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Uniform Convergence to a Left Invariance on Weakly Compact Subsets

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Abstract. Let $\{a_\alpha\}_{\alpha\in I}$ be a bounded net in a Banach algebra A and φ a nonzero multiplicative linear functional on A. In this paper, we deal with the problem of when $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all a in weakly compact subsets of A. We show that Banach algebras associated to locally compact groups such as Segal algebras and L^1-algebras are responsive to this concept. It is also shown that $Wap(A)$ has a left invariant φ-mean if and only if there exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \to 0$ uniformly for all a in weakly compact subsets of A. Other results in this direction are also obtained.

1. Introduction

Let A be an arbitrary Banach algebra and φ a character of A, that is a homomorphism from A onto $\mathbb{C}$. A is called φ-amenable if there exists a bounded linear functional m on A^* satisfying $\langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$ and $f \in A^*$. Approximating m in the weak* topology of A^{**} and then passing to convex combinations, we obtain a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ far all a in A [12]. On the other hand, whenever we have a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$, then each of its weak* accumulation points in A^{**} is a left invariant φ-mean on A^*. For more details on φ-amenability of a Banach algebra the interested reader is referred to [9, 12, 15]. This concept considerably generalizes the notion of left amenability for Lau algebras. Recently

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the notion of α-amenable hypergroups was introduced and studied in [1, 2, 6]. It is clearly that the net $\{a_\alpha\}_{\alpha\in I}$ can be chosen in such a way that $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all a in compact subsets of A. The present paper grew out the attempt to extend the uniform convergence to weakly compact subsets of A.

We shall investigate that this problem is true over a Segal algebra. It has motivated large parts of this paper. In particular, we shall consider the special case $S(G) = L^1(G)$ and it is shown that this problem is equivalent to the amenability of G. Although we are not able to answer for general, we show $Wap(A)$ has a left invariant φ-mean if and only if there exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \to 0$ uniformly for all a in weakly compact subsets of A.

2. Notation and Preliminary

In this paper, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product which is defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f.a$ and $m.f$ of A^* and $mn \in A^{**}$ are defined by

$$\langle f.a, b\rangle = \langle f, ab\rangle, \qquad \langle n.f, a\rangle = \langle n, f.a\rangle, \qquad \langle mn, f\rangle = \langle m, n.f\rangle,$$

respectively. With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**} [3]. A functional $f \in A^*$ for which $\{f.a;\ \|a\| \le 1\}$ is relatively compact in the weak topology of A^* is said to be weakly almost periodic. The set of weakly almost periodic functionals on A is denoted by $Wap(A)$ (see [4, 8]).

Recall that a Segal algebra $S(G)$ on a locally compact group G, is a dense left ideal of $L^1(G)$ that satisfies the following conditions:

(i) $S(G)$ is a Banach space with respect to a norm $\|.\|_S$, called a Segal norm, satisfying $\|\psi\|_1 \le \|\psi\|_S$ for $\psi \in S(G)$, where $\|.\|_1$ denotes the L^1-norm.

(ii) For $\psi \in S(G)$ and $y \in G$, $L_y\psi \in S(G)$, where L_y is the left translation operator defined by $L_y\psi(x) = \psi(y^{-1}x)$, $x \in G$. Moreover, the left translation $L_y\psi$, $y \in G$, is continuous in y for each $\psi \in S(G)$.

(iii) The equality $\|L_y\psi\|_S = \|\psi\|_S$ holds for $\psi \in S(G)$, $y \in G$.

Equipped with the norm $\|.\|_S$ and the convolution product, denoted by $*$, $S(G)$ is a Banach algebra. The inequality $\|h * \psi\|_S \le \|h\|_1\|\psi\|_S$ holds for all $h \in L^1(G)$, and $\psi \in S(G)$. The structure of the Segal algebra has been studied in [17].

