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On Some Characterization of Generalized Representation Wave-Packet Frames Based on Some Dilation Group

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Abstract. In this paper we consider (extended) metaplectic representation of the semidirect product $G_{\mathbb{J}} = \mathbb{R}^{2d} \times \mathbb{J}$ where $\mathbb{J}$ is a closed subgroup of $Sp(d, \mathbb{R})$, the symplectic group. We will investigate continuous representation frame on $G_{\mathbb{J}}$. We also discuss the existence of duals for such frames and give several characterization for them. Finally, we rewrite the dual conditions, by using the Wigner distribution and obtain more reconstruction formulas.

1. Introduction and Preliminaries

The analytic aspects of classical wave-packet systems, as generalization of both Gabor and wavelet systems, in the framework of continuous redundant coherent systems on Euclidean spaces $\mathbb{R}^d$ studied in [20] and in the direction of finite redundant coherent systems introduced in [21, 22]. Abstract harmonic analysis aspects of classcial wave-packet systems in the settings of locally compat Abelian (LCA) groups investigated in [19]. A larger class of unitary opeartors as generalized dilation groups, namely metaplectice operators, suggested which implies much more richer resolution of the identity and reconstruction formulas [17, 18].

The extended metaplectic (Shale- Weil) representation is widely used in many areas of theoretical physics, such as paraxial optic, quantum mechanics [23, 27]. Also it has many interesting applications in the context of harmonic analysis, see [8, 14, 24] and the references therein.

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In recent years several researchers have been interested in frame theory that plays an important role in many problems. Ali, Antoine [2] and Kaiser [26] generalized frames to a family indexed by some locally compact space, endowed with a Radon measure. A large class of such frames which are based on a locally compact group so called representation frames [25]. They have received a lot of interesting applications such as image processing [3, 12, 13]. Characterization of dual frames is a current topic of research [4, 5, 7, 16]. Our main aim is to find necessary and sufficient conditions for dual of representation frame based on dilation groups, by using the extended metaplectic (Shale- Weil) representation. At the first, we review some definitions from frame theory.

Let $\mathcal{H}$ be a separable Hilbert space and X be a locally compact Hausdorff space endowed with a positive Radon measure ν. A mapping $F : X \to \mathcal{H}$ is called a *continuous frame* if the mapping $x \to \langle F(x), \phi\rangle$ is measurable for all $\phi \in \mathcal{H}$ and there exist constants $0 < C, D < +\infty$ such that

$$C\,\|\phi\|^2 \le \int_X |\langle F(x), \phi\rangle|^2\, d\nu(x) \le D\,\|\phi\|^2\,, \quad (\phi \in \mathcal{H})\,. \tag{1.1}$$

A continuous frame is said to be *tight* when $C = D$. The mapping F is called Bessel if the second inequality in (1.1) holds. Suppose that F is Bessel, the operator $T : L^2(X) \to \mathcal{H}$ defined by

$$T\phi = \int_X \phi(x)F(x)d\nu(x)$$

is a bounded linear operator; so called *the synthesis operator*.

The *continuous frame operator*, which is invertible, positive as well as self adjoint, is defined to be $S = TT^*$ [6]. It is useful to reconstruct the elements of $\mathcal{H}$ by

$$\begin{aligned}\phi &= S^{-1}S\phi \\ &= \int_X \langle \phi, F(x)\rangle\, S^{-1}F(x), \quad (\phi \in \mathcal{H})\,.\end{aligned} \tag{1.2}$$

Let G be a locally compact abelian group with the left Haar measure m_G and the dual group $\widehat{G}$.

A continuous unitary representation Γ of G on a separable Hilbert space $\mathcal{H}$ is said to be square integrable if it is irreducible and there exists a nonzero vector $\psi \in \mathcal{H}$ satisfying in the admissibility condition

$$C_\psi^2 := \|\psi\|^{-2} \int_G |\langle \Gamma(g)\psi, \psi\rangle|^2\, dm_G(g) < \infty.$$

For any admissible vector ψ, the continuous wavelet transform on G defined by

$$W_\psi : \mathcal{H} \to L^2(G), \qquad (W_\psi\varphi)\,(g) = C_\psi^{-1}\,\langle \Gamma(g)\psi, \varphi\rangle\,, \quad (\varphi \in \mathcal{H})\,,$$

is a linear isometry onto a (closed) subgroup $\mathcal{H}_\psi$ of $L^2(G)$. As a consequence, every vector $\varphi \in \mathcal{H}$ can be uniquely reconstructed by

$$\begin{aligned} \varphi &= W_\psi^* W_\psi \varphi \\ &= \frac{1}{C_\psi} \int_G \left(W_\psi \varphi\right)(g) \Gamma(g) \psi dm_G(g). \end{aligned} \tag{1.3}$$

