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Weighted Composition Operators Between Extended Lipschitz Algebras on Compact Metric Spaces

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Abstract. In this paper, we provide a complete description of weighted composition operators between extended Lipschitz algebras on compact metric spaces. We give necessary and sufficient conditions for the injectivity and the sujectivity of these operators. We also obtain some sufficient conditions and some necessary conditions for a weighted composition operator between these spaces to be compact.

1. Introduction and Preliminaries

Let X be a Hausdorff space. We denote by $C(X)$ the set of all complex-valued continuous functions on X. Then $C(X)$ is a complex commutative algebra with unit 1_X, the constant function on X with value 1. The set of all bounded function in $C(X)$ is denoted by $C^b(X)$. We know that $C^b(X)$ is a unital commutative complex Banach algebra with unit 1_X when equipped with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\}, \quad \Big(f \in C^b(X)\Big).$$

Note that $C^b(X) = C(X)$ whenever X is compact.

Let X_1 and X_2 be Hausdorff spaces and let A_1 and A_2 be linear subspaces of $C(X_1)$ and $C(X_2)$, respectively. A map $T : A_1 \longrightarrow A_2$ is called a composition operator from A_1 to A_2 if there exists a map $\varphi : X_2 \longrightarrow X_1$ such that $Tf = fo\varphi$ for all $f \in A_1$. Then T is denoted by C_φ and called the composition operator from A_1 to A_2 induced by φ. A map $T : A_1 \longrightarrow A_2$ is called a weighted composition operator

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from A_1 to A_2 if there exist a complex-valued function u on X_2 and a map $\varphi : X_2 \longrightarrow X_1$ such that $Tf = u.(fo\varphi)$ for all $f \in A_1$. Then T is denoted by uC_φ and called the weighted composition operator from A_1 to A_2 induced by u and φ. Clearly, uC_φ is a linear operator. In the case $u = 1_{X_2}$, the weighted composition operator uC_φ reduces to the composition operator C_φ.

Let X be a compact Hausdorff space. A Banach function algebra on X is a complex subalgebra A of $C(X)$ that contains 1_X, separates the points of X and is a unital Banach algebra with an algebra norm $\|\cdot\|$. Let $(A, \|\cdot\|)$ be a Banach function algebra on X. For each $x \in X$, the map $e_{A,X} : A \longrightarrow \mathbb{C}$ defined by $e_{A,X}(f) = f(x)$ for all $f \in A$, is an element of $\Delta(A)$, the character space of A, which is called the evaluation character on A at x. This fact implies that A is semisimple and $\|f\|_X \leq \left\|\hat{f}\right\|_{\Delta(A)} \leq \|f\|$ for all $f \in A$, where $\hat{f}$ is the Gelfand transform of f. Note that the map $x \rightsquigarrow e_{A,X} : X \longrightarrow \Delta(A)$ is an injective continuous mapping from X to $\Delta(A)$ with the Gelfand topology. If this map is surjective, we say that A is natural.

Let (X, d) and (Y, ρ) be metric spaces and let K be a nonempty subset of Y. A map $\varphi : K \longrightarrow X$ is called a Lipschitz mapping from (K, ρ) to (X, d) if there exists a positive constant M such that $d(\varphi(x), \varphi(y)) \leq M\rho(x, y)$ for all $x, y \in K$. A map $\varphi : K \longrightarrow X$ is called a supercontractive mapping from (K, ρ) to (X, d) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$ for all $x, y \in K$ with $0 < \rho(x, y) < \delta$.

Let (X, d) be a metric space. For a complex-valued function f on X, the Lipschitz constant of f in (X, d) is denoted by $p_{(X,d)}(f)$ and defined by

$$p_{(X,d)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in K, x \neq y\right\}.$$

A complex-valued function f is called a Lipschitz function on (X, d) if f is a Lipschitz mapping from (X, d) to the Euclidean metric space $\mathbb{C}$. Clearly, f is a Lipschitz function on (X, d) if and only if $p_{(X,d)}(f) < \infty$. For each $\alpha \in (0, 1]$, the map $d^\alpha : X \times X \longrightarrow \mathbb{R}$ defined by

$$d^\alpha(x, y) = (d(x, y))^\alpha, \quad ((x, y) \in X \times X),$$

is a metric on X and the induced topology on X by d^α coincides with the induced topology on X by d. For $\alpha \in (0, 1]$, we denote by $\mathrm{Lip}(X, d^\alpha)$ the set of all complex-valued bounded Lipschitz functions on (X, d^α). Then $\mathrm{Lip}(X, d^\alpha)$ is a self-adjoint complex subalgebra of $C^b(X)$ containing 1_X

which separates the points of X. Moreover, $\mathrm{Lip}(X, d^\alpha)$ is a unital commutative complex Banach algebra with the α-Lipschitz algebra norm

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f), \quad (f \in \mathrm{Lip}(X, d)).$$

These algebras are called Lipschitz algebras of order α on (X, d) and were first studied by Sherbert in [14] and [15].

Kamowitz and Scheinberg in [10] characterized compact unital endomorphisms of $\mathrm{Lip}(X, d)$ whenever (X, d) is a compact metric space. Jiménez-Vargas and Villegas-Vallecillos in [9] characterized compact composition operators on $\mathrm{Lip}(X, d)$ whenever (X, d) is a metric space, not necessarily compact. Golbaharan and Mahyar in [7] studied weighted composition operators on $\mathrm{Lip}(X, d)$ whenever (X, d) is a compact metric space. They obtained necessary and sufficient conditions for the injectivity, the surjectivity and the compactness of a weighted composition operator on $\mathrm{Lip}(X, d)$. Weighted composition operators between $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(Y, \rho)$ were studied in [2], where (X, d) and (Y, ρ) are metric spaces not necessarily compact.

Let (X, d) be a pointed metric space with a basepoint designated by x_0. We denote by $\mathrm{Lip}_0(X, d)$ the set of all complex-valued Lipschitz functions f on (X, d) for which $f(x_0) = 0$. Then $\mathrm{Lip}_0(X, d)$ is a Banach space with the $p_{(X,d)}(0)$-norm. For further general facts about Lipschitz spaces $\mathrm{Lip}_0(X, d)$, we refer to [16]. Compact composition operators on $\mathrm{Lip}_0(X, d)$ characterized by Jiménez-Vargas and Villegas-Vallecillos in [9] whenever (X, d) is also bounded. Weighted composition operators between $\mathrm{Lip}_0(X, d)$-spaces studied in [6] and the authors obtained some necessary conditions and some sufficient conditions for the injectivity, the surjectivity and the compactness of a weighted composition operator $T = uC_\varphi$ from $\mathrm{Lip}_0(X, d)$ to $\mathrm{Lip}_0(Y, \rho)$.

Let (X, d) be a compact metric space and let K be a nonempty compact subset of X. For $\alpha \in (0, 1]$, we define the extended Lipschitz algebra $\mathrm{Lip}(X, K, d^\alpha)$ as following:

$$\mathrm{Lip}(X, K, d^\alpha) = \{f \in C(X) : f|_K \in \mathrm{Lip}(K, d^\alpha)\}.$$

Then $\mathrm{Lip}(X, K, d^\alpha)$ is a complex subalgebra of $C(X)$ and $\mathrm{Lip}(X, d^\alpha)$ is a complex subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$. Moreover, $\mathrm{Lip}(X, K, d^\alpha) = \mathrm{Lip}(X, d^\alpha)$ whenever $K = X$ and $\mathrm{Lip}(X, K, d^\alpha) = C(X)$ whenever K is finite. It is known [8] that $\mathrm{Lip}(X, K, d^\alpha)$ is natural Banach function algebra on X. We know [5] that $\mathrm{Lip}(X, K, d^\alpha)$ is a regular Banach algebra. Some properties of unital homomorphisms between extended Lipschitz algebras were studied in [5]. For further details of the extended Lipschitz algebras, we refer to [1], [3], [4], [11] and [12].

In this paper we assume that for $j \in \{1, 2\}$, (X_j, d_j) is a compact metric space, K_j is a nonempty compact subset of X_j and $\alpha_j \in (0, 1]$.

In Section 2, we study some properties of weighted composition operators between $\mathrm{Lip}(X_1, K_1, d_1^{\alpha_1})$ and $\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})$. In particular, we show that these operators are bounded. In Section 3, we give some necessary conditions and some sufficient conditions for the injectivity and the surjectivity of a weighted composition operator $T = uC_\varphi$ from $\mathrm{Lip}(X_1, K_1, d_1^{\alpha_1})$ to $\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})$. In Section 4, we give some necessary conditions and some sufficient conditions that a weighted composition operator $T = uC_\varphi$ from $\mathrm{Lip}(X_1, K_1, d_1^{\alpha_1})$ to $\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})$ to be compact.

2. Some Properties of Weighted Composition Operators

Throughout this section we always assume that (X_j, d_j) is a compact metric space, K_j is a nonempty compact subset of X_j, $\alpha_j \in (0, 1]$ and $A_j = \mathrm{Lip}(X_j, K_j, d_j^{\alpha_j})$, where $j \in \{1, 2\}$. For a complex-valued function u on a nonempty set Y, we denote by $\mathrm{coz}(u)$ the set of all $y \in Y$ for which $u(y) \neq 0$.

We first give some sufficient conditions that $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2.

Lemma 2.1. *Let u be a complex-valued continuous function on (X_2, d_2) and let $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(K_2) \subseteq K_1$. Suppose that $\varphi|_{\mathrm{coz}(u)}$ is a continuous mapping from $(\mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and f is a complex-valued continuous function on (X_1, d_1). Then $u \cdot (fo\varphi)$ is a complex-valued continuous function on (X_2, d_2).*

Proof. Assume that $y_0 \in \mathrm{coz}(u)$. Since $\mathrm{coz}(u)$ is an open set in $(X_2, d_2^{\alpha_2})$, φ is a continuous mapping in y_0 and f is a complex-valued continuous function on $(X_1, d_1^{\alpha_1})$. We deduce that $fo\varphi$ is continuous at y_0 and so $u \cdot (fo\varphi)$ is continuous at y_0. Let $y_0 \in X_2 \setminus \mathrm{coz}(u)$ and $\varepsilon > 0$ be given. The continuity of u at y_0 implies that there exists $\delta > 0$ such that

$$|u(y) - u(y_0)| < \frac{\varepsilon}{1 + \|f\|_{X_1}},$$

for all $y \in X_2$ with $d_2(y, y_0) < \delta$. Let $y \in X_2$ with $d_2(y, y_0) < \delta$. Then

$$\begin{aligned} |u \cdot (fo\varphi)(y) - u \cdot (fo\varphi)(y_0)| &= |u(y)|\,|f(\varphi(y))| \\ &= |u(y) - u(y_0)|\,|f(\varphi(y))| \\ &\leq \frac{\varepsilon}{1 + \|f\|_{X_1}} \|f\|_{X_1} \\ &< \varepsilon. \end{aligned}$$

Thus, $u \cdot (fo\varphi)$ is continuous at y_0. Therefore, $u \cdot (fo\varphi)$ is a complex-valued continuous function on (X_2, d_2). □

Theorem 2.2. *Let $u \in A_2$, $\varphi : X_2 \longrightarrow X_1$ be a map for which $\varphi(K_2)$ is a subset of K_1 and let $\varphi|_{\mathrm{coz}(u)}$ be a continuous mapping from $(\mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. If $K_2 \subseteq X_2 \setminus \mathrm{coz}(u)$ or $K_2 \cap \mathrm{coz}(u) \neq \emptyset$ and $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ is a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(K_1, d_1^{\alpha_1})$, then $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2.*

Proof. Let $f \in A_1$. By Lemma 2.1, Tf is a complex-valued continuous function on $(X_2, d_2^{\alpha_2})$. We first assume that $K_2 \subseteq X_2 \setminus \mathrm{coz}(u)$. Then for each $x, y \in K_2$ with $x \neq y$, we have $Tf(x) = Tf(y) = 0$ and so

$$\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x, y)} \leq 1.$$

Therefore, Tf is a complex-valued Lipschitz function on $(K_2, d_2^{\alpha_2})$ and so $Tf \in A_2$.

