ : U \in \mathfrak{U}\}$. We have

$\mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}},\mathcal{P}^*) = \{\alpha U_{U^\circ} : \alpha > 0\}$, since $U_{(\alpha U)^\circ} = U_{\frac{1}{\alpha}U^\circ} = \alpha U_{U^\circ}$. This shows that the convex quasiuniform structure $\hat{\mathfrak{U}} = \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}},\mathcal{P}^*)$ is created by U_{U°. Therefore $(\hat{\mathcal{P}},\hat{\mathfrak{U}})$ is a *uc*-cone. □

Proposition 2.10 ([3, Proposition 2.12])**.** *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones with (SP) and $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ be a continuous linear mapping. Then T has an extension $\hat{T}$ which is a continuous linear mapping of $(\hat{\mathcal{P}},\hat{\mathfrak{U}})$ into $(\hat{\mathcal{Q}},\hat{\mathcal{W}})$.*

Theorem 2.11. *The completion of a bornological locally convex cone with (SP) is bornological.*

Proof. Let $(\mathcal{P},\mathfrak{U})$ be a bornological locally convex cone with (SP). Then $(\mathcal{P},\mathfrak{U})$ is the inductive limit of *uc*-cones $(\mathcal{P}_B,\mathfrak{U}_B)_{B\in\mathcal{B}}$, where $\mathcal{B}$ is the collection of all uniformly convex u-bounded subsets of $(\mathcal{P},\mathfrak{U})$ which have (CP). For every $B \in \mathcal{B}$, $(\mathcal{P}_B,\mathfrak{U}_B)$ is a *uc*-cone. Therefore $(\hat{\mathcal{P}}_B,\hat{\mathfrak{U}}_B)$ is a *uc*-cone by Lemma 2.9 and then it is bornological (see [2]). We know that the locally convex cone $(\mathcal{P},\mathfrak{U})$ is the inductive limit of *uc*-cones $(\mathcal{P}_B,\mathfrak{U}_B)_{B\in\mathcal{B}}$ by the inclusion mappings $I_B : \mathcal{P}_B \to \mathcal{P}$. By the Proposition 2.10, I_B has the extension $\hat{I}_B : \hat{\mathcal{P}}_B \to \hat{\mathcal{P}}$. Now it is enough to show that $(\hat{\mathcal{P}},\hat{\mathfrak{U}})$ is the inductive limit of *uc*-cones $(\hat{\mathcal{P}}_B,\hat{\mathfrak{U}}_B)_{B\in\mathcal{B}}$ by the mappings $\hat{I}_B : \hat{\mathcal{P}}_B \to \hat{\mathcal{P}}$, $B \in \mathcal{B}$. For every $B \in \mathcal{B}$, $\hat{I}_B$ is continuous by the Proposition 2.10. Let $\mathcal{W}$ be an arbitrary convex quasiuniform structure on $\hat{\mathcal{P}}$, that makes all $\hat{I}_B$ continuous. We show that $\hat{\mathfrak{U}}$ is finer than $\mathcal{W}$. Let $W \in \mathcal{W}$. For every $B \in \mathcal{B}$, there is $\alpha_B > 0$ such that $(\hat{I}_B \times \hat{I}_B)(\alpha_B U_{U_B^\circ}) = \alpha_B U_{U_B^\circ} \subseteq W$. We set $V = uch(\bigcup_{B\in\mathcal{B}} \alpha_B U_{U_B^\circ})$. Then we have $V \subseteq W$ and $V \in \hat{\mathfrak{U}}$. Then $(\hat{\mathcal{P}},\hat{\mathfrak{U}})$ is the inductive limit of *uc*-cones $(\hat{\mathcal{P}}_B,\mathfrak{U}_B)_{B\in\mathcal{B}}$ and then it is bornological. □

In the following we suppose $(\mathcal{P},\mathfrak{U})$ is a locally convex cone with (SP) such that its bornologification $(\mathcal{P},\mathfrak{U}_\tau)$ has (SP). Let $(\widehat{\mathcal{P}_\tau},\widehat{\mathfrak{U}_\tau})$ be the completion of $\mathcal{P}$ under the convex quasiuniform structure $\mathfrak{U}_\tau$. Since for each locally convex cone $(\mathcal{P},\mathfrak{U})$ the locally convex cone $(\mathcal{P},\mathfrak{U}_\tau)$ is bornological, Theorem 2.11, shows that $(\widehat{\mathcal{P}_\tau},\widehat{\mathfrak{U}_\tau})$ is a bornological locally convex cone. Now, we prove the main result of this paper.