The *Fourier transform* of any function $f \in L^1(G)$, denoted by $\mathcal{F}(f)$ or $\widehat{f}$, is defined by

$$\begin{aligned} \mathcal{F}(f)(\xi) &= \widehat{f}(\xi) \\ &= \int_G f(x) \overline{\xi(x)} dm_G(x). \end{aligned}$$

As usual, the Fourier transform can be extended to a linear isometry from $L^2(G)$ onto $L^2(\widehat{G})$; the so called Plancherel isomorphism. By the inversion formula, we can recover a function from its Fourier transform. It states that if $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then

$$f(x) = \int_{\widehat{G}} \widehat{f}(\xi) \xi(x) dm_{\widehat{G}}(\xi). \tag{1.4}$$

This formula remains valid in spirit for all $f \in L^2(G)$ [15]. For $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, the translation of f by x is defined by $T_x f(y) = f(y-x)$ for almost all $y \in \mathbb{R}$. For $\omega \in \widehat{\mathbb{R}}$ the modulation of f by ω is defined by $M_\omega f(y) = \overline{\omega(y)} f(y)$ for $y \in \mathbb{R}$. They are unitary operators and their Fourier transforms satisfy the following identities

$$\widehat{M_\omega f} = T_{-\omega} \widehat{f}, \qquad \widehat{T_k f} = M_k \widehat{f}.$$

2. Symplectic Group and Metaplectic Representation

For $d \in \mathbb{N}$ the symplectic group is defined by

$$Sp(d, \mathbb{R}) = \left\{ g \in GL(2d, \mathbb{R}); {}^t g \jmath g = \jmath \right\}, \qquad \jmath = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}.$$

We briefly review the construction of metaplectic representation of this group, see [1] for the basic properties of $Sp(d, \mathbb{R})$.

The Heisenberg group $\mathbb{H}^d$ is the set $\mathbb{R}^{2d} \times \mathbb{R}$ with the operation

$$(z, t).(\acute{z}, \acute{t}) = \left(z + \acute{z}, t + \acute{t} - \frac{1}{2} w\left(z, \acute{z}\right) \right),$$

where $w(z, \acute{z}) = {}^t z \jmath \acute{z}$ is the standard symplectic form and $z = (x, \xi) \in \mathbb{R}^{2d}$. The Schrödinger representation of the group $\mathbb{H}^d$ on $L^2(\mathbb{R}^d)$ is defined by

$$\rho\left(x, \xi, t\right) f(u) = e^{2\pi i t} e^{\pi i \langle x, \xi \rangle} T_x M_\xi f(u),$$

that $\langle x, \xi\rangle$ denotes the usual inner product of $x, \xi \in \mathbb{R}^d$. Let $\mathbb{H}^d \rtimes \mathbb{J}$ be the semidirect product of $\mathbb{H}^d$ and a closed subgroup $\mathbb{J}$ of $Sp(d, \mathbb{R})$. Put

$$\begin{aligned}&\tau : \mathbb{H}^d \to Aut(\mathbb{J})\\&(z,t) \to \tau_{(z,t)}(A) := A.(z,t) = (Az, t)\,.\end{aligned}$$

The group law on $\mathbb{H}^d \rtimes \mathbb{J}$ is given by

$$((z,t)\,, A)\,.\left(\left(\acute{z}, \acute{t}\right), \acute{A}\right) = \left((z,t)\,.\left(A\acute{z}, \acute{t}\right), A\acute{A}\right).$$

For a fixed $A \in \mathbb{J}$ define the representation

$$\rho_A : \mathbb{H}^d \to \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right), \qquad (z,t) \mapsto \rho\,(Az, t)\,.$$

By the Stone-von Neumann theorem ρ_A is equivalent to ρ, that is, for all $(z, t) \in \mathbb{H}$, there exists an intertwining unitary operator $\mu(A) \in \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right)$ such that $\rho_A(z, t) = \mu(A)\rho(z, t)\mu(A)^{-1}$. The representation μ is the famous metaplectic or Shale- Weil representation [8]. By combining the representation ρ and μ, the extended metaplectic representation π on $\mathbb{H}^d \rtimes \mathbb{J}$, for $\psi \in L^2(\mathbb{R}^d)$ is given by

$$\begin{aligned}\pi\left((x, \xi, t)\,, A\right)\psi(y) &= \rho(x, \xi, t)\mu(A)\psi(y)\\&= e^{2\pi it}e^{\pi i\langle x,\xi\rangle}T_xM_\xi\mu(A)\psi(y).\end{aligned}$$

We observe that the reconstruction formula (1.3) is insensitive to phase factors, so the center of the Heisenberg group is irrelevant, hence we consider the underlying group as $G = \mathbb{R}^d \rtimes Sp(d, \mathbb{R})$. The class of semidirect product $\mathbb{R}^d \rtimes D$, where D is a closed matrix group so called the dilation group. From now we consider the representation π on the dilation group $G_{\mathbb{J}} = \mathbb{R}^{2d} \rtimes \mathbb{J}$ which is given by

$$\pi\,(x, \xi, A)\,\psi(y) = e^{\pi i\langle x,\xi\rangle}T_xM_\xi\mu(A)\psi(y), \tag{2.1}$$

for more details see [9].