We now assume that $K_2 \cap \mathrm{coz}(u) \neq \emptyset$ and $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ is a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(K_1, d_1^{\alpha_1})$. Then there exists a positive constant M such that

$$d_1^{\alpha_1}(\varphi(x), \varphi(y)) \leq M d_2^{\alpha_2}(x, y), \tag{2.1}$$

for all $x, y \in K_2 \cap \mathrm{coz}(u)$. We prove that

$$\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x, y)} \leq M \|u\|_{X_2} p_{(K_1, d_1^{\alpha_1})}(f) + \|f\|_{X_1} p_{(K_2, d_2^{\alpha_2})}(u), \tag{2.2}$$

for all $x, y \in K_2$ with $x \neq y$. To this aim, pick $x, y \in K_2$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $x \in K_2$ and $y \in K_2 \setminus \mathrm{coz}(u)$. Then

$$\begin{aligned}\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x, y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x, y)} \\ &= \frac{|u(x)|\,|f(\varphi(x))|}{d_2^{\alpha_2}(x, y)} \\ &= |f(\varphi(x))| \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x, y)} \\ &\leq \|f\|_{X_1} p_{(K_2, d_2^{\alpha_2})}(u).\end{aligned}$$

Case 2. $x \in K_2 \setminus \mathrm{coz}(u)$ and $y \in K_2$. Then

$$\begin{aligned}\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x, y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x, y)} \\ &= \frac{|-u(y)|\,|f(\varphi(y))|}{d_2^{\alpha_2}(x, y)} \\ &= |f(\varphi(y))| \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x, y)}\end{aligned}$$

$$\leq \|f\|_{X_1}\, p_{(K_2,d_2^{\alpha_2})}(u).$$

Case 3. $x, y \in K_2 \cap \mathrm{coz}(u)$ and $\varphi(x) = \varphi(y)$. Then

$$\begin{aligned}\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\ &= |f(\varphi(x))|\,\frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} \\ &\leq \|f\|_{X_1}\, p_{(K_2,d_2^{\alpha_2})}(u).\end{aligned}$$

Case 4. $x, y \in K_2 \cap \mathrm{coz}(u)$ and $\varphi(x) \neq \varphi(y)$. Then

$$\begin{aligned}\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\ &= |u(x)|\,\frac{|f(\varphi(x)) - f(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\ &\quad + |f(\varphi(y))|\,\frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} \\ &= |u(x)|\,\frac{|f(\varphi(x)) - f(\varphi(y))|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))}\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \\ &\quad + |f(\varphi(y))|\,\frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} \\ &\leq M\,\|u\|_{X_2}\, p_{(K_1,d_1^{\alpha_1})}(f) + \|f\|_{X_1}\, p_{(K_2,d_2^{\alpha_2})}(u).\end{aligned}$$

Summarizing, we have proved that (2.2) holds for all $x, y \in K_2$ with $x \neq y$. Therefore, $Tf \in A_2$ and so the proof is complete. □

Corollary 2.3. *Let $u \in A_2$ and let $\varphi : X_2 \longrightarrow X_1$ be a continuous mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ with $\varphi(K_2) \subseteq K_1$. If $\varphi|_{K_2}$ is a Lipschitz mapping from $(K_2, d_2^{\alpha_2})$ to $(K_1, d_1^{\alpha_1})$, then $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2.*

The following example shows that there exists a nonzero weighted composition operator uC_φ from A_1 to A_2 where $K_2 \cap \mathrm{coz}(u) \neq \emptyset$ and φ is not a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(K_1, d_1^{\alpha_1})$.

Example 2.4. Let $X = [-2, 2]$, $K = [-1, 1]$ and d be the Euclidean metric on X. Define the function $u : X \longrightarrow \mathbb{C}$ by

$$u(x) = \begin{cases} 0, & -2 \leq x \leq -1, \\ 1 + x, & -1 \leq x \leq 1, \\ 2, & 1 \leq x \leq 2, \end{cases} \qquad (x \in X).$$

Clearly, $u \in \operatorname{Lip}(X, K, d^1)$. Define the map $\varphi : X \longrightarrow X$ by

$$\varphi(x) = \begin{cases} x, & -2 \le x \le -1, \\ \sqrt{2+2x} - 1, & -1 \le x \le 1, \\ x, & 1 \le x \le 2, \end{cases} \qquad (x \in X).$$

Then φ is a continuous mapping from (X, d^1) to (X, d^1), φ and it is injective on K and $\varphi(K) \subseteq K$. Moreover, $K \cap \operatorname{coz}(u) = (-1, 1]$. Since for each $n \in \mathbb{N}$, $-1 + \frac{1}{n}, -1 + \frac{2}{n} \in K \cap \operatorname{coz}(u)$ and

$$\begin{aligned} \frac{d^1\left(\varphi(-1+\frac{1}{n}), \varphi(-1+\frac{2}{n})\right)}{d^1\left(-1+\frac{1}{n}, -1+\frac{2}{n}\right)} &= \frac{\left|\sqrt{2+2\left(-1+\frac{1}{n}\right)} - \sqrt{2+2\left(-1+\frac{2}{n}\right)}\right|}{\left|\left(-1+\frac{1}{n}\right) - \left(-1+\frac{2}{n}\right)\right|} \\ &= \frac{\left|\sqrt{\frac{2}{n}} - \sqrt{\frac{4}{n}}\right|}{\left|-\frac{1}{n}\right|} \\ &= n\left(\frac{2}{\sqrt{n}} - \frac{\sqrt{2}}{\sqrt{n}}\right) \\ &= \left(2 - \sqrt{2}\right)\sqrt{n}, \end{aligned}$$

we deduce that φ is not a Lipschitz mapping from $(K \cap \operatorname{coz}(u), d^1)$ to (K, d^1).

Now we show that $T = uC_\varphi$ is a weighted composition operator from A to A, where $A = \operatorname{Lip}(X, K, d^1)$. Let $f \in A$. It is clear that Tf is a complex-valued continuous function on (X, d^1). We prove that

$$\frac{|Tf(x) - Tf(y)|}{d^1(x,y)} \le 2p_{(K,d^1)}(f) + p_{(K,d^1)}(u) \|f\|_X, \tag{2.3}$$

for all $x, y \in K$ with $x \neq y$. To this aim, pick $x, y \in K$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $x = -1$ and $y \in K \setminus \{-1\}$. Then

$$\begin{aligned} \frac{|Tf(x) - Tf(y)|}{d^1(x,y)} &= \frac{|-u(y)|}{d^1(x,y)} |f(\varphi(y))| \\ &= \frac{|u(x) - u(y)|}{d^1(x,y)} |f(\varphi(y))| \\ &\le p_{(K,d^1)}(u) \|f\|_X. \end{aligned}$$

Case 2. $x \in K \setminus \{-1\}$ and $y = -1$. Then

$$\frac{|Tf(x) - Tf(y)|}{d^1(x,y)} = \frac{|u(x)|}{d^1(x,y)} |f(\varphi(x))|$$

$$= \frac{|u(x) - u(y)|}{d^1(x,y)} |f(\varphi(x))|$$
$$\leq p_{(K,d^1)}(u) \, \|f\|_X .$$

Case 3. $x, y \in K \setminus \{-1\}$ and $x \neq y$. Then

$$\frac{|Tf(x) - Tf(y)|}{d^1(x,y)} = \frac{|u(x)\,[f(\varphi(x)) - f(\varphi(y))] + [u(x) - u(y)]\, f(\varphi(y))|}{|x-y|}$$

$$\begin{aligned}
&\leq \frac{|u(x)|\,|f(\varphi(x)) - f(\varphi(y))|}{|x-y|} \\
&\quad + \frac{|u(x) - u(y)|}{|x-y|} |f(\varphi(y))| \\
&= |1+x| \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \frac{|f(\varphi(x)) - f(\varphi(y))|}{|\varphi(x) - \varphi(y)|} \\
&\quad + \frac{|u(x) - u(y)|}{|x-y|} |f(\varphi(y))| \\
&\leq |1+x| \frac{|\sqrt{2+2x} - \sqrt{2+2y}|}{|x-y|} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d^1(\varphi(x), \varphi(y))} \\
&\quad + \frac{|u(x) - u(y)|}{d^1(x,y)} |f(\varphi(y))| \\
&\leq |1+x| \frac{|\sqrt{2+2x} - \sqrt{2+2y}|}{|x-y|} p_{(K,d^1)}(f) \\
&\quad + p_{(K,d^1)}(u) \, \|f\|_X \\
&= |1+x| \frac{|\,(2+2x) - (2+2y)\,|}{|x-y| \left|\sqrt{2+2x} - \sqrt{2+2y}\right|} p_{(K,d^1)}(f) \\
&\quad + p_{(K,d^1)}(u) \, \|f\|_X \\
&= \frac{2\,|1+x|}{\sqrt{2+2x} - \sqrt{2+2y}} p_{(K,d^1)}(f) \\
&\quad + p_{(K,d^1)}(u) \, \|f\|_X \\
&= \frac{2\,(1+x)}{\sqrt{2+2x}} p_{(K,d^1)}(f) + p_{(K,d^1)}(u) \, \|f\|_X \\
&= \sqrt{2}\sqrt{1+x}\; p_{(K,d^1)}(f) + p_{(K,d^1)}(u) \, \|f\|_X \\
&\leq 2 p_{(K,d^1)}(f) + p_{(K,d^1)}(u) \, \|f\|_X .
\end{aligned}$$

Summarizing, we have proved that (2.3) holds for all $x, y \in K$ with $x \neq y$. Therefore, $Tf \in \mathrm{Lip}(X, K, d^1) = A$. Hence, $T = uC_\varphi$ is a weighted composition operator from A to A.

We now give some necessary conditions that $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2.

Theorem 2.5. *Let u be a complex-valued function on X_2 and let $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(K_2) \subseteq K_1$. If $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2, then $u \in A_2$ and T is a bounded linear operator from $\Big(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\Big)$ to $\Big(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\Big)$.*

Proof. Let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. Then T is a complex linear operator from A_1 to A_2. Since $1_X \in A_1$ and $T1_X = u$, we deduce that $u \in A_2$. To prove the continuity of T from $\Big(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\Big)$ to $\Big(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\Big)$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_1 with

$$\lim_{n\to\infty} f_n = 0_{X_1}, \quad \Big(\text{in } \Big(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\Big)\Big), \tag{2.4}$$

and $g \in A_2$ with

$$\lim_{n\to\infty} Tf_n = g, \quad \Big(\text{in } \Big(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\Big)\Big). \tag{2.5}$$

Let $y \in X_2$ be given. By (2.4), we deduce that $\lim_{n\to\infty} \|f_n\|_X = 0$ which implies that

$$\lim_{n\to\infty} f_n(\varphi(y)) = 0. \tag{2.6}$$

According to (2.6) and the boundedness of u on X_2, we get

$$\lim_{n\to\infty} u(y) f_n(\varphi(y)) = 0. \tag{2.7}$$

By (2.5), we deduce that $\lim_{n\to\infty} \|u.(f_n o \varphi) - g\|_{X_2} = 0$ which implies that

$$\lim_{n\to\infty} u(y) f_n(\varphi(y)) = g(y). \tag{2.8}$$

From (2.8) and (2.6), we get

$$g(y) = 0. \tag{2.9}$$

Since (2.9) holds for all $y \in X_2$, we deduce that $g = 0_{X_2}$. Therefore, T is a continuous mapping from the Banach space $\Big(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\Big)$ to the Banach space $\Big(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\Big)$ by the closed graph theorem for Banach spaces. Hence, the proof is complete. □

Theorem 2.6. *Let u be a complex-valued function on X_2, let $K_2 \cap \mathrm{coz}(u) \neq \emptyset$ and let $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(K_2) \subseteq K_1$. If $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2, then φ is a continuous mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and a Lipschitz* *mapping from* $(K, d_2^{\alpha_2})$ *to* $(X_1, d_1^{\alpha_1})$ *for all compact subset* K *of* $K_2 \cap \mathrm{coz}(u)$ *in* (X_2, d_2).