Finally, we say that an element a of A is φ-maximal if it satisfies $\|a\| = \varphi(a) = 1$. Let $P_1(A, \varphi)$ denote the collection of all φ-maximal elements of A [11]. When A is an Lau algebra and φ is the identity of the von Neumann algebra A^*, the φ-maximal elements are precisely the positive linear functionals of norm 1 on A^* and hence span A. Let $X(A,\varphi)$ denote the closed linear span of $P_1(A,\varphi)$. Throughout the paper, $\Delta(A)$ will denote the set of all homomorphisms from A onto $\mathbb{C}$.

3. Main Results

Let A be a Banach algebra and let X be a closed subspace of A^*. We say that X is invariant if $f.a \in X$ whenever $f \in X$ and $a \in A$.

Definition 3.1. Let A be a Banach algebra and let X be a closed subspace of A^* with $\varphi \in X$ that is invariant. A continuous functional m on X is called a left invariant φ-mean on X if the following properties hold:

$$\langle m,\varphi\rangle = 1, \qquad \langle m, f.a\rangle = \varphi(a)\langle m,f\rangle, \quad (f\in X,\ a\in A)$$

Definition 3.2. A net $\{a_\alpha\}_{\alpha\in I}$ in $\{a\in A;\ \varphi(a)=1\}$ is said to converges strongly to a left invariance uniformly on weakly compact subsets of A if for every weakly compact set $C\subseteq A$, $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all $a\in C$.

In the following, $P_1((S(G),\|.\|_1),1)$ denotes the collection of all 1-maximal elements of a Segal algebra $S(G)$ with respect to L^1-norm.

Theorem 3.3. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) *There is a net $\psi_\alpha \in P_1((S(G),\|.\|_1),1)$ such that $\|\psi*\psi_\alpha - \psi_\alpha\|_S \to 0$ for each $\psi\in P_1((S(G),\|.\|_1),1)$.*

(ii) *There is a net $\psi_\alpha \in P_1((S(G),\|.\|_1),1)$ such that for each weakly compact subset $C\subseteq P_1((S(G),\|.\|_1),1)$, $\|\psi*\psi_\alpha-\psi_\alpha\|_S \to 0$ uniformly for all $\psi\in C$.*

Proof. (ii) implies (i): This is because the finite subsets in $P_1((S(G),\|.\|_1),1)$ are weakly compact.

(i) implies (ii): Let $\{\psi_\alpha\}_{\alpha\in I}\subseteq P_1((S(G),\|.\|_1),1)$ be as in (i). By definition $\|\psi\|_1 \le \|\psi\|_S$ for all $\psi\in S(G)$, and so $\|\psi*\psi_\alpha - \psi_\alpha\|_1 \to 0$ for each $\psi\in P_1((S(G),\|.\|_1),1)$. We can assume that ψ_α is left equicontinuous (that is, given $\epsilon>0$, there is some neighborhood U of the identity in G such that $\|\delta_x*\psi_\alpha-\psi_\alpha\|_1<\epsilon$ for any α and $x\in U$) otherwise replace ψ_α by $\psi*\psi_\alpha$ where ψ is a fixed element in $P_1((S(G),\|.\|_1),1)$. We claim that for every weakly compact subset C of $P_1((S(G),\|.\|_1),1)$ and $\epsilon\in(0,1)$, there exists α_0 such that $\|\psi*\psi_\alpha-\psi_\alpha\|_1<\epsilon$ for all $\alpha\succeq\alpha_0$ and $\psi\in C$. Let ψ_0 be a fixed element in $P_1((S(G),\|.\|_1),1)$. For the forward implication, note that the weak topology on $S(G)$ is finer than the relative weak topology on $S(G)$ inherited from $L^1(G)$. By Theorem 4.21.2 in [5], there exists a compact set K in G such that $\int_{G\setminus K}\psi(x)dx < \frac{\epsilon}{4\|\psi_o\|_S}$ for all $\psi\in C$. By the above argument, there exists $\alpha_0\in I$ such that $\|\delta_x*\psi_\alpha-\psi_\alpha\|_1<\frac{\epsilon}{2\|\psi_o\|_S}$ for all $\alpha\succeq\alpha_0$ and $x\in K$(see Proposition 6.7 in [16]). For each $\alpha\succeq\alpha_0$ and $\psi\in C$, we have

