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**Sahand Communications in
Mathematical Analysis**

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 17
Number: 4
Pages: 61-69

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2019.109347.625

Volume 17, No. 4, November 2020

Print ISSN 2322-5807
Online ISSN 2423-3900

Sahand Communications
in
Mathematical Analysis



Photo by Farhad Mansoury

Sahand Mountain, Maragheh, Iran.

SCMA, P. O. Box 55181-83111, Maragheh, Iran
<http://scma.maragheh.ac.ir>

Integral Operators on the Besov Spaces and Subclasses of Univalent Functions

Zahra Orouji^{1*} and Ali Ebadian²

ABSTRACT. In this note, we study the integral operators $I_g^{\gamma,\alpha}$ and $J_g^{\gamma,\alpha}$ of an analytic function g on convex and starlike functions of a complex order. Then, we investigate the same operators on H^∞ and Besov spaces.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the plane \mathbb{C} , and $H(\mathbb{D}) := \{g : \mathbb{D} \rightarrow \mathbb{C} \mid g \text{ is analytic}\}$. Also, let A be the subclass of $H(\mathbb{D})$, which its elements are of the form

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Suppose that $S^*(\alpha)$ is the famous subclass of A , which is starlike of order α ($0 \leq \alpha < 1$). Indeed, $g \in S^*(\alpha)$ is equivalent to $\operatorname{Re}(zg'(z)/g(z)) > \alpha$ in \mathbb{D} . Similarly, we have $g \in K(\alpha)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha, \quad (z \in \mathbb{D}),$$

where $K(\alpha)$ is the subclass of A contained in the convex functions of order α .

2020 *Mathematics Subject Classification.* 30C45, 30C80.

Key words and phrases. Integral operators, Besov spaces, Convex functions of complex order, Starlike functions of complex order, Schwarzian norm.

Received: 10 June 2019, Accepted: 22 October 2019.

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As usual, we write $S^* = S^*(0)$ and $K = K(0)$. For $0 \neq b \in \mathbb{C}$, the subclasses of A , S_b^* and K_b , are defined by

$$S_b^* = \left\{ g \in A : \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zg'(z)}{g(z)} - 1 \right) \right\} > 0, \quad (z \in \mathbb{D}) \right\},$$

and

$$K_b = \left\{ g \in A : \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zg''(z)}{g'(z)} \right) \right\} > 0, \quad (z \in \mathbb{D}) \right\}.$$

Then, we can see that for $0 \leq \alpha < 1$,

$$S_{1-\alpha}^* = S^*(\alpha), \quad K_{1-\alpha} = K(\alpha).$$

We refer to [3, 11, 12] for some important results.

For some real number β and non-zero complex number b , we introduce a subclass of $H(\mathbb{D})$, $P(\beta, b)$, as follows:

$$P(\beta, b) := \left\{ g \in H(\mathbb{D}) : \operatorname{Re} \left\{ \frac{1}{b} \left(\frac{zg'(z)}{g(z)} \right) \right\} \geq \beta \text{ and } g(0) = 1 \right\}.$$

For example, $\frac{1}{1-z}$ and $\frac{1}{1+z}$ belong to $P(-\frac{1}{2}, 1)$.

Let $g \in H(\mathbb{D})$ be locally univalent. Let

$$S_g(z) = \left(\frac{g''(z)}{g'(z)} \right)' - \frac{1}{2} \left(\frac{g''(z)}{g'(z)} \right)^2,$$

denote the Schwarzian derivative of g , and let

$$\|S_g\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_g(z)|,$$

which denotes its Schwarzian norm.