Proof. By Theorem 2.5, $u \in A_2$ and T is a bounded linear operator from $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\right)$ to $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. We first show that φ is a continuous mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. Suppose that $y_0 \in K_2 \cap \mathrm{coz}(u)$ and φ is not continuous at y_0. Then there exist a positive number ε and a sequence $\{y_n\}_{n=1}^{\infty}$ in X_2 such that $d_2^{\alpha_2}(y_n, y_0) < \frac{1}{n}$ and $d_1^{\alpha_1}(\varphi(y_n), \varphi(y_0)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Define the function $h_{\varphi(y_0),\varepsilon} : X_1 \longrightarrow \mathbb{C}$ by

$$h_{\varphi(y_0),\varepsilon}(x) = \max\left\{0, \frac{\varepsilon - d_1^{\alpha_1}(x, \varphi(y_0))}{\varepsilon}\right\}, \quad (x \in X_1).$$

Then $h_{\varphi(y_0),\varepsilon} \in \mathrm{Lip}(X_1, d_1^{\alpha_1})$ and so $h_{\varphi(y_0),\varepsilon} \in A_1$. Since $\lim_{n\to\infty} y_n = y_0$ in $(X_2, d_2^{\alpha_2})$ and $Th_{\varphi(y_0),\varepsilon} \in A_2$, we deduce that

$$\lim_{n\to\infty} Th_{\varphi(y_0),\varepsilon}(y_n) = Th_{\varphi(y_0),\varepsilon}(y_0),$$

that is,

$$\lim_{n\to\infty} u(y_n)h_{\varphi(y_0),\varepsilon}(\varphi(y_n)) = u(y_0)h_{\varphi(y_0),\varepsilon}(\varphi(y_0)). \tag{2.10}$$

According to $h_{\varphi(y_0),\varepsilon}(\varphi(y_n)) = 0$ for all $n \in \mathbb{N}$, we get

$$\lim_{n\to\infty} u(y_n)h_{\varphi(y_0),\varepsilon}(\varphi(y_n)) = 0. \tag{2.11}$$

From (2.10) and (2.11), we deduce that

$$\lim_{n\to\infty} u(y_0)h_{\varphi(y_0),\varepsilon}(\varphi(y_0)) = 0. \tag{2.12}$$

By (2.12) and $y_0 \in \mathrm{coz}(u)$, we get $h_{\varphi(y_0),\varepsilon}(\varphi(y_0)) = 0$ which contradicts to $h_{\varphi(y_0),\varepsilon}(\varphi(y_0)) = 1$. Therefore, φ is continuous on $K_2 \cap \mathrm{coz}(u)$.

Let $K \subseteq K_2 \cap \mathrm{coz}(u)$ be a compact set in (X_2, d_2). We show that φ is a Lipschitz mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. Take

$$C = \inf\{|u(y)| : y \in K\}.$$

Since $K \subseteq \mathrm{coz}(u)$ and u is a continuous complex-valued function on $(X_2, d_2^{\alpha_2})$, we deduce that $C > 0$. Let $x, y \in K$ with $x \neq y$. Define the function $f_{\varphi(y)} : X_1 \longrightarrow \mathbb{C}$ by

$$f_{\varphi(y)}(t) = d_1^{\alpha_1}(t, \varphi(y)), \quad (t \in X_1).$$

It is easy to see that $f_{\varphi(y)} \in \mathrm{Lip}(X_1, d_1^{\alpha_1})$, $\left\|f_{\varphi(y)}\right\|_{X_1} \leq (\mathrm{diam}(X_1))^{\alpha_1}$ and $p_{(X_1,d_1^{\alpha_1})}(f_{\varphi(y)}) \leq 1$. Therefore, $f_{\varphi(y)} \in A_1$ and

$$\left\|f_{\varphi(y)}\right\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \leq 1 + (\mathrm{diam}(X_1))^{\alpha_1}. \tag{2.13}$$

By the definition of $f_{\varphi(y)}$, the boundedness of T and (2.13), we get

$$\begin{aligned}
\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{1}{|u(x)|}\,|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \\
&= \frac{1}{|u(x)|}\frac{|u(x)f_{\varphi(y)}(\varphi(x)) - u(y)f_{\varphi(y)}(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\
&= \frac{1}{|u(x)|}\frac{|Tf_{\varphi(y)}(x) - Tf_{\varphi(y)}(y)|}{d_2^{\alpha_2}(x,y)} \\
&\le \frac{1}{C} p_{(X_2,d_2^{\alpha_2})}\left(Tf_{\varphi(y)}\right) \\
&\le \frac{1}{C}\left\|Tf_{\varphi(y)}\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \\
&\le \frac{1}{C}\|T\|\left\|f_{\varphi(y)}\right\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \\
&\le \frac{1}{C}\|T\|\left(1 + (\mathrm{diam}(X_1))^{\alpha_1}\right).
\end{aligned}$$

Since the inequality above holds for all $x, y \in K$ with $x \neq y$, we deduce that φ is a Lipschitz mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. Hence, the proof is complete. □

Theorem 2.7. *Let u be a complex-valued function on X_2, $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(K_2) \subseteq K_1$ and let $\varphi|_{\mathrm{coz}(u)}$ be a continuous mapping from $(\mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. Then T is a weighted composition operator from A_1 to A_2 if and only if $u \in A_2$ and*

$$\sup\left\{|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\} < \infty. \tag{2.14}$$

Proof. We first assume that T is a weighted composition operator from A_1 to A_2. By Theorem 2.5, $u \in A_2$ and T is a bounded linear operator from $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\right)$ to $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. To prove (2.14), let $x, y \in K_2$ with $x \neq y$. Define the function $f_{\varphi(y)} : X_1 \longrightarrow \mathbb{C}$ by

$$f_{\varphi(y)}(t) = d_1^{\alpha_1}(t, \varphi(y)), \quad (t \in X_1).$$

Then $f_{\varphi(y)} \in A_1$ and

$$\|f\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \le (\mathrm{diam}(X_1))^{\alpha_1} + 1. \tag{2.15}$$

According to $Tf = uC_\varphi f \in A_2$ and (2.15), we get

$$\begin{aligned}
|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)d_1^{\alpha_1}(\varphi(x),\varphi(y)) - u(y)d_1^{\alpha_1}(\varphi(y),\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\
&= \frac{\left|u(x)\left(f_{\varphi(y)}o\varphi\right)(x) - u(y)\left(f_{\varphi(y)}o\varphi\right)(y)\right|}{d_2^{\alpha_2}(x,y)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|Tf_{\varphi(y)}(x) - Tf_{\varphi(y)}(y)|}{d_2^{\alpha_2}(x,y)} \\
&\le p_{(K_2,d_2^{\alpha_2})}\left(Tf_{\varphi(y)}\right) \\
&= \left\|Tf_{\varphi(y)}\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \\
&\le \|T\| \left\|f_{\varphi(y)}\right\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \\
&\le \|T\| \left((\mathrm{diam}(X_1))^{\alpha_1} + 1\right).
\end{aligned}
$$

Since the inequality above holds for all $x, y \in K_2$ with $x \neq y$, we deduce that

$$
\sup\left\{|u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\} \le \|T\| \left((\mathrm{diam}(X_1))^{\alpha_1} + 1\right),
$$

and so (2.14) holds.

We now assume that $u \in A_2$ and (2.14) holds. Take

$$
C = \sup\left\{|u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\}. \tag{2.16}
$$

Let $f \in A_1$. By Lemma 2.1, Tf is a complex-valued continuous function on $(X_2, d_2^{\alpha_2})$. We now show that

$$
\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} \le Cp_{(K_1,d_1^{\alpha_1})}(f) + p_{(K_2,d_2^{\alpha_2})}(u) \|f\|_{X_1}, \tag{2.17}
$$

for all $x, y \in K_2$ with $x \neq y$. To this aim, pick $x, y \in K_2$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $\varphi(x) \neq \varphi(y)$. By $\varphi(K_2) \subseteq K_1$, (2.16) and $u \in A_2$, we get

$$
\begin{aligned}
\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\
&\le |u(x)| \frac{|f(\varphi(x)) - f(\varphi(y))|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \\
&\quad + \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} |f(\varphi(y))| \\
&\le Cp_{(K_1,d_1^{\alpha_1})}(f) + p_{(K_2,d_2^{\alpha_2})}(u) \|f\|_{X_1}.
\end{aligned}
$$

Case 2. $\varphi(x) = \varphi(y)$. Then

$$
\begin{aligned}
\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{d_2^{\alpha_2}(x,y)} \\
&= \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} |f(\varphi(x))| \\
&\le p_{(K_2,d_2^{\alpha_2})}(u) \|f\|_{X_1} \\
&\le Cp_{(K_1,d_1^{\alpha_1})}(f) + p_{(K_2,d_2^{\alpha_2})}(u) \|f\|_{X_1}.
\end{aligned}
$$

Therefore, (2.17) holds for all $x, y \in K_2$ with $x \neq y$. This implies that $Tf|_{K_2} \in \mathrm{Lip}(K_2, d_2^{\alpha_2})$.

Summarising, we have shown that Tf is a complex-valued continuous function on $(X_2, d_2^{\alpha_2})$ and $Tf|_{K_2} \in \mathrm{Lip}(K_2, d_2^{\alpha_2})$ for all $f \in A_1$. Therefore, $Tf \in A_2$ for all $f \in A_1$ and so $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2. □

3. Injectivity and Surjectivity

Throughout this section we always assume that (X_j, d_j) is a compact metric space, K_j is a nonempty compact subset of X_j, $\alpha_j \in (0, 1]$ and $A_j = \mathrm{Lip}(X_j, K_j, d_j^{\alpha_j})$, where $j \in \{1, 2\}$.

We first give some necessary and sufficient conditions for the injectivity of weighted composition operators from A_1 to A_2.

Theorem 3.1. *Let u be a complex-valued function on X_2, $\varphi : X_2 \longrightarrow X_1$ be a continuous mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. If T is injective, then $K_1 \subseteq \varphi(X_2)$ and $\varphi(\mathrm{coz}(u))$ is dense in X_1.*

Proof. We first assume that K_1 is not a subset of $\varphi(X_2)$. Then there exists $s \in K_1$ such that $s \notin \varphi(X_2)$. The continuity of φ from (X_2, d_2) to (X_1, d_1) and the compactness of X_2 in (X_2, d_2) imply that $\varphi(X_2)$ is compact in (X_1, d_1). Since A_1 is a regular natural Banach function algebra on X_1, there exists a function $f \in A_1$ such that $f(s) = 1$ and $f(\varphi(X_2)) = \{0\}$. Thus $f \in A_1 \setminus \{0_{X_1}\}$ and

$$Tf(y) = u(y)f(\varphi(y)) = u(y)0 = 0,$$

for all $y \in X_2$. Therefore, T is not injective.

We now assume that T is injective and show that $\varphi(\mathrm{coz}(u))$ is dense in X_1. Define the function $f : X_1 \longrightarrow \mathbb{C}$ by

$$f(x) = \mathrm{dist}_{d_1}(x, \varphi(\mathrm{coz}(u))), \quad (x \in X_1).$$

It is easy to see that $f \in \mathrm{Lip}(X_1, d_1)$ and so $f \in A_1$. If $y \in \mathrm{coz}(u)$, then $\varphi(y) \in \varphi(\mathrm{coz}(u))$ and so $f(\varphi(y)) = 0$ which implies that

$$Tf(y) = u(y)f(\varphi(y)) = 0.$$

If $y \in X_2 \setminus \mathrm{coz}(u)$, then

$$Tf(y) = u(y)f(\varphi(y)) = 0f(\varphi(y)) = 0.$$

Therefore, $Tf(y) = 0$ for all $y \in X_2$ and so $Tf = 0_{X_2}$. The injectivity of T implies that $f = 0_{X_1}$. Therefore,

$$\mathrm{dist}_{d_1}(x, \varphi(\mathrm{coz}(u))) = f(x) = 0$$

for all $x \in X_1$. This implies that $x \in \overline{\varphi(\mathrm{coz}(u))}^{d_1}$, the closure of $\varphi(\mathrm{coz}(u))$ in (X_1, d_1), for all $x \in X_1$. Therefore, $X_1 \subseteq \overline{\varphi(\mathrm{coz}(u))}^{d_1}$ and so $\varphi(\mathrm{coz}(u))$ is dense in X_1. □

Theorem 3.2. *Let u be a complex-valued function on X_2, $\varphi : X_2 \longrightarrow X_1$ be a continuous mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. If $\varphi(\mathrm{coz}(u))$ is dense in X_1, then T is injective and $K_1 \subseteq \varphi(X_2)$.*

Proof. Let $\varphi(\mathrm{coz}(u))$ be dense in X_1. To prove the injectivity of T, let $f \in A_1$ with $Tf = 0_{X_2}$. Assume that $s \in \varphi(\mathrm{coz}(u))$. Then there exists $y \in \mathrm{coz}(u)$ with $s = \varphi(y)$. Therefore, $u(y) \neq 0$ and

$$0 = Tf(y) = u(y)f(\varphi(y)) = u(y)f(s).$$

Hence, $f(s) = 0$. Since $\varphi(\mathrm{coz}(u))$ is dense in $(X_1, d_1^{\alpha_1}$ and $f(\varphi(\mathrm{coz}(u))) = \{0\}$, we deduce that $f(X_1) = \{0\}$ and so $f = 0_{X_1}$. Therefore, T is injective. Hence, $K_1 \subseteq \varphi(X_2)$ by Theorem 3.1 and so the proof is complete. □

Here, we give some sufficient conditions for the surjectivity of weighted composition operators from A_1 to A_2.