$$\begin{aligned}\|\psi*\psi_\alpha-\psi_\alpha\|_1 &= \int\Big|\int\psi_\alpha(y^{-1}x)\psi(y)dy-\psi_\alpha(x)\Big|dx\\ &\le \int\Big|\int_K(\psi_\alpha(y^{-1}x)-\psi_\alpha(x))\psi(y)dy\Big|dx\\ &\quad+\int\int_{G\setminus K}|\psi_\alpha(y^{-1}x)-\psi_\alpha(x)|\psi(y)dydx\\ &< \frac{\epsilon\int_K\psi(y)dy}{2\|\psi_o\|_S}+2\int_{G\setminus K}\psi(y)dy\int\psi_\alpha(x)dx\\ &< \frac{\epsilon}{\|\psi_o\|_S}.\end{aligned}$$

Let us define $\phi_\alpha=\psi_\alpha*\psi_0$. For each $\alpha\succeq\alpha_0$ and $\psi\in C$, we have

$$\begin{aligned}\|\psi*\phi_\alpha-\phi_\alpha\|_S &= \|\psi*\psi_\alpha*\psi_0-\psi_\alpha*\psi_0\|_S\\ &\le \|\psi*\psi_\alpha-\psi_\alpha\|_1\|\psi_o\|_S\\ &< \epsilon.\end{aligned}$$ □

Let G be a locally compact group with left Haar measure and consider the convolution algebra $L^1(G)$ [7]. Note that the group algebra $L^1(G)$ is amenable with respect to the trivial character 1 precisely when G is amenable [10]. The preceding proposition shows that if G is an amenable locally compact group, then $L^1(G)$ has a bounded net which converges strongly to a left invariance uniformly on weakly compact subsets of $L^1(G)$.

As a straightforward application of our main result, we have the following result:

Corollary 3.4. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) *There is a net $\psi_\alpha\in P_1(L^1(G),1)$ such that $\|\psi*\psi_\alpha-\psi_\alpha\|_1\to 0$ for each $\psi\in P_1(L^1(G),1)$, i.e. G is amenable;*

(ii) *There is a net $\psi_\alpha\in P_1(L^1(G),1)$ such that for each weakly compact subset $C\subseteq L^1(G)$, $\|\psi*\psi_\alpha-\int\psi(x)dx\psi_\alpha\|_1\to 0$ uniformly for all $\psi\in C$.*

Proof. As $L^1(G)$ is a Segal algebra, this is just a re-statement of Theorem 3.3. □

Let A be an arbitrary Banach algebra. It remains an open question, to the author's knowledge, whether the existence of a bounded net $\{a_\alpha\}_{\alpha\in I}$ in A which converges strongly to a left invariance uniformly on weakly compact subsets of A is equivalent to $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for each $a \in A$. We show this is the case for,

$$\|a\|_{Wap(A)} = \sup\{|\langle f, a\rangle| : f \in Wap(A), \|f\| \leq 1\}, \quad (a \in A)$$

The reason why we are interested in $Wap(A)$ is the following:

Theorem 3.5. *Let A be a Banach algebra with a bounded approximate identity and $\varphi \in \Delta(A)$. Then the following statements are equivalent:*

(i) *There exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \to 0$ for each $a \in A$;*

(ii) *There exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that for each weakly compact subset $C \subseteq A$, $\|aa_\alpha - \varphi(a)a_\alpha\|_{Wap(A)} \to 0$ uniformly for all $a \in C$.*