Theorem 2.12. *For any locally convex cone $(\mathcal{P},\mathfrak{U})$ with (SP) there exists a unique upper bornological complete locally convex cone $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ with the following properties:*

- (a) *$\mathcal{P}$ is upper bornological dense in $\tilde{\mathcal{P}}$,*
- (b) *Every u-bounded linear mapping T from $(\mathcal{P},\mathfrak{U})$ into an upper bornological complete locally convex cone $(\mathcal{Q},\mathcal{W})$ has a u-bounded linear extension $\tilde{T} : \tilde{\mathcal{P}} \to \mathcal{Q}$.*

Proof. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP). Then the completion $(\widehat{\mathcal{P}_\tau},\widehat{\mathfrak{U}_\tau})$ of $(\mathcal{P},\mathfrak{U}_\tau)$ is an upper complete locally convex cone. Then it is upper bornological complete. We set $\tilde{\mathcal{P}}=\overline{\mathcal{P}}^{ub}$ (the upper bornological closure of $\mathcal{P}$ in $\widehat{\mathcal{P}_\tau}$) and $\tilde{\mathfrak{U}}=\{V\cap\tilde{\mathcal{P}}^2 : V\in\widehat{\mathfrak{U}_\tau}\}$. Then $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ is an upper complete locally convex cone with (SP) by Proposition 2.3. Also $\mathcal{P}$ is upper bornological dense in $\tilde{\mathcal{P}}$ by the definition of $\tilde{\mathcal{P}}$. For the uniqueness of $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$, we note that the completion $(\widehat{\mathcal{P}_\tau},\widehat{\mathfrak{U}_\tau})$ of $(\mathcal{P},\mathfrak{U}_\tau)$ is unique by the Proposition 2.14 from [3]. It is easy to see that if $(\mathcal{P},\mathfrak{U})$ is upper bornological complete, then $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})=(\mathcal{P},\mathfrak{U})$. Now, let $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ be a u-bounded linear mapping, then $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W}_\tau)$ is u-bounded too, since $\mathcal{P}^2$ has the same u-bounded subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$. Then T is continuous, since $(\mathcal{P},\mathfrak{U}_\tau)$ is bornological. By Proposition 2.10, T has the continuous extension $\hat{T}:(\widehat{\mathcal{P}_\tau},\widehat{\mathfrak{U}_\tau})\to(\widehat{\mathcal{Q}_\tau},\widehat{\mathcal{W}_\tau})$. Since $(\mathcal{Q},\mathcal{W})$ is upper bornological complete, we have $(\widehat{\mathcal{Q}_\tau},\widehat{\mathcal{W}_\tau})=(\mathcal{Q},\mathcal{W})$. Now, we suppose $\tilde{T}$ is the restriction of $\hat{T}$ on $\tilde{\mathcal{P}}$. Then it is u-bounded, since it is continuous. □

Definition 2.13. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP). We call $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ constructed in the above theorem, the bornological completion of $(\mathcal{P},\mathfrak{U})$.

Proposition 2.14. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP). Then the locally convex cone $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ is symmetric bornological complete.*

Proof. By the definition, $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ is upper bornological complete. Also, we have $\overline{\overline{\mathcal{P}}^{ub}}^{sb}\subseteq\overline{\overline{\mathcal{P}}^{ub}}^{ub}\subseteq\overline{\mathcal{P}}^{ub}$ by Proposition 2.4. Then $\tilde{\mathcal{P}}=\overline{\mathcal{P}}^{ub}$ is symmetric bornological closed. Now, Proposition 2.3 shows that $(\tilde{\mathcal{P}},\tilde{\mathfrak{U}})$ is symmetric bornological complete. □

Corollary 2.15. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP). Then $\mathcal{P}\subseteq\tilde{\mathcal{P}}\subseteq\widehat{\mathcal{P}_\tau}$.*

Example 2.16. Let X be a countable set and $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP) and $\mathfrak{B}$ be a collection of subsets of X such that $(X,\mathfrak{B})$ is a bornological space. We say that the function $f:X\to\mathcal{P}$ is bounded below whenever $f(B)$ is bounded below in $\mathcal{P}$ for each $B\in\mathfrak{B}$. This means that for every $B\in\mathfrak{B}$ and $U\in\mathfrak{U}$ there is $\lambda>0$ such that $\{0\}\times f(B)\subseteq\lambda U$ for all $B\in\mathfrak{B}$. We suppose $l^\infty(X,\mathcal{P})$ is the cone of all functions $f:X\to\mathcal{P}$, which are bounded below on each $B\in\mathfrak{B}$. For $B\in\mathfrak{B}$ and $U\in\mathfrak{U}$, we set

$$V_{B,U}=\{(f,g):f,g\in l^\infty(X,\mathcal{P}),(f(b),g(b))\in U,\forall b\in B\},$$

and $\mathcal{V}_{\mathfrak{B},\mathfrak{U}}=\{V_{B,U}:B\in\mathfrak{B},U\in\mathfrak{U}\}$. Then $\mathcal{V}_{\mathfrak{B},\mathfrak{U}}$ is a convex quasiuniform structure on $l^\infty(X,\mathcal{P})$ and $(l^\infty(X,\mathcal{P}),\mathcal{V}_{\mathfrak{B},\mathfrak{U}})$ is a locally convex cone.