Theorem 3.3. *Let $u \in A_2$ and let $\varphi : X_2 \longrightarrow X_1$ be a surjective Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ with $\varphi(K_2) = K_1$. If*

$$\inf \left\{ |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} : x, y \in X_2, x \neq y \right\} > 0, \tag{3.1}$$

then $T = uC_\varphi$ is a surjective weighted composition operator from A_1 to A_2.

Proof. Since $u \in A_2$ and $\varphi : X_2 \longrightarrow X_1$ is a Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ with $\varphi(K_2) \subseteq K_1$, we deduce that $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2 by Theorem 2.2. Suppose that (3.1) holds. Then φ is injective on X_2 and $\mathrm{coz}(u) = X_2$. Take

$$C = \inf \left\{ |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} : x, y \in X_2, x \neq y \right\}. \tag{3.2}$$

Since u is a complex-valued continuous function on $(X_2, d_2^{\alpha_2})$ and $\mathrm{coz}(u) = X_2$, there exists $y_0 \in X_2$ such that $u(y_0) \neq 0$ and

$$|u(y)| \leq |u(y_0)|, \tag{3.3}$$

for all $y \in X_2$. Define the function $\rho : X_2 \times X_2 :\longrightarrow \mathbb{R}$ by

$$\rho(x, y) = d_1^{\alpha_1}(\varphi(x), \varphi(y)), \quad (x, y \in X_2).$$

The injectivity of $\varphi : X_2 \longrightarrow X_1$ implies that ρ is a metric on X_2. Let $s, t \in X_1$. Then $\varphi^{-1}(s), \varphi^{-1}(t) \in X_2$. Take $x = \varphi^{-1}(s)$ and $y = \varphi^{-1}(t)$. Then $s = \varphi(x)$ and $t = \varphi(y)$. Thus

$$\rho(\varphi^{-1}(s), \varphi^{-1}(t)) = \rho(x, y) = d_1^{\alpha_1}(\varphi(x), \varphi(y)) = d_1^{\alpha_1}(s, t).$$

Therefore, φ^{-1} is a Lipschitz mapping from $(X_1, d_1^{\alpha_1})$ to (X_2, ρ) and so φ^{-1} is a continuous mapping from $(X_1, d_1^{\alpha_1})$ to (X_2, ρ). Since $\varphi : X_2 \longrightarrow X_1$ is a Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$, there exists a positive constant M such that

$$d_1^{\alpha_1}(\varphi(x), \varphi(y)) \leq M d_2^{\alpha_2}(x, y), \tag{3.4}$$

for all $x, y \in X_2$. From (3.2), (3.3) for $y = x$, the definition of ρ and (3.4), we get

$$\frac{C}{|u(y_0)|} d_2^{\alpha_2}(x, y) \leq \rho(x, y) \leq M d_2^{\alpha_2}(x, y), \tag{3.5}$$

for all $x, y \in X_2$. This implies that the metrics $d_2^{\alpha_2}$ and ρ are boundedly equivalent on X_2. Therefore, $C(X_2, d_2^{\alpha_2}) = C(X_2, \rho)$. Moreover, A_2 and $\mathrm{Lip}(X_2, K_2, \rho)$ have the same elements by [5, Theorem 2.5].

To prove the surjectivity of T, let $g \in A_2$. Then $\frac{g}{u} \in A_2$ and so $\frac{g}{u} \in \mathrm{Lip}(X_2, K_2, \rho)$. Thus $\frac{g}{u}$ is a complex-valued continuous function on (X_2, ρ). Take $f = \frac{g}{u} o \varphi^{-1}$. Then f is a complex-valued continuous function on $(X_1, d_1^{\alpha_1})$. Assume that $s, t \in K_1$. Since $\varphi(K_2) = K_1$, we get

$$\begin{aligned} |f(s) - f(t)| &= \left| \frac{g}{u}(\varphi^{-1}(s)) - \frac{g}{u}(\varphi^{-1}(t)) \right| \\ &\leq p_{(K_2,\rho)}(\frac{g}{u})\ \rho(\varphi^{-1}(s), \varphi^{-1}(t)) \\ &= p_{(K_2,\rho)}(\frac{g}{u})\ d_1^{\alpha_1}\left(\varphi(\varphi^{-1}(s)), \varphi(\varphi^{-1}(t))\right) \\ &= p_{(K_2,\rho)}(\frac{g}{u})\ d_1^{\alpha_1}(s, t). \end{aligned}$$

Thus $f|_{K_1} \in \mathrm{Lip}(K_1, d_1^{\alpha_1})$. Therefore, $f \in A_1$. On the other hand, for each $y \in X_2$ we have

$$Tf(y) = u(y) f(\varphi(y)) = u(y) \frac{g}{u}(\varphi^{-1}(\varphi(y)) = u(y) \frac{g(y)}{u(y)} = g(y).$$

Therefore, $Tf = g$ and so T is surjective. □

We now give some necessary conditions for the surjectivity of composition operators and weighted composition operators between extended Lipschitz algebras.

Theorem 3.4. *Let $\varphi : X_2 \longrightarrow X_1$ be a Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ with $\varphi(K_2) \subseteq K_1$ and let S be the composition operator from A_1 to A_2 induced by φ. If S is surjective, then φ is injective on X_2 and*

$$\inf\left\{\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\} > 0. \tag{3.6}$$

Proof. Let S be surjective. Assume that $y \in X_2$. Define the function $g_y : X_2 \longrightarrow \mathbb{C}$ by

$$g_y(z) = d_2^{\alpha_2}(z, y), \quad (z \in X_2).$$

Then $g_y \in \mathrm{Lip}(X_2, d_2^{\alpha_2})$, $\|g_y\|_{X_2} \leq (\mathrm{diam}_{d_2}(X_2))^{\alpha_2}$ and $p_{(X_2, d_2^{\alpha_2})}(g_y) \leq 1$. Therefore, $g_y \in A_2$ and

$$\|g_y\|_{\mathrm{Lip}}(X_2, K_2, d_2^{\alpha_2}) \leq (\mathrm{diam}_{d_2}(X_2))^{\alpha_2} + 1.$$

To prove the injectivity of φ on X_2, let $x, y \in X_2$ with $\varphi(x) = \varphi(y)$. The surjectivity of S implies that there exists $f \in A_1$ such that $g_y = S(f) = f o \varphi$. It follows that

$$d_2^{\alpha_2}(x, y) = g_y(x) = Sf(x) = f(\varphi(x)) = f(\varphi(y)) = Sf(y) = g_y(y) = 0.$$

Therefore, $x = y$ and so φ is injective.

To prove (3.6), define the map $\rho : X_2 \times X_2 :\longrightarrow \mathbb{R}$ by

$$\rho(x, y) = d_1^{\alpha_1}(\varphi(x), \varphi(y)), \quad (x, y \in X_2).$$

The injectivity of φ on X_2 implies that ρ is a metric on X_2. We claim that

$$\mathrm{Lip}(K_2, d_2^{\alpha_2}) \subseteq \mathrm{Lip}(K_2, \rho). \tag{3.7}$$

Let $g \in \mathrm{Lip}(K_2, d_2^{\alpha_2})$. By Tietze extension theorem [13, Theorem 20.4], there exists a complex-valued continuous function $\tilde{g}$ on $(X_2, d_2^{\alpha_2})$ such that $\tilde{g}|_{K_2} = g$ and $\|\tilde{g}\|_{X_2} \leq 2\|g\|_{K_2}$. Therefore, $\tilde{g} \in A_2$. The surjectivity of S implies that there exists $F \in A_1$ such that $\tilde{g} = SF$. Since $\varphi(K_2)$ is a subset of K_1, we get

$$\begin{aligned}\frac{|g(x) - g(y)|}{\rho(x, y)} &= \frac{|\tilde{g}(x) - \tilde{g}(y)|}{d_1^{\alpha_1}(\varphi(x), \varphi(y))} \\ &= \frac{|SF(x) - SF(y)|}{d_1^{\alpha_1}(\varphi(x), \varphi(y))} \\ &= \frac{|F(\varphi(x)) - F(\varphi(y))|}{d_1^{\alpha_1}(\varphi(x), \varphi(y))} \\ &\leq p_{(K_1, d_1^{\alpha_1})}(F),\end{aligned}$$

for all $x, y \in K_2$ with $x \neq y$. Hence, $g \in \mathrm{Lip}(K_2, \rho)$ and so (3.7) holds. By (3.7), we deduce that the map $g \longmapsto g : \mathrm{Lip}(K_2, d_2^{\alpha_2}) \longrightarrow \mathrm{Lip}(K_2, \rho)$

is an algebra homomorphism from $\mathrm{Lip}(K_2, d_2^{\alpha_2})$ to $\mathrm{Lip}(K_2, \rho)$. Since $\left(\mathrm{Lip}(K_2, \rho), \|\cdot\|_{\mathrm{Lip}(K_2,\rho)}\right)$ is a semisimple commutative Banach algebra, this map is continuous. Therefore, there exists a positive constant M such that $\|g\|_{\mathrm{Lip}(K_2,\rho)} \leq M \|g\|_{\mathrm{Lip}(K_2,d_2^{\alpha_2})}$ for all $g \in \mathrm{Lip}(K_2, d_2^{\alpha_2})$. Let $x, y \in K_2$ such that $x \neq y$. Since $g_y \in A_2$ and $\|g_y\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \leq (\mathrm{diam}_{d_2}(X_2))^{\alpha_2} + 1$, we deduce that $g_y|_{K_2} \in \mathrm{Lip}(K_2, d_2^{\alpha_2})$ and so

$$\begin{aligned}\frac{|g_y(x) - g_y(y)|}{\rho(x,y)} &\leq p_{(K_2,\rho)}(g_y)\\ &\leq \|g_y|_{K_2}\|_{\mathrm{Lip}(K_2,\rho)}\\ &\leq M \|g_y|_{K_2}\|_{\mathrm{Lip}(K_2,d_2^{\alpha_2})}\\ &\leq M \|g_y\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\\ &\leq M \left((\mathrm{diam}_{d_2}(X_2))^{\alpha_2} + 1\right).\end{aligned}$$

Take

$$M' = \frac{1}{M\left((\mathrm{diam}_{d_2}(X_2))^{\alpha_2} + 1\right)}.$$

Then $M' > 0$ and

$$\begin{aligned}\frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{\rho(x,y)}{d_2^{\alpha_2}(x,y)}\\ &= \frac{\rho(x,y)}{|g_y(x) - g_y(y)|}\\ &\geq M'.\end{aligned}$$

Since the inequality above holds for all $x, y \in K_2$ with $x \neq y$, we deduce that

$$\inf\left\{\frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\} \geq M'.$$

Hence, (3.6) holds and the proof is complete. □

Theorem 3.5. *Let $u \in A_2$, $\varphi : X_2 \longrightarrow X_1$ be a Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ with $\varphi(K_2) \subseteq K_1$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. If T is surjective, then φ is injective on X_2 and*

$$\inf\left\{|u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x,y)} : x, y \in K_2, x \neq y\right\} > 0.$$

Proof. Since $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2, by Theorem 2.5, $u \in A_2$. Let T be surjective. Since $1_{X_2} \in A_2$, there exists a function $f_1 \in A_1$ such that $Tf_1 = 1_{X_2}$. This implies that

$$1 = Tf_1(y) = u(y) f_1(\varphi(y)),$$

for all $y \in Y$. Therefore, $u(y) \neq 0$ for all $y \in Y$. This implies that $\frac{1}{u} \in A_2$. It follows that $\frac{1}{u}Tf \in A_2$ for all $f \in A_1$. Thus, $fo\varphi \in A_2$ for all $f \in A_1$. Therefore, $C_\varphi : A_1 \longrightarrow A_2$ is a composition operator from A_1 to A_2. Take

$$M_1 = \inf\left\{\frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} : x, y \in K_2, x \neq y\right\}.$$

We claim that C_φ is surjective. Let $g \in A_2$. Then $ug \in A_2$. By the surjectivity of T, there exists $f \in A_1$ such that

$$u.(fo\varphi) = Tf = ug.$$

This implies that $C_\varphi f = fo\varphi = g$ since $u(y) \neq 0$ for all $y \in X_2$. Hence, our claim is justified. Therefore, φ is surjective and $M_1 > 0$ by Theorem 3.4. Since K_2 is a nonempty compact subset of X_2 in $(X_2, d_2^{\alpha_2})$ and u is continuous on K_2, there exists $y_1 \in K_2$ such that

$$|u(y_1)| = \inf\{|u(y)| : y \in K_2\}.$$

By the argument above, $|u(y_1)| > 0$. Let $x, y \in K_2$ with $x \neq y$. Then

$$|u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} \geq |u(y_1)| M_1.$$

Therefore,

$$\inf\left\{|u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} : x, y \in K_2, x \neq y\right\} \geq |u(y_1)| M_1.$$

Hence, the proof is complete. □

4. Compactness of Weighted Composition Operators

This section is devoted to the compactness of weighted composition operators between extended Lipschitz algebras. We first give a generalized of [7, Theorem 4.1] that its proof can be done similarly.