Lemma 2.1. *Let $\mathbb{J}$ be a closed subgroup of $Sp(d, \mathbb{R})$. Then the left Haar measure of $G_{\mathbb{J}} = \mathbb{R}^{2d} \rtimes \mathbb{J}$ is given by $dm_{G_{\mathbb{J}}}(z, A) := dm_{\mathbb{R}^{2d}}(z)dm_{\mathbb{J}}(A)$ and $\triangle_{G_{\mathbb{J}}}(z, A) := \triangle_{\mathbb{J}}(A)$ is its modular function.*

Let $\mathbb{J}$ be a closed subgroup of $Sp(d, \mathbb{R})$ and $\psi \in L^2(\mathbb{R}^d)$ a window function, i.e. a function that is zero-valued outside of some chosen interval. The metaplectic transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ is given by

$$\begin{aligned}\nu_\psi f\,(x, \xi, A) &= \langle f, \pi\,(x, \xi, A)\,\psi\rangle\\&= \left\langle f, e^{\pi i\langle x,\xi\rangle}T_xM_\xi\mu(A)\psi\right\rangle.\end{aligned}$$

Using Theorem 3.2.1 of [24] follows that

$$\int_{\mathbb{R}^{2d}} |\langle f, M_\xi T_x g\rangle|^2 \, dm_{\mathbb{R}^d}(x) dm_{\mathbb{R}^d}(\xi) = \|f\|_2^2 \, \|g\|_2^2 \, , \quad \left(f, g \in L^2\left(\mathbb{R}^d\right)\right).$$

The above result can be stated for the representation $\left(\pi, L^2\left(\mathbb{R}^d\right)\right)$ on the semidirect product group $G_{\mathbb{J}}$ in the following theorem [10].

Theorem 2.2. *Let $\mathbb{J}$ be a closed subgroup of $Sp(d,\mathbb{R})$ and $G_{\mathbb{J}} = \mathbb{R}^{2d} \rtimes \mathbb{J}$. Then for every $\psi \in L^2(\mathbb{R}^d)$, the family $\tau(\psi) := \{\pi(z, A)\psi\}_{(z,A)\in G(\mathbb{J})}$ is a tight representation frame with the bound $\|\psi\|_2^2$ if and only if $\mathbb{J}$ is compact.*

Corollary 2.3. *Let $\mathbb{J}$ be a compact subgroup of $Sp(d,\mathbb{R})$ and $\psi \in L^2\left(\mathbb{R}^d\right)$ be a non-zero window function,*

(i) *The following orthogonal relation holds,*

(2.2)
$$\langle \nu_\psi f, \nu_\varphi g\rangle_{L^2(G_{\mathbb{J}})} = \langle \psi, \varphi\rangle_{L^2\left(\mathbb{R}^d\right)} \langle f, g\rangle_{L^2\left(\mathbb{R}^d\right)} \, , \quad \left(f, g \in L^2\left(\mathbb{R}^d\right)\right).$$

(ii) *Every $f \in L^2(\mathbb{R}^2)$ can be reconstructed continuously in the weak sense as*

$$f = \|\psi\|_2^{-2} \int_{\mathbb{R}^{2d}} \int_{\mathbb{J}} \langle f, T_x M_\xi \mu(A)\psi\rangle \, T_x M_\xi \mu(A)\psi dm_{\mathbb{R}^{2d}}(x,\xi) \, dm_{\mathbb{J}}(A).$$

Proof. Applying Theorem 2.2, we obtain

$$\|\nu_\psi f\|^2_{L^2(G_{\mathbb{J}})} = \|\psi\|^2_{L^2\left(\mathbb{R}^d\right)} \|f\|^2_{L^2\left(\mathbb{R}^d\right)} \, , \quad \left(f \in L^2(\mathbb{R}^2)\right).$$

So, using the polarization identity, (2.2) follows immediately. Hence, the family $\{\pi(z, A)\psi\}_{(z,A)\in G_{\mathbb{J}}}$ is a tight representation frame with the bound $\|\psi\|_2^2$ by Theorem 2.2. Thus, the desired result follows by (1.2). □

Corollary 2.4. *Let $\mathbb{J}$ be a compact group of $Sp(d,\mathbb{R})$ and $G_{\mathbb{J}} = \mathbb{R}^{2d} \rtimes \mathbb{J}$. Then the representation $\left(\pi, L^2\left(\mathbb{R}^d\right)\right)$ given by (2.1) is irreducible.*