For each continuous linear functional μ on $\mathcal{P}$ and $x \in X$, the function $\mu_x : l^\infty(X, \mathcal{P}) \to \overline{\mathbb{R}}$ acting as $\mu_x(f) = \mu(f(x))$ is a continuous linear functional on $l^\infty(X, \mathcal{P})$. The linearity of μ_x is clear. We prove that it is continuous. There is $B \in \mathfrak{B}$ and $U \in \mathfrak{U}$ such that $x \in B$ and $\mu \in U^\circ$. We claim that $\mu_x \in (V_{B,U})^\circ$. If $(f, g) \in V_{B,U}$, then $(f(x), g(x)) \in U$. Now, since $\mu \in U^\circ$, we have $\mu(f(x)) \leq \mu(g(x)) + 1$. Then $\mu_x(f) \leq \mu_x(g) + 1$. Also the locally convex cone $(l^\infty(X, \mathcal{P}), \mathcal{V}_{\mathfrak{B},\mathfrak{U}})$ has (SP). Indeed, if $(f, g) \notin \rho V_{B,U}$ for some $\rho > 1$, $B \in \mathfrak{B}$ and $U \in \mathfrak{U}$, then there is $b \in B$ such that $(f(b), g(b)) \notin \rho U$. Since $(\mathcal{P}, \mathfrak{U})$ has (SP), there is $\mu \in \mathcal{P}^*$ such that $\mu(f(b) > \mu(g(b))+1$. This shows that $\mu_b(f) > \mu_b(g)+1$.

Let $(\mathcal{P}, \mathfrak{U})$ be a *uc*-cone and $C_c(X, \mathcal{P})$ be the cone of all $f \in l^\infty(X, \mathcal{P})$ such that the support of f i.e. the set $\{x \in X : f(x) \neq 0\}$ is finite. Also we suppose $\mathfrak{U} = \{\varepsilon U : \varepsilon > 0\}$. For each $B \in \mathfrak{B}$ we define

$$C_\circ(B, \mathcal{P}) = \{f \in l^\infty(B, \mathcal{P}) : \forall \varepsilon > 0, \{x \in B : f(x) \notin (0)(\varepsilon U)\} \text{ is finite}\}.$$

We set

$$C_\circ(X, \mathcal{P}) = \bigcap_{B \in \mathfrak{B}} I_B^{-1}(C_\circ(B, \mathcal{P})),$$

where $I_B : l^\infty(X, \mathcal{P}) \to l^\infty(B, \mathcal{P})$ is the restriction map. We claim that $C_\circ(X, \mathcal{P})$ is the bornological completion of $C_c(X, \mathcal{P})$. We prove that $C_\circ(X, \mathcal{P}) = \overline{C_c(X, \mathcal{P})}^{ub}$. Let $f \in C_\circ(X, \mathcal{P})$ and $supp(f) = \{x_1, x_2, \ldots\}$. Then for each $B \in \mathfrak{B}$, $I_B(f) \in C_\circ(B, \mathcal{P})$. This shows that for each $B \in \mathfrak{B}$ and $n \in \mathbb{N}$, $E_n = \{x \in X : f(x) \notin (0)(\frac{1}{n}U)\}$ is finite. Also, since $U \supseteq \dfrac{1}{2}U \supseteq \dfrac{1}{3}U \supseteq \cdots$, we have $E_n \subseteq E_{n+1}$ for each $n \in \mathbb{N}$. Now, we set $f_n := f.\chi_{E_n}$. Since E_n is finite, the support of f_n is finite. This shows that $f_n \in C_c(X, \mathcal{P})$ for each $n \in \mathbb{N}$. We claim that $\{(nf_n, nf) : n \in \mathbb{N}\}$ is u-bounded in $(l^\infty(X, \mathcal{P}), \mathcal{V}_{\mathfrak{B},\mathfrak{U}})$. Let $U \in \mathfrak{U}$, $B \in \mathfrak{B}$ and $b \in B$. For $n \in \mathbb{N}$, if $b \in E_n$, then $(nf_n(b), nf(b)) = (nf(b), nf(b)) \in \triangle \subseteq U$. If for $n \in \mathbb{N}$, $b \notin E_n$, then $(nf_n(b), nf(b)) = (0, nf(b)) \in U$ by the definition of E_n. Then for each $b \in B$, $(nf_n(b), nf(b)) \in U$. This shows that $(nf_n, nf) \in V_{B,U}$ for all $n \in \mathbb{N}$. Therefore $\{(nf_n, nf) : n \in \mathbb{N}\}$ is u-bounded. We set $M = uch(\{(nf_n, nf) : n \in \mathbb{N}\} \bigcup \{(0, f)\})$. Then M is a uniformly convex u-bounded subset of $(l^\infty(X, \mathcal{P}))^2$. We have $(f_n, f) \in \dfrac{1}{n}M$ for each $n \in \mathbb{N}$. This shows that $(f_n)_{n\in\mathbb{N}}$ is upper convergent to f in the *uc*-cone $((l^\infty(X, \mathcal{P}))_M, (\mathcal{V}_{\mathfrak{B},\mathfrak{U}})_M)$. Therefore $(f_n)_{n\in\mathbb{N}}$ is upper bornological-convergent to f.

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