Theorem 4.1. *For $j \in \{1, 2\}$, let X_j be a compact Hausdorff space and $\left(A_j, \|\cdot\|_j\right)$ be a compact Banach function algebra on X_j. Suppose that T is a linear operator from A_1 to A_2 which is bounded from $\left(A_1, \|\cdot\|_{X_1}\right)$ to $\left(A_2, \|\cdot\|_{X_2}\right)$. If T is compact, then $\{Tf_n\}_{n=1}^{\infty}$ converges to the function 0_{X_2} in $(A_2, \|\cdot\|_2)$ for each bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $(A_1, \|\cdot\|_1)$ which converges uniformly to the function 0_{X_1}. The converse is true if the closed unit ball of $(A_1, \|\cdot\|_1)$ is relatively compact in $\left(A_1, \|\cdot\|_{X_1}\right)$.*

In the rest of this section, we always assume that (X_j, d_j) is a compact metric space, K_j is a compact subset of X_j, $\alpha_j \in (0, 1]$ and $A_j = \mathrm{Lip}(X_j, K_j, d_j^{\alpha_j})$, where $j \in \{1, 2\}$.

We give some necessary conditions for the compactness of weighted composition operators from A_1 to A_2.

Theorem 4.2. *Let u be a complex-valued function on X_2 and let $\varphi : X_2 \longrightarrow X_1$ be a map such that $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2. If T is compact, then $\{Tf_n\}_{n=1}^{\infty}$ converges to the function 0_{X_2} in $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$ for all bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\right)$ which converges uniformly to the function 0_{X_1}.*

Proof. Let $T = uC_\varphi$ is compact. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\right)$ which converges uniformly to the function 0_{X_1}. Since $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2, we deduce that $u \in A_2$ and for each $f \in A_1$ we have

$$|Tf(y)| = |u(y)f(\varphi(y))| = |u(y)|\,|f(\varphi(y))| \leq \|u\|_{X_2}\,\|f\|_{X_1},$$

for all $y \in X_2$. Therefore,

$$\|Tf\|_{X_2} \leq \|u\|_{X_2}\,\|f\|_{X_1},$$

for all $f \in A_1$. This implies that

$$\|Tf_n\|_{X_2} \leq \|u\|_{X_2}\,\|f_n\|_{X_1} \tag{4.1}$$

for all $n \in \mathbb{N}$. Since u is a complex-valued bounded function on X_2, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function 0_{X_1} and (4.1) holds for all $n \in \mathbb{N}$. We deduce that $\{Tf_n\}_{n=1}^{\infty}$ converges uniformly to the function 0_{X_2}. Since $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\right)$ and $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$ are complex Banach function algebras on X_1 and X_2, respectively, we deduce that

$$\lim_{n\to\infty} \|Tf_n\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} = 0,$$

by Theorem 4.1. Hence, the proof is complete. □

Theorem 4.3. *Let u be a complex-valued function on X_2, $\varphi : X_2 \longrightarrow X_1$ be a mapping and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. If T is compact, then*

$$\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} = 0,$$

when $x, y \in K_2$ and $d_1(\varphi(x),\varphi(y))$ tends to 0.

Proof. Let T be compact but $\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \neq 0$ when $x, y \in K_2$ and $d_1(\varphi(x),\varphi(y))$ tends to 0. Then there exist $\varepsilon > 0$ and two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in K_2 with $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d_1\left(\varphi(x_n), \varphi(y_n)\right) = 0$ but

$$|u(x_n)| \frac{d_1^{\alpha_1}(\varphi(x_n), \varphi(y_n))}{d_2^{\alpha_2}(x_n, y_n)} \geq \varepsilon \tag{4.2}$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Define the function $h_n : X_1 \longrightarrow \mathbb{C}$ by

$$h_n(t) = \begin{cases} d_1^{\alpha_1}(t, \varphi(y_n)), & d_1(t, \varphi(y_n)) \leq d_1(\varphi(x_n), \varphi(y_n)), \\ d_1^{\alpha_1}(\varphi(x_n), \varphi(y_n)), & d_1(t, \varphi(y_n)) \geq d_1(\varphi(x_n), \varphi(y_n)). \end{cases}$$

It is easy to see that $h_n \in \mathrm{Lip}(X_1, d_1^{\alpha_1})$ and $p_{(X_1, d_1^{\alpha_1})}(h_n) \leq 1$. Moreover, we can easily show that

$$\|h_n\|_{X_1} \leq d_1^{\alpha_1}(\varphi(x_n), \varphi(y_n)) \leq \left(\mathrm{diam}_{d_1}(X_1)\right)^{\alpha_1}.$$

Therefore, $\{h_n\}_{n=1}^{\infty}$ is a bounded sequence in $\left(A_1, \|\cdot\|_{\mathrm{Lip}(X_1, K_1, d_1^{\alpha_1})}\right)$ and $\{h_n\}_{n=1}^{\infty}$ converges uniformly on X_1 to the function 0_{X_1}. Since T is compact, by Theorem 4.2 we deduce that

$$\lim_{n\to\infty} \|Tf\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})} = 0. \tag{4.3}$$

By (4.3), there exists $N \in \mathbb{N}$ such that

$$\|Tf_N\|_{X_2} + p_{(K_2, d_2^{\alpha_2})}(Th_N) < \frac{\varepsilon}{2}. \tag{4.4}$$

On the other hand,

$$\begin{aligned} |u(x_N)| \frac{d_1^{\alpha_1}(\varphi(x_N), \varphi(y_N))}{d_2^{\alpha_2}(x_N, y_N)} &= \frac{|u(x_N) h_N(\varphi(x_N))|}{d_2^{\alpha_2}(x_N, y_N)} \\ &= \frac{|u(x_N) h_N(\varphi(x_N)) - u(y_N) h_N(\varphi(y_N))|}{d_2^{\alpha_2}(x_N, y_N)} \\ &= \frac{|Th_N(x_N) - Th_N(y_N)|}{d_2^{\alpha_2}(x_N, y_N)} \\ &\leq p_{(K_2, d_2^{\alpha_2})}(Th_N). \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we get

$$|u(x_N)| \frac{d_1^{\alpha_1}(\varphi(x_N), \varphi(y_N))}{d_2^{\alpha_2}(x_N, y_N)} < \frac{\varepsilon}{2},$$

which contradicts to (4.2) for $n = N$. Hence, the proof is complete. □

Theorem 4.4. *Let u be a complex-valued function on X_2, $K_2 \cap \mathrm{coz}(u) \neq \emptyset$, let $\varphi : X_2 \longrightarrow X_1$ be a mapping such that $\varphi|_{K_2}$ is a uniformly continuous from $(K_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. If T is compact, then φ is a supercontractive mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ for all nonempty compact subset K of $K_2 \cap \mathrm{coz}(u)$ in $(X_2, d_2^{\alpha_2})$.*

Proof. Let T be compact. By Theorem 4.3,

$$\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}=0, \tag{4.6}$$

when $x,y\in K_2$ and $d_1(\varphi(x),\varphi(y))$ tends to 0. Let K be a nonempty compact subset of $K_2\cap \mathrm{coz}(u)$. Let $\varepsilon>0$ be given. Take

$$C=\inf\{|u(y)|:y\in K\}. \tag{4.7}$$

Then $C>0$. By (4.6), there exists $\delta_1>0$ such that

$$|u(x)|\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}<C\varepsilon, \tag{4.8}$$

for all $x,y\in K_2$ with $0<d_1(\varphi(x),\varphi(y))<\delta_1$. Since $\varphi|_{K_2}$ is a uniformly continuous from $(K_2,d_2^{\alpha_2})$ to $(X_1,d_1^{\alpha_1})$, there exists $\delta>0$ such that

$$d_1^{\alpha_1}(\varphi(x),\varphi(y))<\delta_1^{\alpha_1}, \tag{4.9}$$

for all $x,y\in K_2$ with $d_2^{\alpha_2}(x,y)<\delta$. Assume that $x,y\in K$ with $0<d_2^{\alpha_2}(x,y)<\delta$. Then, by (4.9), $d_1(\varphi(x),\varphi(y))<\delta_1$ since $K\subseteq K_2$. If $\varphi(x)=\varphi(y)$, then

$$\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}<\varepsilon.$$

If $\varphi(x)\neq\varphi(y)$, then $0<d_1(\varphi(x),\varphi(y))<\delta_1$ and so

$$\begin{aligned}\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}&\leq\frac{1}{C}|u(x)|\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}\\&<\frac{1}{C}C\varepsilon\\&=\varepsilon,\end{aligned}$$

by (4.7) and (4.8). Hence, the proof is complete. □

The following example shows that the converse of Theorem 4.4 does not hold in general.

Example 4.5. Let $X_1=X_2=\left\{\frac{1}{n}:n\in\mathbb{Z}\setminus\{0\}\right\}\cup\{0\}$ and d_1 and d_2 be the Euclidean metric on X_1 and X_2, $K_1=\left\{\frac{(-1)^{n+1}}{|n|}:n\in\mathbb{Z}\setminus\{0\}\right\}\cup\{0\}$, $K_2=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\cup\{0\}$ and $\alpha_1=\alpha_2=1$. Clearly, (X_j,d_j) is a compact metric space and K_j is a compact set in (X_j,d_j) for $j\in\{1,2\}$. Define the function $u:X_2\longrightarrow\mathbb{C}$ by

$$u(x)=x,\quad (x\in X_2).$$

Then $u\in A_2$, $\mathrm{coz}(u)=\left\{\frac{1}{n}:n\in\mathbb{Z}\setminus\{0\}\right\}$ and $K_2\cap\mathrm{coz}(u)=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$. Define the function $\varphi:X_2\longrightarrow X_1$ by

$$\varphi(0)=0,\qquad \varphi(\frac{1}{n})=\frac{(-1)^{n+1}}{|n|},\quad (n\in\mathbb{Z}\setminus\{0\}).$$

Clearly, $\varphi(K_2)$ is a subset of K_1. Since $\left|\varphi(\frac{1}{n})-\varphi(0)\right| = \left|\frac{1}{n}\right| = \left|\frac{1}{n}-0\right|$ for all $n \in \mathbb{Z}\setminus\{0\}$, we deduce that φ is continuous at 0. On the other hand, 0 is the only limit point of X_2 in $(X_2, d_2^{\alpha_2})$. This implies that φ is continuous at $\frac{1}{n}$ for all $n \in \mathbb{Z}\setminus\{0\}$. Therefore, φ is a continuous mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$. The compactness of $(X_2, d_2^{\alpha_2})$ implies that φ is uniformly continuous from $(K_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$.