Proof. By the Banach Alaoghlu's Theorem [18], without loss of generality we may assume that $a_\alpha \to m$ in the weak* topology of A^{**}. Then $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$, for all $f \in Wap(A)$, $a \in A$ [12]. Let $T_f : A \longmapsto A^*$ be a bounded linear mapping specified by $T_f(a) = f.a$. Define the map $\kappa_A : A^* \longmapsto B(A, A^*)$ by $\kappa_A(f) = T_f$. Take $f \in Wap(A)$ and consider $\{a_\alpha f\}_{\alpha\in I}$. The corresponding net $\{T_{a_\alpha f}\}_{\alpha\in I}$ converges to T_{mf} in the weak operator topology. This is immediate from the fact that the weak topology and weak* topology coincide on weak closure $\overline{\{a.f : \|a\| \leq \|m\|\}}$ of $\{a.f : \|a\| \leq \|m\|\}$. The equicontinuity of $\{T_{a_\alpha f}\}_{\alpha\in I}$ is now an exercise in functional analysis. Let C be any weakly compact subset of A. C is weakly bounded, and so C is norm bounded (see Theorem 3.18 in [18]). Let $M = \sup\{\|c\| : c \in C\}$. The net $\{T_{a_\alpha f}\}_{\alpha\in I}$ converges uniformly to $T_{m.f}$ in the weak operator topology on C. This latter fact is crucial for our argument, so we give a proof.

Let W be a weak neighborhood of zero in A^*. Choose a weak neighborhood V of zero in A^* such that $V + V + V \subseteq W$ and a symmetric weak neighborhood U of zero in A such that $T_{a_\alpha f}(U) \subseteq V$ for all $\alpha \in I$ and $T_{mf}(U) \subseteq V$. C is weakly compact, and therefore $C \subseteq S_0 + U$ for some finite set $S_0 = \{a_1, a_2 ..., a_n\}$. It is a routine matter to see that there exists $\alpha_0 \in I$ such that $T_{a_\alpha f}(a_i) - T_{mf}(a_i) \in V$ for all $\alpha \succeq \alpha_0$ and $a_i \in S_0$. For $\alpha \succeq \alpha_0$ and $a \in C$, we have

$$(T_{a_\alpha f} - T_{mf})(a) \in \bigcup_{i=1}^{n} (T_{a_\alpha f} - T_{mf})(a_i) + (T_{a_\alpha f} - T_{mf})(U)$$

$$\subseteq \bigcup_{i=1}^{n}(T_{a_\alpha f} - T_{mf})(a_i) + T_{a_\alpha f}(U) - T_{mf}(U)$$
$$\subseteq V + V + V \subseteq W.$$

By the above argument, for any given $\epsilon > 0$ and any $n \in A^{**}$, there exists $\alpha_0 \in I$ such that

$$|\langle n, T_{a_\alpha f}(a) - T_{mf}(a)\rangle| < \frac{\epsilon}{2}.$$

for all $\alpha \succeq \alpha_0$ and $a \in C$. On the other hands, A has an approximate identity $\{e_\alpha\}_{\alpha\in I}$. Any weak*-lim E of $\{e_\alpha\}_{\alpha\in I}$ is a right identity of Banach algebra A^{**}. Hence for all $\alpha \succeq \alpha_0$ and $a \in C$,

$$\begin{aligned}|\langle aa_\alpha - am, f\rangle| &= |\langle a_\alpha f - mf, a\rangle| \\ &= |\langle E, a_\alpha f.a - mf.a\rangle| \\ &< \frac{\epsilon}{2}.\end{aligned}$$

We also have $|\langle a_\alpha, f\rangle - \langle m, f\rangle| < \frac{\epsilon}{2M}$ for all $\alpha \succeq \alpha_0$. Consequently

$$\begin{aligned}|\langle aa_\alpha - \varphi(a)a_\alpha, f\rangle| &\leq |\langle aa_\alpha - am, f\rangle| + |\varphi(a)||\langle m, f\rangle - \langle a_\alpha, f\rangle| \\ &< \epsilon.\end{aligned}$$

This means that $aa_\alpha - \varphi(a)a_\alpha \longrightarrow 0$ uniformly in the weak topology of $Wap(A)$ for all $a \in C$. An argument similar to that in the proof of Theorem 1.2 in [12] shows that we can find a bounded net $\{u_\alpha\}_{\alpha\in I}$ consisting of convex combination of elements in $\{a_\alpha\}_{\alpha\in I}$ such that $\|au_\alpha - \varphi(a)u_\alpha\|_{Wap(A)} \to 0$ uniformly for all $a \in C$. □

A special interesting case is that there exists a left invariant φ-mean on A^*. We obtain:

Theorem 3.6. *Let $\{a_\alpha\}_{\alpha\in I}$ be a bounded net in $\{a \in A;\ \varphi(a) = 1\}$ which converges strongly to a left invariance uniformly on weakly compact subsets of A and let m be a left invariant φ-mean on A^*. Then there is a net $\{b_\beta\}_{\beta\in J}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $b_\beta \to m$ in the weak* topology and $\{b_\beta\}_{\beta\in J}$ converges strongly to a left invariance uniformly on weakly compact subsets of A.*

Proof. Let such a net $\{a_\alpha\}_{\alpha\in I}$ exists. Choose a net $\{b_\beta\}_{\beta\in J}$ in A with the property that $b_\beta \to m$ in the weak* topology of A^{**} and $\|b_\beta\| \leq \|m\|$ for all $\beta \in J$ [18]. Since $\langle b_\beta, \varphi\rangle \to \langle m, \varphi\rangle = 1$, after passing to a subnet and replacing b_β by $\frac{1}{\varphi(b_\beta)}b_\beta$, we can assume that $\varphi(b_\beta) = 1$ and $\|b_\beta\| \leq \|m\| + 1$ for all $\beta \in J$. For each (α, f) in the product directed set $I \times \prod\{J;\ \alpha \in I\}$, we define $R(\alpha, f) = (\alpha, f(\alpha))$, $\alpha \in I$, $f \in \prod\{J;\ \alpha \in I\}$ and let $S(\alpha, \beta) = a_\alpha b_\beta$. The iterated limit $\lim_\alpha \lim_\beta a_\alpha b_\beta$

(in the weak* topology of A^{**}) exists and is equal to m. Indeed, for $f \in A^*$

$$\begin{aligned}\lim_\beta \langle f, a_\alpha b_\beta\rangle &= \lim_\beta \langle f a_\alpha, b_\beta\rangle \\ &= \lim_\beta \langle b_\beta, f a_\alpha\rangle \\ &= \langle m, f a_\alpha\rangle \\ &= \langle m, f\rangle.\end{aligned}$$

By the Iterated Limit Theorem, see p.69 in [13],

$$\begin{aligned}\lim_{(\alpha,f)} SoR(\alpha, f) &= \lim_{(\alpha,f)} a_\alpha b_{f(\alpha)} \\ &= m\end{aligned}$$

in the weak* topology of A^{**} (with respect to (α, f)). It remains to show that $SoR(\alpha, f)$ converges strongly to a left invariance uniformly on weakly compact subsets C of A. Let $\epsilon > 0$ be given. For every weakly compact subset C of A, there exists $\alpha_0 \in I$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| < \frac{\epsilon}{\|m\|+1}$ for all $\alpha \succeq \alpha_0$ and $a \in C$. If $\alpha \succeq \alpha_0$ and $a \in C$, then

$$\begin{aligned}\|aSoR(\alpha, f) - \varphi(a)SoR(\alpha, f)\| &= \|aa_\alpha b_{f(\alpha)} - \varphi(a)a_\alpha b_{f(\alpha)}\| \\ &\le \|aa_\alpha - \varphi(a)a_\alpha\|(\|m\| + 1) \\ &< \epsilon.\end{aligned}$$

This completes the proof. □

Proposition 3.7. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following statements are equivalent:*

(i) *There exists a net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\{a_\alpha\}_{\alpha\in I}$ converges to some left invariant φ-mean m with $\|m\| = 1$ in the weak* topology and $\{a_\alpha\}_{\alpha\in I}$ converges strongly to a left invariance uniformly on weakly compact subsets of A;*

(ii) *For every weakly compact subset C of A and $\epsilon > 0$,*

$$\inf \{\sup \{\|ca\|;\ c \in C\},\ \varphi(a) = 1,\ \|a\| \le 1 + \epsilon\} \le (1+\epsilon) \sup \{|\varphi(c)|;\ c \in C\};$$

(iii) *There exists a net $\{a_\alpha\}_{\alpha\in I}$ in A with the following properties: $\varphi(a_\alpha) = 1$ for all $\alpha \in I$, $\|a_\alpha\| \to 1$ and $\lim_\alpha \|aa_\alpha\| = |\varphi(a)|$ uniformly on weakly compact subsets of A.*

