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Integral Operators on the Besov Spaces and Subclasses of Univalent Functions

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Abstract. In this note, we study the integral operators $I_g^{\gamma,\alpha}$ and $J_g^{\gamma,\alpha}$ of an analytic function g on convex and starlike functions of a complex order. Then, we investigate the same operators on H^∞ and Besov spaces.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the plane $\mathbb{C}$, and $H(\mathbb{D}) := \{g : \mathbb{D} \longrightarrow \mathbb{C} \mid g \text{ is analytic}\}$. Also, let A be the subclass of $H(\mathbb{D})$, which its elements are of the from

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Suppose that $S^*(\alpha)$ is the famous subclass of A, which is starlike of order α $(0 \leq \alpha < 1)$. Indeed, $g \in S^*(\alpha)$ is equivalent to $Re(zg'(z)/g(z)) > \alpha$ in $\mathbb{D}$. Similarly, we have $g \in K(\alpha)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha, \quad (z \in \mathbb{D}),$$

where $K(\alpha)$ is the subclass of A contained in the convex functions of order α.

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As usual, we write $S^* = S^*(0)$ and $K = K(0)$. For $0 \neq b \in \mathbb{C}$, the subclasses of A, S_b^* and K_b, are defined by

$$S_b^* = \left\{ g \in A : \mathrm{Re}\left\{ 1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1 \right) \right\} > 0, \quad (z \in \mathbb{D}) \right\},$$

and

$$K_b = \left\{ g \in A : \mathrm{Re}\left\{ 1 + \frac{1}{b}\left(\frac{zg''(z)}{g'(z)} \right) \right\} > 0, \quad (z \in \mathbb{D}) \right\}.$$

Then, we can see that for $0 \leq \alpha < 1$,

$$S_{1-\alpha}^* = S^*(\alpha), \qquad K_{1-\alpha} = K(\alpha).$$

We refer to [3, 11, 12] for some important results.

For some real number β and non-zero complex number b, we introduce a subclass of $H(\mathbb{D})$, $P(\beta, b)$, as follows:

$$P(\beta, b) := \left\{ g \in H(\mathbb{D}) : \mathrm{Re}\left\{ \frac{1}{b}\left(\frac{zg'(z)}{g(z)} \right) \right\} \geq \beta \text{ and } g(0) = 1 \right\}.$$

For example, $\frac{1}{1-z}$ and $\frac{1}{1+z}$ belong to $P(-\frac{1}{2}, 1)$.

Let $g \in H(\mathbb{D})$ be locally univalent. Let

$$S_g(z) = \left(\frac{g''(z)}{g'(z)} \right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)} \right)^2,$$

denote the Schwarzian derivative of g, and let

$$\|S_g\| = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^2 |S_g(z)|,$$

which denotes its Schwarzian norm.

If $g \in K$ and $h(z) = 1 + \frac{zg''(z)}{g'(z)}$, then $Reh(z) > 0$ $(z \in \mathbb{D})$, so h is subordinate to $\lambda(z) = \frac{1+z}{1-z}$, where λ is the half-plan mapping. Therefore, $h(z) = \lambda(\varphi(z))$ for some Schwarz function φ, and we have

$$\begin{aligned} \frac{zg''(z)}{g'(z)} &= \frac{1 + \varphi(z)}{1 - \varphi(z)} - 1 \\ &= \frac{2\varphi(z)}{1 - \varphi(z)}, \end{aligned}$$

with the notation $\psi(z) = \frac{\varphi(z)}{z}$, where ψ is analytic and satisfies $|\psi(z)| \leq 1$ in $\mathbb{D}$. Then it can be written as follows:

$$\frac{g''(z)}{g'(z)} = \frac{2\psi(z)}{1 - z\psi(z)}. \tag{1.1}$$

Hence, the Schwarzian derivative of g can be written in the following form:

$$\begin{aligned} S_g(z) &= \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2 \\ &= \frac{2\psi'(z)}{(1-z\psi(z))^2}. \end{aligned}$$

Then, we obtain

$$|S_g(z)| \leq \frac{2}{(1-|z|^2)^2}. \tag{1.2}$$

By the inequality (1.2), the Schwarzian norm $\|S_g\|$ of the convex mapping is not greater than 2. If the convex mapping g is bounded, then $\|S_g\| < 2$ (see [10]).