We now show that

$$|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \leq 3, \tag{4.10}$$

for all $x, y \in K_2$ with $x \neq y$. To this aim, pick $x, y \in K_2$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $x = 0$ and $y \in K_2 \setminus \{0\}$. Then

$$\begin{aligned} |u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= 0 \\ &< 3. \end{aligned}$$

Case 2. $x = \frac{1}{n}$ with $n \in \mathbb{N}$ and $y = 0$. Then

$$\begin{aligned} |u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \left|\frac{1}{n}\right| \frac{\left|\varphi(\frac{1}{n})-\varphi(0)\right|}{\left|\frac{1}{n}-0\right|} \\ &= \left|\frac{(-1)^{n+1}}{n}-0\right| \\ &= \frac{1}{n} \\ &< 3. \end{aligned}$$

Case 3. $x = \frac{1}{2j}$ and $y = \frac{1}{2k}$, where $j, k \in \mathbb{N}$ with $j \neq k$. Then

$$\begin{aligned} |u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{1}{2j}\,\frac{\left|-\frac{1}{2j}+\frac{1}{2k}\right|}{\left|\frac{1}{2j}-\frac{1}{2k}\right|} \\ &= \frac{1}{2j} \\ &< 3. \end{aligned}$$

Case 4. $x = \frac{1}{2j-1}$ and $y = \frac{1}{2k-1}$, where $j, k \in \mathbb{N}$ with $j \neq k$. Then

$$\begin{aligned} |u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{1}{2j-1}\,\frac{\left|\frac{1}{2j-1}-\frac{1}{2k-1}\right|}{\left|\frac{1}{2j-1}-\frac{1}{2k-1}\right|} \\ &= \frac{1}{2j-1} \end{aligned}$$

$$< 3.$$

Case 5. $x = \frac{1}{2j}$ and $y = \frac{1}{2k-1}$, where $j, k \in \mathbb{N}$ with $2j < 2k - 1$. Then

$$\begin{aligned} |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} &= \frac{2k - 1 + 2j}{2j(2k - 1 - 2j)} \\ &= \frac{1}{2j} + \frac{2}{2k - 1 - 2j} \\ &< 3. \end{aligned}$$

Case 6. $x = \frac{1}{2j}$ and $y = \frac{1}{2k-1}$, where $j, k \in \mathbb{N}$ with $2j > 2k - 1$. Then

$$\begin{aligned} |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} &= \frac{2k - 1 + 2j}{2j(2j - 2k + 1)} \\ &< \frac{4j}{2j(2j - 2k + 1)} \\ &= \frac{2}{2j - 2k + 1} \\ &< 3. \end{aligned}$$

Case 7. $x = \frac{1}{2j-1}$ and $y = \frac{1}{2k}$, where $j, k \in \mathbb{N}$ with $2j - 1 < 2k$. Then

$$\begin{aligned} |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} &= \frac{2k + 2j - 1}{(2j - 1)(2k - 2j + 1)} \\ &= \frac{2k - 2j + 1 + 2(2j - 1)}{(2j - 1)(2k - 2j + 1)} \\ &= \frac{1}{2j - 1} + \frac{2}{2k - 2j + 1} \\ &< 3. \end{aligned}$$

Case 8. $x = \frac{1}{2j-1}$ and $y = \frac{1}{2k}$, where $j, k \in \mathbb{N}$ with $2j - 1 > 2k$. Then

$$\begin{aligned} |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} &= \frac{2k + 2j - 1}{(2j - 1)(2j - 1 - 2k)} \\ &\leq \frac{2(2j - 1)}{(2j - 1)(2j - 1 - 2k)} \\ &= \frac{2}{2j - 1 - 2k} \\ &< 3. \end{aligned}$$

Therefore, (4.10) holds for all $x, y \in K_2$ with $x \neq y$. This implies that

$$\sup \left\{ |u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} : x, y \in K_2, x \neq y \right\} \leq 3.$$

Hence, $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2 by Theorem 2.7.

It is clear that, $K_2 \cap \mathrm{coz}(u) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Since the relative topology on $K_2 \cap \mathrm{coz}(u)$ is the discrete topology on $K_2 \cap \mathrm{coz}(u)$, the compact subsets of $K_2 \cap \mathrm{coz}(u)$ are finite. Therefore, φ is a supercontractive mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ for all nonempty compact subset K of $K_2 \cap \mathrm{coz}(u)$.

We now show that $T = uC_\varphi$ is not a compact operator from A_1 to A_2. Let $n \in \mathbb{N}$. Define the function $f_n : X_1 \longrightarrow \mathbb{C}$ by

$$f_n(x) = \begin{cases} -\frac{1}{n}, & x \leq -\frac{1}{n}, \\ x, & -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ \frac{1}{n}, & x \geq \frac{1}{n}. \end{cases}$$

Clearly, f_n is a complex-valued continuous function on $(X_1, d_1^{\alpha_1})$ and

$$\|f_n\|_{X_1} \leq \frac{1}{n}. \tag{4.11}$$

We claim that

$$p_{(K_1, d_1^{\alpha_1})}(f_n) \leq 1. \tag{4.12}$$

To this aim, pick $x, y \in K_1$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $x = 0$ and $y = \frac{(-1)^{j+1}}{|j|}$, where $j \in \mathbb{Z}$ and $|j| \geq n$. Then

$$\begin{aligned} \frac{|f_n(x) - f_n(y)|}{d_1^{\alpha_1}(x, y)} &= \frac{|0 - y|}{|0 - y|} \\ &\leq 1. \end{aligned}$$

Case 2. $x = 0$ and $y = \frac{(-1)^{j+1}}{|j|}$, where $j \in \mathbb{Z}$ and $|j| \leq n$. Then

$$\begin{aligned} \frac{|f_n(x) - f_n(y)|}{d_1^{\alpha_1}(x, y)} &= \frac{\frac{1}{n}}{\frac{1}{|j|}} \\ &\leq 1. \end{aligned}$$

Case 3. $x = \frac{(-1)^{j+1}}{|j|}$ and $y = 0$, where $j \in \mathbb{Z}$ and $|j| \geq n$. Then

$$\begin{aligned} \frac{|f_n(x) - f_n(y)|}{d_1^{\alpha_1}(x, y)} &= \frac{|x - 0|}{|x - 0|} \\ &\leq 1. \end{aligned}$$

Case 4. $x = \frac{(-1)^{j+1}}{|j|}$ and $y = 0$, where $j \in \mathbb{Z}$ and $|j| \leq n$. Then

$$\frac{|f_n(x) - f_n(y)|}{d_1^{\alpha_1}(x, y)} = \frac{\frac{1}{n}}{\frac{1}{|j|}}$$

$$\leq 1.$$

Hence, our claim is justified.

From (4.11) and (4.12), we get

$$\|f_n\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \leq 2.$$

Therefore, $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\Big(A_1, \|\cdot\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})}\Big)$, which converges uniformly to the function 0_{X_1}. On the other hand, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
p_{(K_2,d_2^{\alpha_2})}(Tf_n) &\geq \frac{\left|Tf_n(\frac{1}{n}) - Tf_n(\frac{1}{n+1})\right|}{d_2^{\alpha_2}(\frac{1}{n}, \frac{1}{n+1})} \\
&= \frac{\left|u(\frac{1}{n})f_n(\varphi(\frac{1}{n})) - u(\frac{1}{n+1})f_n(\varphi(\frac{1}{n+1}))\right|}{\left|\frac{1}{n} - \frac{1}{n+1}\right|} \\
&= n(n+1)\left|\frac{1}{n}f_n(\frac{(-1)^{n+1}}{n}) - \frac{1}{n+1}f_n(\frac{(-1)^{n+2}}{n+1})\right| \\
&= n(n+1)\left|\frac{(-1)^{n+1}}{n^2} - \frac{(-1)^{n+2}}{(n+1)^2}\right| \\
&= n(n+1)\left|\frac{1}{n^2} + \frac{1}{(n+1)^2}\right| \\
&= n(n+1)\frac{(n+1)^2+n^2}{n^2(n+1)^2} \\
&= \frac{(n+1)^2+n^2}{n(n+1)} \\
&> \frac{2n^2+2n}{n(n+1)} \\
&= 2.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|Tf_n\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \neq 0$. Hence, T is not compact by Theorem 4.2.

We now give some sufficient conditions for the compactness of weighted composition operators from A_1 to A_2.

Theorem 4.6. *Let u be a complex-valued function on X_2, $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(X_2) \subseteq K_1$, $K_2 \subseteq X_2 \setminus \mathrm{coz}(u)$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. Then T is compact.*

Proof. To prove the compactness of T, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_1 with $\|f_n\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \leq 1$. Then $\|f_n\|_{X_1} \leq 1$ and $p_{(K_1,d_1^{\alpha_1})}(f_n) \leq 1$ for all $n \in \mathbb{N}$. Thus, $\{f_n|_{K_1}\}_{n=1}^{\infty}$ is a uniformly bounded sequence of complex-valued functions on K_1 and an equicontinuous sequence of complex-valued functions on compact metric space $(K_1, d_1^{\alpha_1})$. By Arzela-Ascoli theorem, there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{f_{n_j}|_{K_1}\}_{j=1}^{\infty}$ converges uniformly on K_1. Since T is a weighted composition operator from A_1 to A_2, we deduce that $u \in A_2$ by Theorem 2.5. We claim that $\{Tf_{n_j}\}_{j=1}^{\infty}$ is a Cauchy sequence in $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. Let $\varepsilon > 0$ be given. Since $\varphi(X_2) \subseteq K_1$ and $\{f_{n_j}|_{K_1}\}_{j=1}^{\infty}$ converges uniformly on K_1, we deduce that $\{f_{n_j} o \varphi\}_{j=1}^{\infty}$ converges uniformly on X_2. This implies that $\{u.(f_{n_j} o\varphi)\}_{j=1}^{\infty}$ converges uniformly on X_2, since u is a complex-valued bounded function on X_2. Therefore, there exists $N \in \mathbb{N}$ such that

$$\left|u(x)f_{n_j}(\varphi(x)) - u(x)f_{n_k}(\varphi(x))\right| < \frac{\varepsilon}{2}, \tag{4.13}$$

for all $j, k \in \mathbb{N}$ with $j \geq N$ and $k \geq N$ and for each $x \in X_2$. Let $j, k \in \mathbb{N}$ with $j \geq N$ and $k \geq N$. Then

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{X_2} \leq \frac{\varepsilon}{2}. \tag{4.14}$$

Since $K_2 \subseteq X_2 \setminus \mathrm{coz}(u)$ for each $x, y \in K_2$ with $x \neq y$ we have

$$\frac{|(Tf_{n_j} - Tf_{n_k})(x) - (Tf_{n_j} - Tf_{n_k})(y)|}{d_2^{\alpha_2}(x,y)} = 0.$$

This implies that

$$p_{(K_2, d_2^{\alpha_2})}(Tf_{n_j} - Tf_{n_k}) = 0. \tag{4.15}$$

From (4.14) and (4.15), we get

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} < \varepsilon.$$

Hence, our claim is justified. Since $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$ is a Banach space, we deduce that there exists $g \in A_2$ such that $\{Tf_{n_j}\}_{j=1}^{\infty}$ converges to g in $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. Therefore, T is compact and so the proof is complete. □

Example 4.7. Let $X_1 = [-1, 1]$, d_1 be the Euclidean metric on X_1, $K_1 = [0, 1]$, $\alpha_1 = 1$, $X_2 = [0, 2]$, $K_2 = [0, 1]$, d_2 be the Euclidean metric on X_2 and $\alpha_2 = 1$. Define the function $u : X_2 \longrightarrow \mathbb{C}$ by

$$u(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1 - x, & 1 \leq x \leq 2. \end{cases}$$

Clearly, $u \in \mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})$ and $K_2 \cap \mathrm{coz}(u) = \emptyset$. Define the function $\varphi : X_2 \longrightarrow X_1$ by

$$\varphi(x) = \frac{x}{2}, \quad (x \in X_2).$$

It is obvious that φ is a Lipschitz mapping from $(X_2, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and $\varphi(X_2) = [0,1] \subseteq K_1$. Therefore, $T = uC_\varphi$ is a compact weighted composition operator from A_1 to A_2 by Theorem 2.2 and Theorem 4.6.

Theorem 4.8. *Let u be a complex-valued function on X_2, $K_2 \cap \mathrm{coz}(u) \neq \emptyset$, $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(X_2) \subseteq K_1$, $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ be a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and let $T = uC_\varphi$ be a weighted composition operator from A_1 to A_2. Then T is compact if $\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} = 0$ when $x, y \in K_2$ and $d_1(\varphi(x), \varphi(y))$ tends to 0.*

Proof. Let $\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} = 0$ when $x, y \in K_2$ and $d_1(\varphi(x), \varphi(y))$ tends to 0. Since $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ is a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$, there exists $M > 0$ such that

$$\frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} \leq M, \tag{4.16}$$

for all $x, y \in K_2 \cap \mathrm{coz}(u)$ with $x \neq y$.