Proof. Let M be a nonzero closed invariant subspace of $L^2\left(\mathbb{R}^d\right)$. We claim that $M = L^2\left(\mathbb{R}^d\right)$, or equivalently $M^\perp = 0$. Suppose $\varphi \in M^\perp$ and choose a fixed nonzero vector $\psi \in M$, then $\langle \varphi, \pi(x,\xi,A)\psi\rangle = 0$ for all $(x,\xi,A) \in G_{\mathbb{J}}$. By Theorem 2.2, $\{\pi(x,\xi,A)\psi; \quad (x,\xi,A) \in G_{\mathbb{J}}\}$ is a tight representation frame with frame bound C, so

$$\begin{aligned} 0 &= \int_{G_{\mathbb{J}}} |\langle \varphi, \pi(x,\xi,A)\psi\rangle|^2 \, dm_{G_{\mathbb{J}}}(x,\xi,A) \\ &= C\|\varphi\|_2^2. \end{aligned}$$

Therefore $C\|\varphi\|_2 = 0$, hence $\varphi = 0$. □

3. Duality of Metaplectic Representation Frames

In general, the canonical dual frames have not the structure of the original frames [11]. Hence, it is worthwhile to obtain more characterizations for alternate duals. In this section, we consider the representation frames based on the dilation group $\mathbb{R}^{2d} \rtimes \mathbb{J}$ where $\mathbb{J}$ is closed subgroup of $Sp(d,\mathbb{R})$. We are also interested in finding out conditions under which these frames are dual of each other.

Let G be a locally compact group with the left Haar measure m_G and let $(\pi,\mathcal{H})$ be a unitary representation on G. The frame operator for two Bessel families generated by elements φ and ψ in Hilbert space $\mathcal{H}$ is given by

$$S_{\varphi,\psi}f(g) = \int_G \langle f, \pi(g)\varphi\rangle \pi(g)\psi dm_G(g).$$

If $\varphi = \psi$ we recover the frame operator $S_\psi = S_{\psi,\psi}$. Bessel families $\{\pi(g)\varphi\}_{g\in G}$ and $\{\pi(g)\psi\}_{g\in G}$ are called dual of each other if $S_{\varphi,\psi} = I$. Also, they are called approximate dual frames if

$$\|S_{\varphi,\psi} - I_{\mathcal{H}}\| < 1.$$

Note that if φ and ψ generate approximate dual frames, then the operator $S_{\varphi,\psi}$ is invertible and hence each $f\in\mathcal{H}$ can be rewritten as

$$f = \int_G \langle f, \pi(g)\varphi\rangle {S_{\varphi,\psi}}^{-1}\pi(g)\psi dm_G(g).$$

In particular, $\{\pi(g)\varphi\}_{g\in G}$ and $\{{S_{\varphi,\psi}}^{-1}\pi(g)\varphi\}_{g\in G}$ are dual pair frames.

From now, let $G_{\mathbb{J}} = \mathbb{R}^{2d}\rtimes\mathbb{J}$ that $\mathbb{J}$ is a compact subgroup of $Sp(d,\mathbb{R})$ and π is the extended metaplectic representation on $G_{\mathbb{J}}$.

Theorem 3.1. *Let $\mathbb{J}$ be a compact subgroup of $Sp(d,\mathbb{R})$ and $\psi,\widetilde{\psi}\in L^2(\mathbb{R}^d)$. Then the following are equivalent:*

(i) *$\tau(\psi)$ and $\tau(\widetilde{\psi})$ are dual frame.*
(ii) *$\left\langle \psi,\widetilde{\psi}\right\rangle = 1$.*
(iii) *$\nu_\psi\nu^*_{\widetilde{\psi}} = 1$.*
(iv) *$S_{\psi,\widetilde{\psi}}f = \left\langle \psi,\widetilde{\psi}\right\rangle f$.*

Proof. Using the orthogonality relation (2.2), for every $f,g\in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\left\langle S_{\psi,\widetilde{\psi}}f, g\right\rangle &= \left\langle \int_{G_{\mathbb{J}}} \langle f,\pi(z,A)\psi\rangle \pi(z,A)\widetilde{\psi} dm_{\mathbb{R}^{2d}}(z)dm_{\mathbb{J}}(A), g\right\rangle \\ &= \int_{G_{\mathbb{J}}} \langle f,\pi(z,A)\psi\rangle \left\langle \pi(z,A)\widetilde{\psi}, g\right\rangle dm_{\mathbb{R}^{2d}}(z)dm_{\mathbb{J}}(A)\end{aligned}$$

$$= \left\langle \nu_\psi f, \nu_{\widetilde{\psi}} g \right\rangle$$
$$= \left\langle \psi, \widetilde{\psi} \right\rangle \langle f, g \rangle .$$

The results follow immediately. □

As a consequence, we can obtain more pairs of dual frames.