Finally, let $g \in H(\mathbb{D})$. We consider two integral operators on $H(\mathbb{D})$, as follows:

$$I_g^{\gamma,\alpha}(h)(z) = \int_0^z h'(w)g^\gamma(w)w^{\alpha-1}dw, \quad (z \in \mathbb{D}),$$

and

$$J_g^{\gamma,\alpha}(h)(z) = \int_0^z h(w)(g'(w))^\gamma w^{\alpha-1}dw, \quad (z \in \mathbb{D}),$$

where $\gamma, \alpha > 0$.

Integral operators play an important role in various fields (see [4, 8]). If $\gamma = \alpha = 1$, then $I_g^{1,1}(h) = I_g(h)$ and $J_g^{1,1}(h) = J_g(h)$, which are Alexander operators. These integral operators have been investigated by many authors [7, 9, 13, 14].

In this note, we study $I_g^{\gamma,\alpha}$ and $J_g^{\gamma,\alpha}$ operators on K, $K(\alpha)$ and S_b^*. Here, we obtain the necessary and sufficient conditions such that $I_g^{\gamma,\alpha}(\mathbb{D})$ and $J_g^{\gamma,\alpha}(\mathbb{D})$ are bounded, Furthermore, we obtain the sufficient conditions such that $|S_{I_g^{\gamma,\alpha}}| < 2$.

2. Integral Operators on $K(\alpha)$ and $S^*(\alpha)$

Now, we verify the integral operators, $I_g^{\gamma,\alpha}$ and $J_g^{\gamma,\alpha}$, on $K(\alpha)$ and $S^*(\alpha)$.

Lemma 2.1. (i) *Let $\alpha > 0, \gamma > 0$, $\beta \geq 0$ and $0 \neq b \in \mathbb{C}$, where $(\alpha-1)\mathrm{Re}b \geq 0$. If $g \in P(\beta, b)$, then $I_g^{\gamma,\alpha}$ is an operator on K_b.*

(ii) *Let $\gamma > 0$, $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha + \beta\gamma < 2$. If $g \in P(\beta, 1)$, then $I_g^{\gamma,\alpha}$ is an operator from $K(\alpha)$ to $K(2\alpha+\beta\gamma-1)$.*

Proof. (i) Let $h \in K_b$. Then, for $z \in \mathbb{D}$,

(2.1)

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{1}{b}\frac{z(I_g^{\gamma,\alpha}h)''(z)}{(I_g^{\gamma,\alpha}h)'(z)}\right\}\\
&\quad=\mathrm{Re}\left\{1+\frac{1}{b}\frac{z(z^{\alpha-1}h''(z)g^{\gamma}(z)+\gamma z^{\alpha-1}h'(z)g'(z)g^{\gamma-1}(z))}{z^{\alpha-1}h'(z)g^{\gamma}(z)}\right\}\\
&\qquad+\mathrm{Re}\left\{\frac{1}{b}\frac{z(\alpha-1)z^{\alpha-2}h'(z)g^{\gamma}(z)}{z^{\alpha-1}h'(z)g^{\gamma}(z)}\right\}\\
&\quad=(\alpha-1)\mathrm{Re}\left(\frac{1}{b}\right)+\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zh''(z)}{h'(z)}\right)\right\}+\gamma\mathrm{Re}\left\{\frac{1}{b}\left(\frac{zg'(z)}{g(z)}\right)\right\}.
\end{aligned}$$

By this hypothesis and (2.1), we obtain

$$\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(I_g^{\gamma,\alpha}h)''(z)}{(I_g^{\gamma,\alpha}h)'(z)}\right)\right\}>0.$$

Therefore, $I_g^{\gamma,\alpha}h \in K_b$ for all $h \in K_b$.

(ii) We can prove this part in a similar manner as the proof of part (i).