To prove the compactness of T, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_1 with $\|f_n\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \leq 1$. Then $\|f_n\|_{X_1} \leq 1$ and $p_{(K_1,d_1^{\alpha_1})}(f_n) \leq 1$ for all $n \in \mathbb{N}$. Thus, $\{f_n|_{K_1}\}_{n=1}^{\infty}$ is a uniformly bounded sequence of complex-valued functions on K_1 and an equicontinuous sequence of complex-valued functions on compact metric space $(K_1, d_1^{\alpha_1})$. By Arzela-Ascoli theorem, there exists a subsequence $\left\{f_{n_j}\right\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\left\{f_{n_j}|_{K_1}\right\}_{j=1}^{\infty}$ converges uniformly on K_1. Since T is a weighted composition operator from A_1 to A_2, we deduce that $u \in A_2$ by Theorem 2.5. We claim that $\left\{Tf_{n_j}\right\}_{j=1}^{\infty}$ is a Cauchy sequence in $\Big(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\Big)$. Let $\varepsilon > 0$ be given. By hypothesis, there exists $\delta > 0$ such that

$$|u(x)| \frac{d_1^{\alpha_1}(\varphi(x), \varphi(y))}{d_2^{\alpha_2}(x, y)} < \frac{\varepsilon}{2\,\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} + 4}, \tag{4.17}$$

where $x, y \in K_2$ and $0 < d_1(\varphi(x), \varphi(y)) \leq \delta$. Since $\varphi(X_2) \subseteq K_1$ and $\left\{f_{n_j}|_{K_1}\right\}_{j=1}^{\infty}$ converges uniformly on K_1, we deduce that $\left\{f_{n_j} o\varphi\right\}_{j=1}^{\infty}$ converges uniformly on X_2. Thus, there exists $N_1 \in \mathbb{N}$ such that

$$\left|f_{n_j}(\varphi(x)) - f_{n_k}(\varphi(x))\right| < \frac{\varepsilon}{2\,\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} + 4}, \tag{4.18}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_1$ and $k \geq N_1$ and each $x \in X_2$. Since $\left\{f_{n_j} o\varphi\right\}_{j=1}^{\infty}$ converges uniformly on X_2 and u is a complex-valued bounded function on X_2, we deduce that $\{u.(f_{n_j} o\varphi)\}_{j=1}^{\infty}$ converges uniformly on X_2. It follows that there exists $N_2 \in \mathbb{N}$ such that

$$|u(x)f_{n_j}(\varphi(x)) - u(x)f_{n_k}(\varphi(x))| < \min\left\{\frac{\varepsilon}{3}, \frac{\delta^{\alpha_1}\varepsilon}{4M}\right\}, \tag{4.19}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_2$ and $k \geq N_2$ and for each $x \in X_2$. Take $N = \max\{N_1, N_2\}$ and let $j, k \in \mathbb{N}$ with $j \geq N$ and $k \geq N$. Since (4.19) holds for all $x \in X_2$, we get

$$\|Tf_{n_j} - Tf_{n_k}\|_{X_2} \leq \frac{\varepsilon}{3}. \tag{4.20}$$

We now show that

$$\frac{|(Tf_{n_j} - Tf_{n_k})(x) - (Tf_{n_j} - Tf_{n_k})(y)|}{d_2^{\alpha_2}(x,y)} \leq \frac{\varepsilon}{2}, \tag{4.21}$$

for all $x, y \in K_2$ with $x \neq y$. To this aim, take $h_{j,k} = Tf_{n_j} - Tf_{n_k}$ and pick $x, y \in K_2$ with $x \neq y$. Let us distinguish the following cases.

Case 1. $\varphi(x) = \varphi(y)$. Then, by (4.18) and $u \in A_2$ we get

$$\begin{aligned}
\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} |f_{n_j}(\varphi(x)) - f_{n_k}(\varphi(x))| \\
&\leq p_{(K_2,d_2^{\alpha_2})}(u) \frac{\varepsilon}{2\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} + 4} \\
&\leq \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \frac{\varepsilon}{2\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} + 4} \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

Case 2. $0 < d_1(\varphi(x), \varphi(y)) \leq \delta$. Then by (4.16), (4.17), (4.19) and (4.18) we get

$$\begin{aligned}
\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x,y)} &\leq |u(x)| \frac{f_{n_j}(\varphi(x)) - f_{n_j}(\varphi(y))}{d_2^{\alpha_2}(x,y)} \\
&\quad + |u(x)| \frac{f_{n_k}(\varphi(x)) - f_{n_k}(\varphi(y))}{d_2^{\alpha_2}(x,y)} \\
&\quad + \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} |f_{n_j}(\varphi(y)) - f_{n_k}(\varphi(y))| \\
&= |u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \frac{f_{n_j}(\varphi(x)) - f_{n_j}(\varphi(y))}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \\
&\quad + |u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \frac{f_{n_k}(\varphi(x)) - f_{n_k}(\varphi(y))}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \\
&\quad + \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x,y)} |f_{n_j}(\varphi(y)) - f_{n_k}(\varphi(y))|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varepsilon}{2\left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}+4} p_{(K_1,d_1^{\alpha_1})}(f_{n_j}) \\
&\quad + \frac{\varepsilon}{2\left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}+4} p_{(K_1,d_1^{\alpha_1})}(f_{n_k}) \\
&\quad + \left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \frac{\varepsilon}{2\left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}+4} \\
&\leq \frac{\left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}+2}{2\left\|u\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}+4}\varepsilon \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Case 3. $x, y \in K_2 \cap \mathrm{coz}(u)$ and $d_1(\varphi(x), \varphi(y)) \geq \delta$. Since $j \geq N_2$ and $k \geq N_2$ and $x, y \in X_2$, by (4.19) we get

$$\left|u(x)f_{n_j}(\varphi(x)) - u(x)f_{n_k}(\varphi(x))\right| < \frac{\delta^{\alpha_1}\varepsilon}{4M}, \tag{4.22}$$

and

$$\left|u(y)f_{n_j}(\varphi(y)) - u(y)f_{n_k}(\varphi(y))\right| < \frac{\delta^{\alpha_1}\varepsilon}{4M}. \tag{4.23}$$

Now, from (4.16), (4.22) and (4.23) we obtain

$$\begin{aligned}
\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \\
&\leq M \frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \\
&\leq M \left[\frac{\left|u(x)f_{n_j}(\varphi(x)) - u(x)f_{n_k}(\varphi(x))\right|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \right. \\
&\quad \left. + \frac{\left|u(y)f_{n_j}(\varphi(y)) - u(y)f_{n_k}(\varphi(y))\right|}{d_1^{\alpha_1}(\varphi(x),\varphi(y))} \right] \\
&\leq M \left[\frac{\left|u(x)f_{n_j}(\varphi(x)) - u(x)f_{n_k}(\varphi(x))\right|}{\delta^{\alpha_1}} \right. \\
&\quad \left. + \frac{\left|u(y)f_{n_j}(\varphi(y)) - u(y)f_{n_k}(\varphi(y))\right|}{\delta^{\alpha_1}} \right] \\
&\leq \frac{M}{\delta^{\alpha_1}} \left[\frac{\delta^{\alpha_1}\varepsilon}{4M} + \frac{\delta^{\alpha_1}\varepsilon}{4M} \right] \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Case 4. $x, y \in K_2 \setminus \mathrm{coz}(u)$. Then

$$\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x, y)} < \frac{\varepsilon}{2}.$$

Case 5. $x \in K_2 \setminus \mathrm{coz}(u)$, $y \in K_2$ and $d_1(\varphi(x), \varphi(y)) \geq \delta$. Then

$$\begin{aligned}\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x, y)} &= \frac{|u(y)|}{d_2^{\alpha_2}(x, y)}|f_{n_j}(\varphi(y)) - f_{n_k}(\varphi(y))| \\ &= \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x, y)}|f_{n_j}(\varphi(y)) - f_{n_k}(\varphi(y))| \\ &\leq p_{(K_2, d_2^{\alpha_2})}(u)\frac{\varepsilon}{2\|u\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})} + 4} \\ &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Case 6. $x \in K_2$, $y \in K_2 \setminus \mathrm{coz}(u)$ and $d_1(\varphi(x), \varphi(y)) \geq \delta$. Then

$$\begin{aligned}\frac{|(h_{j,k})(x) - (h_{j,k})(y)|}{d_2^{\alpha_2}(x, y)} &= \frac{|u(x)|}{d_2^{\alpha_2}(x, y)}|f_{n_j}(\varphi(x)) - f_{n_k}(\varphi(x))| \\ &= \frac{|u(x) - u(y)|}{d_2^{\alpha_2}(x, y)}|f_{n_j}(\varphi(x)) - f_{n_k}(\varphi(x))| \\ &\leq p_{(K_2, d_2^{\alpha_2})}(u)\frac{\varepsilon}{2\|u\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})} + 4} \\ &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Summarising, we have proved that (4.23) holds for all $x, y \in K_2$ with $x \neq y$. This implies that

$$p_{(K_2, d_2^{\alpha_2})}\left(Tf_{n_j} - Tf_{n_k}\right) \leq \frac{\varepsilon}{2}. \tag{4.24}$$

By (4.20) and (4.24), we deduce that

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})} < \varepsilon.$$

Hence, our claim is justified. Since $(A_2, \|\cdot\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})})$ is a Banach space, we deduce that there exists $g \in A_2$ such that $\left\{Tf_{n_j}\right\}_{j=1}^{\infty}$ converges to g in $(A_2, \|\cdot\|_{\mathrm{Lip}(X_2, K_2, d_2^{\alpha_2})})$. Therefore, T is compact and so the proof is complete. $\square$

Corollary 4.9. *Let (X, d) be compact metric space and let K be a nonempty clopen proper subset of X in (X, d). Suppose that $0 < \beta <$*

$\alpha \leq 1$ and $u \in \mathrm{Lip}(X, K, d^\beta)$. Let $y_0 \in K$ and let the map $\varphi_{y_0} : X \longrightarrow X$ defined by

$$\varphi_{y_0}(y) = \begin{cases} y, & y \in K, \\ y_0, & y \in X \setminus K. \end{cases}$$

Then $T_{y_0} = uC_{\varphi_{y_0}} : \mathrm{Lip}(X, K, d^\alpha) \longrightarrow \mathrm{Lip}(X, K, d^\beta)$ is a compact weighted composition operator.

Proof. Clearly, φ_{y_0} is a continuous mapping from (X, d^β) to (X, d^α). Since

$$\begin{aligned}\frac{d^\alpha(\varphi_{y_0}(x), \varphi_{y_0}(y))}{d^\beta(x, y)} &= \frac{d^\alpha(x, y)}{d^\beta(x, y)} \\ &= d^{\alpha-\beta}(x, y) \\ &\leq (\mathrm{diam}_d(X))^{\alpha-\beta},\end{aligned}$$

for all $x, y \in K$ with $x \neq y$, we deduce that φ_{y_0} is a Lipschitz mapping from (X, d^β) to (X, d^α). According to $\varphi_{y_0}(K) \subseteq K$ and $u \in \mathrm{Lip}(X, K, d^\beta)$, we conclude that $T_{y_0} = uC_{\varphi_{y_0}}$ is a weighted composition operator from $\mathrm{Lip}(X, K, d^\alpha)$ to $\mathrm{Lip}(X, K, d^\beta)$ by Theorem 2.2. Since $\alpha - \beta > 0$ and

$$\frac{d^\alpha(\varphi_{y_0}(x), \varphi_{y_0}(y))}{d^\beta(x, y)} = d^{\alpha-\beta}(x, y) = (d(\varphi_{y_0}(x), \varphi_{y_0}(y)))^{\alpha-\beta},$$

for all $x, y \in K$ with $x \neq y$, we deduce that $\lim \frac{d^\alpha(\varphi_{y_0}(x), \varphi_{y_0}(y))}{d^\beta(x,y)} = 0$ when $x, y \in K$ and $d(\varphi_{y_0}(x), \varphi_{y_0}(y))$ tends to 0. Therefore, the boundedness of u on X implies that

$$\lim u(x) \frac{d^\alpha(\varphi_{y_0}(x), \varphi_{y_0}(y))}{d^\beta(x, y)} = 0,$$

when $x, y \in K$ and $d(\varphi_{y_0}(x), \varphi_{y_0}(y))$ tends to 0. Hence, $T_{y_0} = uC_{\varphi_{y_0}}$ is compact by Theorem 4.8. □

Theorem 4.10. *Let u be a complex-valued function on X_2, $K_2 \cap \mathrm{coz}(u) \neq \emptyset$, $\varphi : X_2 \longrightarrow X_1$ be a map with $\varphi(X_2) \subseteq K_1$ and let $\varphi|_K$ be a supercontractive mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ for all nonempty compact subset K of $K_2 \cap \mathrm{coz}(u)$. Suppose that $T = uC_\varphi$ is a nonzero weighted composition operator from A_1 to A_2. If $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ is a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$, then T is compact.*

Proof. Let $\varphi|_{K_2 \cap \mathrm{coz}(u)}$ be a Lipschitz mapping from $(K_2 \cap \mathrm{coz}(u), d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ and $\varphi|_K$ is a supercontractive mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ for all nonempty compact subset K of $K_2 \cap \mathrm{coz}(u)$. To prove the compactness of T, by Theorem 4.8, it is sufficient to show that

$$\lim u(x)\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}=0, \tag{4.25}$$

when $x,y\in K_2$ and $d_1(\varphi(x),\varphi(y))$ tends to 0. Let $\varepsilon>0$ be given. Since $\varphi|_{K_2\cap\mathrm{coz}(u)}$ is a Lipschitz mapping from $(K_2\cap\mathrm{coz}(u),d_2^{\alpha_2})$ to $(X_1,d_1^{\alpha_1})$, there exists $M>0$ such that

$$\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}\leq M, \tag{4.26}$$

for all $x,y\in K_2\cap\mathrm{coz}(u)$ with $x\neq y$. Set

$$K=\left\{x\in K_2:|u(x)|\geq\frac{\varepsilon}{2\,(M+1)}\right\}. \tag{4.27}$$

Clearly, K is a compact subset of $K_2\cap\mathrm{coz}(u)$. We first assume that $K=\emptyset$. Then

$$|u(x)|<\frac{\varepsilon}{2\,(M+1)}, \tag{4.28}$$

for all $x\in K_2$. Take $\delta=\left(\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}}\right)^{\frac{1}{\alpha_1}}$. Let $x,y\in K_2$ with $0<d_1(\varphi(x),\varphi(y))<\delta$. If $x,y\in K_2\cap\mathrm{coz}(u)$, then by (4.26) and (2.17) we get