Corollary 3.2. *Let* $\mathbb{J}$ *be a compact subgroup of* $Sp(d, \mathbb{R})$ *and* $\psi, \widetilde{\psi} \in L^2\left(\mathbb{R}^d\right)$ *such that* $\left|1 - \left\langle \psi, \widetilde{\psi} \right\rangle\right| < 1$. *Then* $S_{\psi,\widetilde{\psi}}$ *is invertible and*

$$f = 1/\langle \psi, \widetilde{\psi} \rangle \int_{G_{\mathbb{J}}} \left(f.T_x \overline{\mu(A)\psi} \right)^{\wedge}(\xi)\, T_x M_\xi \mu(A) \widetilde{\psi} dm_{\mathbb{R}^{2d}}(x, \xi)\, dm_{\mathbb{J}}(A),$$

for every $f \in L^2\left(\mathbb{R}^d\right)$. *In particular,* $\tau(\psi)$ *and* $\tau\left(\widetilde{\psi}\right)$ *are dual frames.*

Proof. Notice that $\|I - S_{\psi,\widetilde{\psi}}\| = \left|1 - \left\langle \psi, \widetilde{\psi} \right\rangle\right| < 1$, therefore $S_{\psi,\widetilde{\psi}}$ is invertible. Hence

$$f = S_{\psi,\widetilde{\psi}} S_{\psi,\widetilde{\psi}}^{-1} f$$
$$= \int_{G_{\mathbb{J}}} \left\langle S_{\psi,\widetilde{\psi}}^{-1} f, \pi(x,\xi,A)\psi \right\rangle \pi(x,\xi,A) \widetilde{\psi} dm_{G_{\mathbb{J}}}(x,\xi,A)$$
$$= 1/\langle \psi, \widetilde{\psi} \rangle \int_{G_{\mathbb{J}}} \langle f, T_x M_\xi \mu(A)\psi \rangle\, T_x M_\xi \mu(A) \widetilde{\psi} dm_{\mathbb{R}^{2d}}(x, \xi)\, dm_{\mathbb{J}}(A).$$

□

Corollary 3.3. *Let* $\mathbb{J}$ *be a compact subgroup of* $Sp(d, \mathbb{R})$ *and* $\psi, \widetilde{\psi} \in L^2(\mathbb{R}^2)$.

(i) *If* $\|\psi\|_2 \|\widetilde{\psi}\|_2 < 1$, *then* $\tau(\psi)$ *and* $\tau\left(\widetilde{\psi}\right)$ *cannot be dual frames.*

(ii) *If* $\langle \psi, \widetilde{\psi} \rangle \neq 0$ *then* $\tau(\psi)$ *and* $\tau(\frac{1}{\langle \psi, \widetilde{\psi} \rangle}\psi)$ *are dual frames.*

(iii) *The following are equivalent:*
  (a) $\tau(\psi)$ *and* $\tau(\widetilde{\psi})$ *are dual frames.*
  (b) $\tau(\psi)$ *and* $\tau\left(S_\psi^{-1}\psi + S_\psi^2 \widetilde{\psi} - S_\psi \psi\right)$ *are dual frames.*
  (c) $\tau(\psi)$ *and* $\tau\left(\widetilde{\psi} + \varphi\right)$ *are dual frames, for some* φ *with* $\langle \varphi, \psi \rangle = 0$.
  (d) $\tau(\psi)$ *and* $\tau\left(2S_\psi^{-1}\psi + S_\psi S_{\psi,\widetilde{\psi}}^{-1} \widetilde{\psi} - \psi - \widetilde{\psi}\right)$ *are dual frames, if* $S_{\psi,\widetilde{\psi}}$ *is invertible.*
  (e) $\tau(\psi)$ *and* $\tau\left(2\widetilde{\psi} + \varphi - \frac{1}{\|\psi\|^2}\psi\right)$ *are dual frames, for some* φ *with* $\langle \varphi, \psi \rangle = 0$.

To illustrate our results we present the following example.

Example 3.4. The closed subgroup

$$\begin{aligned}\mathbb{K} &= Sp(d,\mathbb{R}) \cap So(2d)\\ &\cong U(d)\end{aligned}$$

is the unique maximal compact subgroup of $Sp(d,\mathbb{R})$ up to conjugation. The elements of $\mathbb{K}$ are easily seen to be of the form

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix},$$

where X and Y are $d\times d$ matrices that satisfy the two equations ${}^tXY = {}^tYX$ and ${}^tXX + {}^tYY = I_d$. The mapping $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \to X + iY$ establishes an isomorphism of $\mathbb{K}$ onto the unitary group $U(d)$ [9]. Putting $G_{\mathbb{K}} = \mathbb{R}^{2d}\times\mathbb{K}$, since $\mathbb{K}$ is compact, applying Theorem 2.2 shows that, the family $\{\pi(z,A)\psi\}_{(z,A)\in G_{\mathbb{K}}}$ is a tight representation frame with the bound $\|\psi\|_2^2$ and

$$f = \|\psi\|_2^{-2}\int_{G_{\mathbb{K}}} \langle f, T_xM_\xi\mu(A)\psi\rangle\, T_xM_\xi\mu(A)\psi dm_{\mathbb{R}^{2d}}(x,\xi)dm_{\mathbb{K}}(A),$$

for every $\psi, f\in L^2(\mathbb{R}^d)$.