Proof. (i) implies (ii): Let C be a weakly compact subset of A, $\epsilon > 0$ and let $\delta > 0$ be given. By hypothesis there exists $\alpha_0 \in I$ such that $\|ca_\alpha - \varphi(c)a_\alpha\| < \delta$, $\|a_\alpha\| \le 1 + \epsilon$ for all $\alpha \succeq \alpha_0$ and $c \in C$. Thus for every $c \in C$,

$$\begin{aligned}\|ca_{\alpha_0}\| &\le |\varphi(c)|\|a_{\alpha_0}\| + \delta \\ &< (1 + \epsilon)|\varphi(c)| + \delta.\end{aligned}$$

Since $\delta > 0$ may be chosen arbitrarily, the property holds.

(ii) implies (i): We claim that for every weakly compact subset C of A and $\epsilon > 0$, there exists $a_{C,\epsilon}$ such that $\varphi(a_{C,\epsilon}) = 1$, $\|a_{C,\epsilon}\| \leq 1+\epsilon$ and $\|ca_{C,\epsilon} - \varphi(c)a_{C,\epsilon}\| < \epsilon$ for all $c \in C$. Choose $\delta > 0$ such that $(1+\delta)^2 < 1+\epsilon$. Take $b_{C,\epsilon} \in A$ such that $\varphi(b_{C,\epsilon}) = 1$ and $\|b_{C,\epsilon}\| \leq 1+\delta$. Obviously

$$\{c - \varphi(c)b_{C,\epsilon};\ c \in C\} \cup \{cb_{C,\epsilon} - c;\ c \in C\}$$

is weakly compact and also $\varphi(c - \varphi(c)b_{C,\epsilon}) = \varphi(cb_{C,\epsilon} - c) = 0$ for all $c \in C$. By assumption, there exists $a_{C,\epsilon}{}' \in A$ with $\|a_{C,\epsilon}{}'\| \leq 1+\delta$, $\varphi(a_{C,\epsilon}{}') = 1$ such that $\|(c - \varphi(c)b_{C,\epsilon})a_{C,\epsilon}{}'\| < \frac{\epsilon}{2}$ and $\|cb_{C,\epsilon}a_{C,\epsilon}{}' - ca_{C,\epsilon}{}'\| < \frac{\epsilon}{2}$ for all $c \in C$. Put $a_{C,\epsilon} = b_{C,\epsilon}a_{C,\epsilon}{}'$. Thus $\|a_{C,\epsilon}\| = \|b_{C,\epsilon}a_{C,\epsilon}{}'\| \leq (1+\delta)^2 \leq 1+\epsilon$ and $\varphi(a_{C,\epsilon}) = 1$. For every $c \in C$, we have

$$\begin{aligned}\|ca_{C,\epsilon} - \varphi(c)a_{C,\epsilon}\| &= \|cb_{C,\epsilon}a_{C,\epsilon}{}' - \varphi(c)b_{C,\epsilon}a_{C,\epsilon}{}'\| \\ &\leq \|cb_{C,\epsilon}a_{C,\epsilon}{}' - ca_{C,\epsilon}{}'\| + \|ca_{C,\epsilon}{}' - \varphi(c)b_{C,\epsilon}a_{C,\epsilon}{}'\| \\ &< \epsilon.\end{aligned}$$

Now, order the pairs (C,ϵ), $C \subseteq A$ weakly compact, $\epsilon > 0$, in the obvious manner, and let m be a weak* cluster point of the net $\{a_{C,\epsilon}\}$ in A. Then $\|m\| \leq 1$, $\langle m, \varphi\rangle = 1$ and hence $\|m\| = 1$. So $\{a_{C,\epsilon}\}_{C,\epsilon}$ is the required net.