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}&\leq\frac{\varepsilon}{2\,(M+1)}M\\&<\varepsilon.\end{aligned}$$

If $x\in K_2\setminus\mathrm{coz}(u)$ and $y\in K_2$, then

$$|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}<\varepsilon.$$

If $x\in K_2$ and $y\in K_2\setminus\mathrm{coz}(u)$, then

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)}&=\frac{|u(x)-u(y)|}{d_2^{\alpha_2}(x,y)}d_1^{\alpha_1}(\varphi(x),\varphi(y))\\&\leq p_{(K_2,d_2^{\alpha_2})}(u)\delta^{\alpha_1}\\&\leq\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}}\\&<\varepsilon.\end{aligned}$$

We now assume that $K\neq\emptyset$. Then φ is a supercontractive mapping from $(K,d_2^{\alpha_2})$ to $(X_1,d_1^{\alpha_1})$ and so there exists δ_0 with $0<\delta_0<$

$\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}}$ such that

$$\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} < \frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)}, \tag{4.29}$$

for all $x \in K$ with $0 < d_2^{\alpha_2}(x,y) < \delta_0$. Take

$$\delta = \min\left\{\left(\frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)}\right)^{\frac{1}{\alpha_1}}, \delta_0^{\frac{2}{\alpha_1}}\right\}.$$

We prove that

$$|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} < \varepsilon, \tag{4.30}$$

for all $x, y \in K_2$ with $0 < d_1(\varphi(x),\varphi(y)) < \delta$. To this aim, pick $x, y \in K_2$ with $0 < d_1(\varphi(x),\varphi(y)) < \delta$. Let us distinguish the following cases.

Case 1. $x, y \in K$ with $0 < d_2^{\alpha_2}(x,y) < \delta_0$. Then by (4.29) we get

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &\le \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)}\\ &< \varepsilon.\end{aligned}$$

Case 2. $x, y \in K$ with $d_2^{\alpha_2}(x,y) \ge \delta_0$. Then

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &\le \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\frac{\delta^{\alpha_1}}{\delta_0}\\ &\le \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\delta_0\\ &\le \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}}\\ &< \varepsilon.\end{aligned}$$

Case 3. $x \in K_2 \setminus K$ and $y \in K_2$. Then

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &\le \frac{\varepsilon}{2\,(M+1)}M\\ &< \varepsilon.\end{aligned}$$

Case 4. $x \in K$ and $y \in K_2 \setminus (K \cup \mathrm{coz}(u))$. Then

$$\begin{aligned}|u(x)|\,\frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)-u(y)|}{d_2^{\alpha_2}(x,y)}d_1^{\alpha_1}(\varphi(x),\varphi(y))\\ &\le p_{(K_2,d_2^{\alpha_2})}(u)\delta^{\alpha_1}\\ &\le \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\,\frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)}\end{aligned}$$

$$< \varepsilon.$$

Case 5. $x \in K$ and $y \in (K_2 \cap \mathrm{coz}(u)) \setminus K$. Then

$$\begin{aligned}
|u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &\leq \frac{|u(x)-u(y)|}{d_2^{\alpha_2}(x,y)} d_1^{\alpha_1}(\varphi(x),\varphi(y)) \\
&\quad + |u(y)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} \\
&\leq p_{(K_2,d_2^{\alpha_2})}(u)\delta^{\alpha_1} + \frac{\varepsilon}{2\,(M+1)} M \\
&< \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Case 6. $x \in K$ and $y \in K \setminus \mathrm{coz}(u)$. Then

$$\begin{aligned}
|u(x)| \frac{d_1^{\alpha_1}(\varphi(x),\varphi(y))}{d_2^{\alpha_2}(x,y)} &= \frac{|u(x)-u(y)|}{d_2^{\alpha_2}(x,y)} d_1^{\alpha_1}(\varphi(x),\varphi(y)) \\
&\leq p_{(K_2,d_2^{\alpha_2})}(u)\delta^{\alpha_1} \\
&\leq \|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} \frac{\varepsilon}{2\left(1+\|u\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)} \\
&< \varepsilon.
\end{aligned}$$

Summarizing, we have shown that (4.30) holds for all $x, y \in K_2$ with $0 < d_1(\varphi(x),\varphi(y)) < \delta$. Hence, (4.25) holds and the proof is complete. □

The following example shows that the converse of Theorem 4.10 is not valid.

Example 4.11. Let $X_j = [-2,2]$, d_j be the Euclidean metric on X_j, $K_j = [-1,1]$ and $\alpha_j = 1$ for $j \in \{1,2\}$. Define the function $u : X_2 \longrightarrow \mathbb{C}$ by

$$u(x) = x, \quad (x \in X_2).$$

Then $u \in A_2$, $K_2 \cap \mathrm{coz}(u) = [-1,1] \setminus \{0\}$. Define the map $\varphi : X_2 \longrightarrow X_1$ by

$$\varphi(x) = \mathrm{sgn}(x), \quad (x \in X_2).$$

It is easy to see that for each nonempty compact subset K of $K_2 \cap \mathrm{coz}(u)$, there exists a $\gamma \in (0,1)$ such that $K \subset [-1,-\gamma] \cup [\gamma,1]$. On the other hand, it is clear that φ is a supercontractive mapping from $[-1,-\gamma] \cup [\gamma,1]$ for all $\gamma \in (0,1)$. Therefore, $\varphi|_K$ is a supercontractive mapping from $(K, d_2^{\alpha_2})$ to $(X_1, d_1^{\alpha_1})$ for all nonempty compact subset of $K_2 \cap \mathrm{coz}(u)$.

Since

$$\frac{d_1^{\alpha_1}(\varphi(\frac{1}{n}),\varphi(-\frac{1}{n}))}{d_2^{\alpha_2}(\frac{1}{n},-\frac{1}{n})}=\frac{\left|\varphi(\frac{1}{n})-\varphi(-\frac{1}{n})\right|}{\left|\frac{1}{n}-\frac{-1}{n}\right|}=n,$$

for all $n\in\mathbb{N}$, we deduce that $\varphi|_{K_2\cap\operatorname{coz}(u)}$ is not a Lipschitz mapping from $(K_2\cap\operatorname{coz}(u),d_2^{\alpha_2})$ to $(X_1,d_1^{\alpha_1})$.

Let $f\in A_1$. We show that

$$p_{(X_2,d_2^{\alpha_2})}(Tf)\le 3\,\|f\|_{K_1}\,. \tag{4.31}$$

To prove (4.31), it is sufficient to show that

$$\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}\le 3\,\|f\|_{K_1}\,, \tag{4.32}$$

for all $x,y\in X_2$ with $x\neq y$. To this aim, pick $x,y\in X_2$ with $x\neq y$. Let us distinguish the following cases.

Case 1. $x=0$ and $y\neq 0$. Then

$$\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}=\frac{|y|\,|f(\varphi(y))|}{|y|}\le\|f\|_{K_1}\,.$$

Case 2. $x\neq 0$ and $y=0$. Then

$$\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}=\frac{|x|\,|f(\varphi(x))|}{|x|}\le\|f\|_{K_1}\,.$$

Case 3. $x>0$ and $y>0$. Then

$$\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}=\frac{|x-y|\,|f(1)|}{|x-y|}\le\|f\|_{K_1}\,.$$

Case 4. $x<0$ and $y<0$. Then

$$\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}=\frac{|x-y|\,|f(-1)|}{|x-y|}\le\|f\|_{K_1}\,.$$

Case 5. $x>0$ and $y<0$. Then

$$\begin{aligned}\frac{|Tf(x)-Tf(y)|}{d_2^{\alpha_2}(x,y)}&=\frac{|xf(1)-yf(-1)|}{|x-y|}\\&=\frac{|x[f(1)-f(-1)]+(x-y)f(-1)|}{|x-y|}\\&\le\frac{x}{x-y}\,|f(1)-f(-1)|+|f(-1)|\\&\le|f(1)-f(-1)|+|f(-1)|\\&\le 3\,\|f\|_{K_1}\,.\end{aligned}$$

Case 6. $x < 0$ and $y > 0$. Then

$$\begin{aligned}\frac{|Tf(x) - Tf(y)|}{d_2^{\alpha_2}(x,y)} &= \frac{|xf(-1) - yf(1)|}{|x-y|} \\ &\leq |f(1)| + \frac{y}{y-x}\,|f(1) - f(-1)| \\ &\leq |f(1)| + |f(1) - f(-1)| \\ &\leq 3\,\|f\|_{K_1}.\end{aligned}$$

Thus, (4.32) holds for all $x, y \in X_2$ with $x \neq y$ and so (4.31) holds. Therefore, $Tf \in A_2$. Since $f \in A_1$ was chosen arbitrary, we deduce that $T = uC_\varphi$ is a weighted composition operator from A_1 to A_2.

We now show that T is compact. Let $\{f_n\}_{n=1}^\infty$ be a sequence in A_1 with $\|f_n\|_{\mathrm{Lip}(X_1,K_1,d_1^{\alpha_1})} \leq 1$ for all $n \in \mathbb{N}$. Then the sequence $\{f_n|_{K_1}\}_{n=1}^\infty$ is uniformly bounded on K_1 and $p_{(X_1,d_1^{\alpha_1})(f_n)} \leq 1$ for all $n \in \mathbb{N}$ which implies that $\{f_n|_{K_1}\}_{n=1}^\infty$ is equicontinuous on the compact metric space $(K_1, d_1^{\alpha_1})$. By Arzela-Ascoli theorem, there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $\{f_{n_j}|_{K_1}\}_{j=1}^\infty$ converges uniformly on K_1. We now claim that $\{Tf_{n_j}\}_{j=1}^\infty$ is a Cauchy sequence in $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. Let $\varepsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that

$$\left\|f_{n_j} - f_{n_k}\right\|_{K_1} < \frac{\varepsilon}{6},$$

for all $j, k \in \mathbb{N}$ with $j \geq N_1$ and $k \geq N_1$. Since $\{f_{n_j}\}_{j=1}^\infty$ converges uniformly on K_1, $\varphi(X_2) \subseteq K_1$ and u is a complex-valued bounded function on X_2, we deduce that $\{Tf_{n_j}\}_{j=1}^\infty$ converges uniformly on X_2. Thus, there exists N_2 such that

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{X_2} < \frac{\varepsilon}{2},$$

for all $j, k \in \mathbb{N}$ with $j \geq N_2$ and $k \geq N_2$. Take $N = \max\{N_1, N_2\}$ and let $j, k \in \mathbb{N}$ with $j \geq N$ and $k \geq N$. Then

$$\left\|f_{n_j} - f_{n_k}\right\|_{K_1} < \frac{\varepsilon}{6}, \tag{4.33}$$

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{X_2} < \frac{\varepsilon}{2}. \tag{4.34}$$

By the argument above and applying (4.33), we deduce that

$$\begin{aligned}p_{(K_2,d_2^{\alpha_2})}\left(Tf_{n_j} - Tf_{n_k}\right) &= p_{(K_2,d_2^{\alpha_2})}\left(T(f_{n_j} - f_{n_k})\right) \\ &\leq 3\left\|f_{n_j} - f_{n_k}\right\|_{K_1}\end{aligned}$$

$$< \frac{\varepsilon}{2}. \tag{4.35}$$

From (4.34) and (4.35), we get

$$\left\|Tf_{n_j} - Tf_{n_k}\right\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})} < \varepsilon.$$

Hence, our claim is justified. Since $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$ is a Banach space, we deduce that $\left\{Tf_{n_j}\right\}_{j=1}^{\infty}$ converges in $\left(A_2, \|\cdot\|_{\mathrm{Lip}(X_2,K_2,d_2^{\alpha_2})}\right)$. Therefore, T is compact.

5. Conclusions

In this paper,we study weighted composition operators between extended Lipschitz algebras on compact metric spaces. In particular, we show that every weighted composition operator between extended Lipschitz algebras is automatically continuous. We also give some necessary conditions and some sufficient conditions for the injectivity, the surjectivity and the compactness of these operators. Our results extend some of the obtained results in [5] and [7].

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