4. Repesentation Frames Based on $G_{\mathbb{J}}$ via Wigner Distribution

In this section, we first review some facts of the short-time Fourier transform (STFT) and cross-Wigner distribution on $L^2(\mathbb{R}^d)$. Using these results, we discuss on the existence of representation frames based on $G_{\mathbb{J}}$. To obtain information about local properties of a signal, restrict it to an interval and take the Fourier transform. For the window function $g\neq 0$, the short-time Fourier transform of a function f with respect to g is defined as

$$V_gf(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{2\pi it.\omega}dm_{\mathbb{R}^d}(t), \quad \left(x,\omega\in\mathbb{R}^d\right).$$

Also, the cross-Wigner distribution $W_{f,g}$ of $f,g\in L^2(\mathbb{R}^d)$ is given by

$$W_{f,g}(x,\omega)\,d\omega = \int_{\mathbb{R}^d} e^{-2\pi i\langle\omega,y\rangle} f\left(x+\frac{y}{2}\right)\overline{g\left(x-\frac{y}{2}\right)}dm_{\mathbb{R}^d}(y).$$

At the rest, we review some well-known properties that make the cross-Wigner distribution most popular time frequency representation in signal analysis [24].

Lemma 4.1. *The Wigner distribution of $f,g\in L^2(\mathbb{R}^d)$ satisfies:*

(i) *$W_{f,g}$ is uniformly continuous on $\mathbb{R}^{2d}$ and $\|W_{f,g}\|_\infty \leq 2^d\|f\|_2\|g\|_2$.*

(ii) $W_{f,g} = \overline{W}_{g,f}$.

(iii) *Moyal's identity:* $\langle W_{f_1,g_1}, W_{f_2,g_2}\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}$.

(iv) *For all* $f, g \in L^2(\mathbb{R}^d)$

$$W_{f,g}(x,\omega) = 2^d e^{4\pi i x.\omega} V_{\zeta g} f(2x, 2\omega) \\ = 2^d e^{4\pi i x.\omega} \left(\widehat{f} * M_{-2x}\widehat{\zeta g}^*\right)(2\omega)$$

where $\zeta g(x) = g(-x)$ *is the reflection operator.*

Proposition 4.2. (i) *Let* $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. *Then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_{f,g}(x,\omega)\, dm_{\mathbb{R}^d}(\omega)\, dm_{\mathbb{R}^d}(x) = \langle f, g\rangle .$$

(ii) *Let* $R(x,\xi)$ *be a measurable function on* $\mathbb{R}^{2d}$ *such that*

$$\int_{\mathbb{R}^{2d}} R(x,\xi)\, W_{f,g}(x,\xi)\, dm_{\mathbb{R}^d}(\xi)\, dm_{\mathbb{R}^d}(x) = 0, \quad \left(f, g \in L^2\left(\mathbb{R}^d\right)\right).$$

Then $R(x,\xi) = 0$ *for a.e.* $(x,\xi) \in \mathbb{R}^{2d}$.

Proof. (i) Using Lemma 4.1 follows that

$$|W_{f,g}(x,\omega)| = 2^d \left|V_{\zeta_g} f(2x, 2\omega)\right| \\ = 2^d \left|\widehat{f} * M_{-2x}\widehat{g}^*\right|.$$

Let $T_s F(x,t) = F\left(x + \frac{t}{2}, x - \frac{t}{2}\right)$ be the symmetric coordinate transform and

$$\mathcal{F}_2 F(x,\xi) = \int_{\mathbb{R}^d} F(x,t) e^{-2\pi i\langle \xi, t\rangle} dm_{\mathbb{R}^d}(t),$$

be the Fourier transform in the second variable. Then $W_{f,g} = \mathcal{F}_2 T_s(f \otimes \overline{g})$. Since $\widehat{f}, \widehat{g} \in L^1(\mathbb{R}^d)$ and $W_{f,g}(x,.) \in L^1(\mathbb{R}^d)$ for each fixed $x \in \mathbb{R}^d$, by the inversion formula (1.4) we have

$$\int_{\mathbb{R}^d} W_{f,g}(x,\omega)\, dm_{\mathbb{R}^d}(\omega) = \mathcal{F}_2^{-1}\left[(\mathcal{F}_2 T_s)(f \otimes \overline{g})\right](x,0) \\ = T_s(f \otimes \overline{g})(x,0) \\ = f(x)\overline{g(x)}.$$

Therefore,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_{f,g}(x,\omega)\, dm_{\mathbb{R}^d}(\omega)\, dm_{\mathbb{R}^d}(x) = \int_{\mathbb{R}^d} f(x)\overline{g(x)} dm_{\mathbb{R}^d}(x) \\ = \langle f, g\rangle .$$