$\square$

Lemma 2.2. *Let $\alpha > 0$, $0 \neq b \in \mathbb{C}$ and $0 < \gamma < \alpha\mathrm{Re}(\frac{1}{b})$. If $g \in K_b$, then $J_g^{\gamma,\alpha}$ is an operator from S_b^* to K_b.*

Proof. Let $h \in S_b^*$, then for $z \in \mathbb{D}$,

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{1}{b}\frac{z(J_g^{\gamma,\alpha}h)''(z)}{(J_g^{\gamma,\alpha}h)'(z)}\right\}\\
&\quad=\mathrm{Re}\left\{1+\frac{1}{b}\frac{z(z^{\alpha-1}h'(z)(g'(z))^{\gamma}+\gamma z^{\alpha-1}h(z)g''(z)(g'(z))^{\alpha-1})}{z^{\alpha-1}h(z)(g'(z))^{\gamma}}\right\}\\
&\qquad+\mathrm{Re}\left\{\frac{1}{b}\frac{z(\alpha-1)z^{\alpha-2}h(z)(g'(z))^{\gamma}}{z^{\alpha-1}h(z)(g'(z))^{\gamma}}\right\}\\
&\quad=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zh'(z)}{h(z)}\right)+\frac{\gamma}{b}\left(\frac{zg''(z)}{g'(z)}\right)+\frac{\alpha-1}{b}\right\}\\
&\quad=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zh'(z)}{h(z)}-1\right)\right\}+\gamma\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zg''(z)}{g'(z)}\right)\right\}+\alpha\mathrm{Re}\left(\frac{1}{b}\right)-\gamma.
\end{aligned}$$

By this hypothesis, we obtain $J_g^{\gamma,\alpha}h \in K_b$. $\square$

Theorem 2.3. *Let $\gamma > 0$, $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha+\beta\gamma < 2$. Also, let $g \in P(\beta, 1)$ and $h \in K(\alpha)$. Then, the image $(I_g^{\gamma,\alpha}h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)\left|\frac{zh''(z)}{h'(z)}+\frac{\gamma zg'(z)}{g(z)}+\alpha+1\right|<1.$$

Proof. By using Part (2) of Lemma 2.1, we have $I_g^{\gamma,\alpha}h \in K$. By replacing $g = I_g^{\gamma,\alpha}h$ in (1.1), then there exists $\psi \in H(\mathbb{D})$ such that

$$\frac{(I_g^{\gamma,\alpha}h)''(z)}{(I_g^{\gamma,\alpha}h)'(z)} = \frac{2\psi(z)}{1 - z\psi(z)}.$$

Therefore

$$\begin{aligned}\psi(z) &= \frac{\frac{(I_g^{\gamma,\alpha}h)''(z)}{(I_g^{\gamma,\alpha}h)'(z)}}{2 + \frac{z(I_g^{\gamma,\alpha}h)''(z)}{(I_g^{\gamma,\alpha}h)'(z)}} \\ &= \frac{\frac{h''(z)}{h'(z)} + \frac{\gamma g'(z)}{g(z)} + \frac{\alpha-1}{z}}{1 + \alpha + \frac{zh''(z)}{h'(z)} + \frac{\gamma z g'(z)}{g(z)}}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{1-|z|}{|1-z\psi(z)|} &= \frac{1-|z|}{\frac{2}{\left|\frac{zh''(z)}{h'(z)} + \frac{\gamma z g'(z)}{g(z)} + \alpha + 1\right|}} \\ &= \frac{1}{2}(1-|z|)\left|1 + \alpha + \frac{zh''(z)}{h'(z)} + \frac{\gamma z g'(z)}{g(z)}\right|.\end{aligned} \tag{2.2}$$

By Theorem 2 in [10], we can conclude that the image $(I_g^{\gamma,\alpha}h)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z|\to 1} \frac{1-|z|}{|1-z\psi(z)|} < \frac{1}{2},$$

and so by (2.2), the proof is complete. □

Similarly, by using Lemma 2.2, the following theorem is achieved:

Theorem 2.4. *Let $0 < \gamma < \alpha$, $g \in K$ and $h \in S^*$. Then, the image $(J_g^{\gamma,\alpha}h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)\left|\frac{zh'(z)}{h(z)} + \frac{\gamma z g''(z)}{g'(z)} + \alpha + 1\right| < 1.$$

By using part 2 of Lemma 2.1, we can obtain the below result:

Corollary 2.5. *Let $\gamma > 0$, $0 \le \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \le 2\alpha + \beta\gamma < 2$. If $g \in P(\beta, 1)$ and $h \in K(\alpha)$, then $\|S_{I_g^{\gamma,\alpha}h}\| \le 2$.*

And also, by Theorem 2.3, the below conclusion is gained:

Corollary 2.6. *Let $\gamma > 0$, $0 \le \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \le 2\alpha + \beta\gamma < 2$. If*

$$\limsup_{|z|\to 1}(1-|z|)\left|\frac{zh''(z)}{h'(z)} + \frac{\gamma z g'(z)}{g(z)} + \alpha + 1\right| < 1,$$

then $\|S_{I_g^{\gamma,\alpha}h}\| < 2$.