(iii) implies (ii): Let $\epsilon > 0$ and let C be a weakly compact subset of A. For every $\delta > 0$, there exists $\alpha_0 \in I$ such that $|\|ca_\alpha\| - |\varphi(c)|| < \delta$ and $\|a_\alpha\| \leq 1+\epsilon$ for every $\alpha \succeq \alpha_0$ and $c \in C$. Then

$$\begin{aligned}&\inf\{\sup\{\|ca\|;\ c \in C\},\ \varphi(a) = 1,\ \|a\| \leq 1+\epsilon\} \\ &\qquad \leq \inf\{\sup\{\|ca_\alpha\|;\ c \in C\},\ \alpha \in I\} \\ &\qquad \leq (1+\epsilon)\sup\{|\varphi(c)|;\ c \in C\} + \delta.\end{aligned}$$

Since $\delta > 0$ may be chosen arbitrarily, the property holds.

(i) implies (iii): By hypothesis there exists a net $\{a_\alpha\}_{\alpha\in I}$ in A such that $\varphi(a_\alpha) = 1$ for all $\alpha \in I$, $\|a_\alpha\| \to 1$ and $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly on weakly compact subsets of A. Let $\epsilon > 0$ and let C be a weakly compact subset of A. Since C is a weakly compact subset of A, C is weakly bounded and so $\{|\varphi(c)|;\ c \in C\}$ is bounded [18]. Let $k = \sup\{|\varphi(c)|;\ c \in C\}$. For every $\alpha \in I$ and $c \in C$, we have

$$\begin{aligned}||\|aa_\alpha\| - |\varphi(a)|| &\leq ||\|aa_\alpha\| - |\varphi(a)|\|a_\alpha\|| + |\varphi(a)|\,|\|a_\alpha\| - 1| \\ &\leq \|aa_\alpha - \varphi(a)a_\alpha\| + k|\|a_\alpha\| - 1|.\end{aligned}$$

This shows that $\lim_\alpha \|aa_\alpha\| = |\varphi(a)|$ uniformly on weakly compact subsets of A.

□

Let A be a Lau algebra. The identity of A^* will be denoted by e. Also $P(A)$ will denote the cone of all positive functionals in A and $P_1(A)$ will denote the set of all $f \in P(A)$ such that $f(e) = 1$. Lau in [14] proved that A is left amenable if and only if there exists a net $f_\alpha \in P_1(A)$ such that $\lim_\alpha \|f.f_\alpha\| = |f(e)|$ for each $f \in A$. Note that a Banach algebra A^{**} has a left invariant φ-mean if any one of the conditions in Proposition 1 hold.

Definition 3.8. Let A be a Banach algebra and let Z be a compact convex subset of a locally convex Hausdorff topological vector space E. The pair (A, Z) is called a *flow*, if;

- (i) There exists a map $\rho : A \times E \to E$ such that for each $z \in Z$, the map $\rho(-, z) : A \to E$ is continuous and linear when A has the weak topology;
- (ii) For any $a, b \in A$ and $z \in Z$, $\rho(a, \rho(b, z)) = \rho(ab, z)$.

If $\varphi \in \Delta(A)$, we say that Z is $P_1(A, \varphi)$-invariant under ρ if $\rho(a, z) \in Z$ for any $a \in P_1(A, \varphi)$ and $z \in Z$. In this case ρ induces a map $\rho : P_1(A, \varphi) \times Z \to Z$ of $P_1(A, \varphi)$ on the compact convex subset Z (as affine maps now).

Theorem 3.9. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Among the following two properties, the implication (i) → (ii) hold. If $X(A, \varphi) = A$, then (ii) → (i).*

- (i) *There exists a left invariant φ-mean m in $\overline{P_1(A, \varphi)}^{w^*}$;*
- (ii) *Every flow (A, Z) admits a $P_1(A, \varphi)$-invariant element $z \in Z$, that is, for all $a \in P_1(A, \varphi)$, $\rho(a, z) = z$.*