(ii) It is enough to show that $\{V_g f; \quad f, g \in L^2(\mathbb{R}^d)\}$ is dense in $L^2(\mathbb{R}^d)$, see Lemma 4.1 (iv). To see this, suppose that $h \in L^2(\mathbb{R}^d)$ and

$$\langle h, V_g f\rangle = 0,$$

for all $f, g \in L^2(\mathbb{R}^d)$. Then $V_g^* h = 0$, for all $g \in L^2(\mathbb{R}^d)$ and hence, $h = 0$ since V_g is unitary. Thus, for every $F \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\int_{\mathbb{R}^{2d}} R(x,\xi)\, F(x,\xi)\, dm_{\mathbb{R}^d}(x) dm_{\mathbb{R}^d}(\xi) &= \langle R, \overline{F}\rangle \\ &= \left\langle \lim_{n\to\infty} \sum_{k=0}^{n} c_k W_{f_k,g_k}, R\right\rangle \\ &= \lim_{n\to\infty} \left\langle \sum_{k=0}^{n} c_k W_{f_k,g_k}, R\right\rangle \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} c_k \langle W_{f_k,g_k}, R\rangle \\ &= 0.\end{aligned}$$

So that $R(x,\xi) = 0$, for a.e. $(x,\xi) \in \mathbb{R}^{2d}$.

□

We now return to the group $G_{\mathbb{J}} = \mathbb{R}^{2d} \rtimes \mathbb{J}$, where $\mathbb{J}$ is a closed subgroup of $Sp(d,\mathbb{R})$ and consider the extended metaplectic representation on $G_{\mathbb{J}}$. One of our aims is to find concrete frame condition of representation frames based on $G_{\mathbb{J}}$ and to characterize their alternate duals with respect to cross-Wigner distribution.

Theorem 4.3. *Consider $\varphi, \psi \in L^2(\mathbb{R}^d)$ and suppose the mapping*

$$(z, A) \mapsto W_{\pi(z,A)\varphi,\pi(z,A)\psi}(x,\xi) \tag{4.1}$$

is in $L^1(G_{\mathbb{J}})$ for a.e. $(x,\xi) \in \mathbb{R}^{2d}$. Also

$$\int_{G_{\mathbb{J}}} \left|W_{\pi(z,A)\varphi,\pi(z,A)\psi}(x,\xi)\right| dm_G \leq M \quad a.e.\,(x,\xi) \in \mathbb{R}^{2d}. \tag{4.2}$$

Then Bessel families $\{\pi(z,A)\varphi\}_{(z,A)\in G_{\mathbb{J}}}$ and $\{\pi(z,A)\psi\}_{(z,A)\in G_{\mathbb{J}}}$ are dual frames i.e.

$$f = \int_{G_{\mathbb{J}}} \langle f, \pi(z,A)\varphi\rangle\, \pi(z,A)\psi dm_{G_{\mathbb{J}}}(z,A)$$

if and only if

$$\int_{G_{\mathbb{J}}} W_{\pi(z,A)\varphi,\pi(z,A)\psi}(x,\xi)\, dm_{G_{\mathbb{J}}}(z,A) = 1 \quad a.e.\,(x,\xi) \in \mathbb{R}^{2d}. \tag{4.3}$$

Proof. Let $\{\pi(z,A)\varphi\}_{(z,A)\in G_{\mathbb{J}}}$ and $\{\pi(z,A)\psi\}_{(z,A)\in G_{\mathbb{J}}}$ be dual representation frames. Using Lemma 4.1 we obtain

$$\begin{aligned}\langle f,g\rangle &= \int_{G_{\mathbb{J}}} \langle f,\pi(z,A)\varphi\rangle\langle \pi(z,A)\psi,g\rangle\, dm_{G_{\mathbb{J}}}(z,A)\\ &= \int_{G_{\mathbb{J}}} \left\langle W_{f,g}, W_{\pi(z,A)\varphi,\pi(z,A)\psi}\right\rangle dm_{G_{\mathbb{J}}}(z,A)\\ &= \int_{\mathbb{R}^{2d}}\int_{G_{\mathbb{J}}} \left(W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, dm_{G_{\mathbb{J}}}(z,A)\right) W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi).\end{aligned}$$

Due to Proposition 4.2 (i) we have

$$\int_{\mathbb{R}^{2d}}\int_{G_{\mathbb{J}}} \left(W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, dm_{G_{\mathbb{J}(F,\mathbb{A})}} - 1\right) W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi) = 0.$$