Finally, a similar corollary to those above is also true for the operator $J_f^{\gamma,\alpha}h$.

3. Integral Operators on Besov Spaces

We use $dA(z)$ to denote the area measure of $\mathbb{D}$ which is normalized, so the area of $\mathbb{D}$ is 1. We have

$$\begin{aligned} dA(z) &= \frac{1}{\pi}dxdy \\ &= \frac{r}{\pi}drd\theta, \end{aligned}$$

where $z = x + iy = re^{i\theta}$ and we set

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),$$

where $\alpha > -1$. It is clear that if α is a real number then

$$\int_{\mathbb{D}} (1-|z|^2)^\alpha dA(z) < \infty,$$

if and only if $\alpha > -1$.

For $1 < p < \infty$ and $\delta \geq 1$, the Besov space B_δ^p is defined as the set of all $g \in H(\mathbb{D})$ such that

$$\begin{aligned} \|g\|_{B_\delta^p} &:= |g(0)| + \left\{(p-1)\int_{\mathbb{D}} (1-|z|^2)^{p-2}|g'(z)|^p dA_\delta(z)\right\}^{\frac{1}{p}} \\ &< \infty. \end{aligned}$$

For simplicity, the space B_1^p will be denoted by B^p. Many authors have studied the properties of the Besov spaces [1, 2]. The space H^∞ consists of bounded analytic functions g in $\mathbb{D}$ where

$$\|g\|_{H^\infty} := \lim_{r\to 1^-}(\max_{|z|\leq r}|g(z)|) < \infty.$$

In this section, we study two operators $I_g^{\gamma,\alpha}$ and $J_g^{\gamma,\alpha}$ on H^∞ and Besov space B_δ^p.

Theorem 3.1. *Let $1 < p < \infty$ and $\delta \geq 1$. If $g \in B_\delta^p$ then $J_g^{\gamma,\alpha}$ is bounded on H^∞ and $\|J_g^{\gamma,\alpha}\|_{B_\delta^p} \leq \|g\|_{B_\delta^p}$ where $\gamma \leq 1$ and $\alpha \geq 1$.*

Proof. Let $\|h\|_{H^\infty} = 1$. Therefore,

$$\begin{aligned} \|J_g^{\gamma,\alpha}\|_{B_\delta^p}^p &= (p-1)\int_{\mathbb{D}} (1-|z|^2)^{p-2}|h(z)(g'(z))^\gamma z^{\alpha-1}|^p dA_\delta(z) \\ &\leq (p-1)\|h\|_{H^\infty}^p \int_{\mathbb{D}} (1-|z|^2)^{p-2}|(g'(z))^\gamma|^p dA_\delta(z) \\ &\leq \Big(\|g\|_{B_\delta^p} - |g(0)|\Big)^p < \infty, \end{aligned}$$

since $\gamma \leq 1$ and $\alpha \geq 1$. $\square$

Theorem 3.2. *Let $1 < p < \infty$, $\delta \geq 1$ and $g \in H^\infty$. Then $I_g^{\gamma,\alpha} \in B_\delta^p$, where $\alpha + \gamma \geq 1$. Moreover, $\|I_g^{\gamma,\alpha}\|_{B_\delta^p} \leq \|Ig^\gamma\|_{H^\infty}$, where $I(z) = z^{\alpha-1}$ ($z \in \mathbb{D}$).*

Proof. Suppose that $g \in H^\infty$. There exists a number $N > 0$ such that $\frac{|g(z)|}{N} < |z|$ ($z \in \mathbb{D}$), thus,

$$|z^{\alpha-1}g^\gamma(z)| \leq N^\gamma, \quad (z \in \mathbb{D}),$$

where $\gamma + \alpha - 1 \geq 0$.