Proof. Assume that A^{**} has a left invariant φ-mean $m \in \overline{P_1(A, \varphi)}^{w^*}$. Let Z be a compact convex subset of a locally convex Hausdorff topological vector space E and let (A, Z) be a flow. If $f \in E^*$ and $z \in Z$, we may define a functional f^z on A by putting $\langle f^z, a\rangle = \langle f, \rho(a, z)\rangle$, $a \in A$. Since the map $a \mapsto \rho(a, z)$ is continuous, we have $f^z \in A^*$. We embed E into the algebric dual $(E^*)'$ of E^* with the topology $\sigma((E^*)', E^*)$. If Λ is a $\sigma((E^*)', E^*)$-cluster point of Z, then there exists a net $\{z_\alpha\}_{\alpha\in I}$ in Z such that $z_\alpha \to \Lambda$ in the $\sigma((E^*)', E^*)$-topology. Since Z is compact in E, without loss of generality, we may assume that $z_\alpha \to z$ for some $z \in Z$. For every $f \in E^*$, we have $\langle z_\alpha, f\rangle \to \langle \Lambda, f\rangle$ and also $\langle f, z_\alpha\rangle \to \langle f, z\rangle$. We conclude that $\Lambda = z \in Z$, and so Z is a closed subset in $(E^*)'$.

Let z_0 be a fixed element in Z and let $n \in \overline{P_1(A, \varphi)}^{w^*}$. Define $\Lambda_n : E^* \to \mathbb{C}$ by $\Lambda_n(f) = \langle n, f^{z_0}\rangle$. It is easily checked that Λ_n is linear, and so $\Lambda_n \in (E^*)'$. Define $\Lambda : \overline{P_1(A, \varphi)}^{w^*} \to (E^*)'$ by $\Lambda(n) = \Lambda_n$. The mapping Λ from $\overline{P_1(A,\varphi)}^{w^*}$ equipped with the weak* topology into $(E^*)'$ equipped with the $\sigma((E^*)', E^*)$-topology is continuous. In particular, if $a \in P_1(A,\varphi)$, $P_1(A,\varphi)$-invariance of Z imply that $\Lambda(a) = \Lambda_a \in Z$. Indeed, $\Lambda_a = \rho(a, z_0)$. Since $P_1(A,\varphi)$ is weak* dense in $\overline{P_1(A,\varphi)}^{w^*}$ and Z is closed in $(E^*)'$, we conclude that $\Lambda_m \in Z$. We shall show that Λ_m is the required fixed point. Let $a \in P_1(A,\varphi)$ and $f \in E^*$. We consider the mapping $\rho_a : Z \to Z$ defined by $\rho_a(z) = \rho(a, z)$. We have

$$\begin{aligned}\langle f, \Lambda_m\rangle &= \langle m, f^{z_0}\rangle \\ &= \langle m, f^{z_0}a\rangle \\ &= \langle m, (fo\rho_a)^{z_0}\rangle \\ &= \langle fo\rho_a, \Lambda_m\rangle \\ &= \langle f, \rho(a, \Lambda_m)\rangle.\end{aligned}$$

This shows that $\rho(a, \Lambda_m) = \Lambda_m$, that is, Λ_m is a fixed point under the map ρ.

Conversely, assume (ii). Let $E = A^{**}$ with weak* topology and $Z = \overline{P_1(A,\varphi)}^{w^*}$. By the Banach-Alaoglu's theorem [18], Z is weak* compact. Define a map ρ of $A \times A^{**}$ into A^{**} by $\rho(a, p) = ap$ for each $a \in A$ and $p \in A^{**}$. Let p be a fixed element in A^{**} and let $\{a_\alpha\}_{\alpha\in I}$ be a net in A converging to $a \in A$ in the weak topology of A. Then, for $f \in A^*$,

$$\begin{aligned}\lim_\alpha \langle a_\alpha p, f\rangle &= \lim_\alpha \langle a_\alpha, pf\rangle \\ &= \lim_\alpha \langle pf, a_\alpha\rangle \\ &= \langle pf, a\rangle \\ &= \langle ap, f\rangle.\end{aligned}$$

This shows that the mapping $a \mapsto \rho(a, p)$ is continuous. By hypothesis there exists $m \in Z = \overline{P_1(A,\varphi)}^{w^*}$ that is fixed under the map ρ, that is, for every $a \in P_1(A,\varphi)$, $am = m$. Hence m is a left invariant φ-mean. $\square$

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