Applying Proposition 4.2 (ii) for

$$R(x,\xi) = \int_{G_{\mathbb{J}}} W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, dm_{G_{\mathbb{J}}}(z,A) - 1$$

follows the desired results. Conversely, assume that (4.3) holds. Then

$$\begin{aligned}&\int_{\mathbb{R}^{2d}}\int_{G_{\mathbb{J}}} \left|W_{\pi(z,A)\varphi,\pi(z,A)\psi}(x,\xi)\, W_{f,g}(x,\xi)\right| dm_{G_{\mathbb{J}}}(z,A)\, dm_{\mathbb{R}^{2d}}(x,\xi)\\ &= \int_{\mathbb{R}^{2d}} \left(\int_{G_{\mathbb{J}}} \left|W_{\pi(z,A)\varphi,\pi(z,A)\psi}(x,\xi)\right| dm_{G_{\mathbb{J}}}(z,A)\right) |W_{f,g}(x,\xi)|\, dm_{\mathbb{R}^{2d}}(x,\xi)\\ &\le M\int_{\mathbb{R}^{2d}} |W_{f,g}(x,\xi)|\, dm_{\mathbb{R}^{2d}}(x,\xi) < \infty.\end{aligned}$$

Hence, by using the Fubini's theorem for every $f,g\in L^2\left(\mathbb{R}^{2d}\right)$ we have

$$\begin{aligned}\langle f,g\rangle &= \int_{\mathbb{R}^{d}}\int_{\mathbb{R}^{d}} W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi)\\ &= \int_{\mathbb{R}^{2d}} \left(\int_{G_{\mathbb{J}}} W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, dm_{G_{\mathbb{J}}}(z,A)\right) W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi)\\ &= \int_{G_{\mathbb{J}}} \left(\int_{\mathbb{R}^{2d}} W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi)\right) dm_{G_{\mathbb{J}}}(z,A).\end{aligned}$$

Using Moyal's identity follows that

$$\begin{aligned}\langle f,g\rangle &= \int_{G_{\mathbb{J}}} \left(\int_{\mathbb{R}^{2d}} W_{\pi(z,A)\psi,\pi(z,A)\varphi}(x,\xi)\, W_{f,g}(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi)\right) dm_{G_{\mathbb{J}}}(z,A)\\ &= \int_{G_{\mathbb{J}}} \left\langle W_{f,g}, W_{\pi(z,A)\varphi,\pi(z,A)\psi}\right\rangle dm_{G_{\mathbb{J}}}(z,A)\end{aligned}$$

$$= \int_{G_{\mathbb{J}}} \langle f, \pi(z,A)\varphi\rangle \langle \pi(z,A)\psi, g\rangle \, dm_{G_{\mathbb{J}}}(z,A).$$

This completes the proof. □

The following result is obtained immediately from Theorem 2.2. We denote the family $\{\pi(z,A)\psi\}$ by $\tau(\psi)$.

Corollary 4.4. *Let $G_{\mathbb{J}} = \mathbb{R}^{2d} \times \mathbb{J}$ and $\psi \in L^2(\mathbb{R}^d)$. Assuming the preceding Theorem $\mathbb{J}$ is compact if and only if*

$$\int_{G_{\mathbb{J}}} W_{\tau(\psi)}(x,\xi)\, dm_{G_{\mathbb{J}}} = 1 \quad a.e.$$

We end this section by a perturbation result with respect to the Wigner distribution.

Proposition 4.5. *Let $\tau(\psi)$, $\tau\left(\widetilde{\psi}\right)$ and $\tau(\varphi)$ be Bessel mappings and B be a Bessel bound of $\tau\left(\widetilde{\psi}\right)$ such that*

$$\int_{\mathbb{J}} W_{\tau(\psi)-\tau(\varphi)}(x,\xi)\, dm_{\mathbb{J}} < 1/B.$$

If $\tau(\psi)$ and $\tau\left(\widetilde{\psi}\right)$ are dual pairs, then $\tau\left(\widetilde{\psi}\right)$ and $\tau(\varphi)$ are approximately dual.

Proof. By using the assumption for any $f \in L^2(\mathbb{R}^d)$ we obtain

$$\begin{aligned}\left\|\left(T^*_{\tau(\psi)} - T^*_{\tau(\varphi)}\right) f\right\|_2^2 &= \int_{G_{\mathbb{J}}} |\langle f, \tau(\psi) - \tau(\varphi)\rangle|^2 \, dm_{G_{\mathbb{J}}} \\ &= \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{J}} W_{\tau(\psi)-\tau(\varphi)}(x,\xi)\, dm_{\mathbb{J}}\right) W_f(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi) \\ &< 1/B \int_{\mathbb{R}^{2d}} W_f(x,\xi)\, dm_{\mathbb{R}^{2d}}(x,\xi) = 1/B\|f\|_2^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|I - S_{\varphi,\widetilde{\psi}}\| &= \left\|T_{\tau(\widetilde{\psi})}\left(T^*_{\tau(\psi)} - T^*_{\tau(\varphi)}\right)\right\| \\ &\leq \|T_{\tau(\widetilde{\psi})}\|\|T^*_{\tau(\psi)} - T^*_{\tau(\varphi)}\| < 1.\end{aligned}$$

Namely, $\tau\left(\widetilde{\psi}\right)$ and $\tau(\varphi)$ are approximately dual. □

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