Therefore, $z^{\alpha-1}g^\gamma(z) \in H^\infty$. Set $I(z) = z^{\alpha-1}$ ($z \in \mathbb{D}$), so there exists a number $c > 0$ such that $\|Ig^\gamma\|_{H^\infty} = c$. Now, for any $\|h\|_{B_\delta^p} = 1$, we have

$$\begin{aligned}
\|I_g^{\gamma,\alpha}h\|_{B_\delta^p}^p &= (p-1)\int_{\mathbb{D}}(1-|z|^2)^{p-2}\left|h'(z)z^{\alpha-1}g^\gamma(z)\right|^p dA_\delta(z)\\
&\leq c^p(p-1)\int_{\mathbb{D}}(1-|z|^2)^{p-2}|h'(z)|^p dA_\delta(z)\\
&\leq c^p\|h\|_{B_\delta^p}^p\\
&= c^p,
\end{aligned}$$

and the proof is complete. $\square$

Let $\lambda > 0$ and g be a locally univalent function. Also let

$$B(\lambda) = \{g \in H(\mathbb{D}); \|\frac{g''}{g'}\| \leq 2\lambda\},$$

where

$$\left\|\frac{g''}{g'}\right\| = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right|,$$

is the norm of the pre-Schwarzian derivative $\frac{g''}{g'}$ of g. Kim and Sugawa [5, 6] investigated the properties of the class $B(\lambda)$.

Theorem 3.3. *Let $1 < p < \infty$, $\delta > 1$ and $\lambda < 1$. Therefore, $B(\lambda) \subseteq B_\delta^p$.*

Proof. Let $|z| = t < 1$ and $g \in B(\lambda)$. Then we have

$$\begin{aligned}
\log\left|\frac{g'(z)}{g'(0)}\right| &\leq \left|\log\frac{g'(z)}{g'(0)}\right|\\
&= \left|\int_0^z \frac{g''(w)}{g'(w)}dw\right|\\
&\leq t\int_0^1 \left|\frac{g''(rz)}{g'(rz)}\right| dr\\
&\leq t\int_0^1 \frac{2\lambda}{1-t^2r^2}dr
\end{aligned}$$

$$= 2\lambda \log \sqrt{\frac{1+t}{1-t}}.$$

This implies

$$|g'(z)| \leq |g'(0)| \left(\frac{1+t}{1-t}\right)^{\lambda}, \quad (|z| = t < 1), \tag{3.1}$$

therefore by using relationship (3.1), we can obtain

$$\begin{aligned} \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)| &\leq |g'(0)| \sup_{0<t<1}(1-t^2)\left(\frac{1+t}{1-t}\right)^{\lambda} \\ &\leq 2^{1+\lambda}|g'(0)| \sup_{0<t<1}(1-t)^{1-\lambda} \\ &= 2^{1+\lambda}|g'(0)|. \end{aligned} \tag{3.2}$$

If we set $m = 2^{1+\lambda}|g'(0)|$ and use relationship (3.2), it is deduced that

$$\begin{aligned} \int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p-2} dA_\delta(z) &= (\delta+1)\int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p+\delta-2} dA(z) \\ &\leq (\delta+1)m^p \int_{\mathbb{D}} (1-|z|^2)^{\delta-2} dA(z) < \infty, \end{aligned}$$

where $\delta > 1$ and finally it is concluded that $g \in B^p_\delta$. □

Theorem 3.4. *Assume that $\alpha+\gamma \geq 1$ and $g \in H^\infty$. Then the integral operator $I_g^{\gamma,\alpha}$ is compact from B^p_δ space to B^p_δ space where $1 < p < \infty$ and $\delta \geq 1$.*

Proof. Let $g \in H^\infty$ and (h_n) be a sequence in B^p_δ such that $h_n \to 0$. For $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \|I_g^{\gamma,\alpha} h_n\|^p_{B^p_\delta} &= (p-1)\int_{\mathbb{D}} (1-|z|^2)^{p-2} |h_n'(z) g^\gamma(z) z^{\alpha-1}|^p dA_\delta(z) \\ &\leq \|g\|^{\gamma p}_{H^\infty} \cdot \|h_n\|^p_{B^p_\delta}. \end{aligned}$$

Since for $h_n \to 0$ on $\overline{\mathbb{D}}$, we have $\|h_n\|_{B^p_\delta} \to 0$ and by considering $n \to \infty$ in the last inequality, we obtain that

$$\lim_{n\to\infty} \|I_g^{\gamma,\alpha} h_n\|_{B^p_\delta} = 0.$$

Therefore, $I_g^{\gamma,\alpha}$ is compact. □

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