If $g \in K$ and $h(z) = 1 + \frac{zg''(z)}{g'(z)}$, then $\operatorname{Re} h(z) > 0$ ($z \in \mathbb{D}$), so h is subordinate to $\lambda(z) = \frac{1+z}{1-z}$, where λ is the half-plan mapping. Therefore, $h(z) = \lambda(\varphi(z))$ for some Schwarz function φ , and we have

$$\begin{aligned} \frac{zg''(z)}{g'(z)} &= \frac{1 + \varphi(z)}{1 - \varphi(z)} - 1 \\ &= \frac{2\varphi(z)}{1 - \varphi(z)}, \end{aligned}$$

with the notation $\psi(z) = \frac{\varphi(z)}{z}$, where ψ is analytic and satisfies $|\psi(z)| \leq 1$ in \mathbb{D} . Then it can be written as follows:

$$(1.1) \quad \frac{g''(z)}{g'(z)} = \frac{2\psi(z)}{1 - z\psi(z)}.$$

Hence, the Schwarzian derivative of g can be written in the following form:

$$\begin{aligned} S_g(z) &= \left(\frac{g''(z)}{g'(z)} \right)' - \frac{1}{2} \left(\frac{g''(z)}{g'(z)} \right)^2 \\ &= \frac{2\psi'(z)}{(1 - z\psi(z))^2}. \end{aligned}$$

Then, we obtain

$$(1.2) \quad |S_g(z)| \leq \frac{2}{(1 - |z|^2)^2}.$$

By the inequality (1.2), the Schwarzian norm $\|S_g\|$ of the convex mapping is not greater than 2. If the convex mapping g is bounded, then $\|S_g\| < 2$ (see [10]).

Finally, let $g \in H(\mathbb{D})$. We consider two integral operators on $H(\mathbb{D})$, as follows:

$$I_g^{\gamma, \alpha}(h)(z) = \int_0^z h'(w)g^\gamma(w)w^{\alpha-1}dw, \quad (z \in \mathbb{D}),$$

and

$$J_g^{\gamma, \alpha}(h)(z) = \int_0^z h(w)(g'(w))^\gamma w^{\alpha-1}dw, \quad (z \in \mathbb{D}),$$

where $\gamma, \alpha > 0$.

Integral operators play an important role in various fields (see [4, 8]). If $\gamma = \alpha = 1$, then $I_g^{1,1}(h) = I_g(h)$ and $J_g^{1,1}(h) = J_g(h)$, which are Alexander operators. These integral operators have been investigated by many authors [7, 9, 13, 14].

In this note, we study $I_g^{\gamma, \alpha}$ and $J_g^{\gamma, \alpha}$ operators on K , $K(\alpha)$ and S_b^* . Here, we obtain the necessary and sufficient conditions such that $I_g^{\gamma, \alpha}(\mathbb{D})$ and $J_g^{\gamma, \alpha}(\mathbb{D})$ are bounded, Furthermore, we obtain the sufficient conditions such that $|S_{I_g^{\gamma, \alpha}}| < 2$.

2. INTEGRAL OPERATORS ON $K(\alpha)$ AND $S^*(\alpha)$

Now, we verify the integral operators, $I_g^{\gamma, \alpha}$ and $J_g^{\gamma, \alpha}$, on $K(\alpha)$ and $S^*(\alpha)$.

- Lemma 2.1.** (i) Let $\alpha > 0, \gamma > 0, \beta \geq 0$ and $0 \neq b \in \mathbb{C}$, where $(\alpha - 1)\text{Re}b \geq 0$. If $g \in P(\beta, b)$, then $I_g^{\gamma, \alpha}$ is an operator on K_b .
 (ii) Let $\gamma > 0, 0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha + \beta\gamma < 2$. If $g \in P(\beta, 1)$, then $I_g^{\gamma, \alpha}$ is an operator from $K(\alpha)$ to $K(2\alpha + \beta\gamma - 1)$.

Proof. (i) Let $h \in K_b$. Then, for $z \in \mathbb{D}$,

$$\begin{aligned}
(2.1) \quad & \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(I_g^{\gamma, \alpha} h)''(z)}{(I_g^{\gamma, \alpha} h)'(z)} \right\} \\
&= \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(z^{\alpha-1} h''(z) g^\gamma(z) + \gamma z^{\alpha-1} h'(z) g'(z) g^{\gamma-1}(z))}{z^{\alpha-1} h'(z) g^\gamma(z)} \right\} \\
&\quad + \operatorname{Re} \left\{ \frac{1}{b} \frac{z(\alpha-1) z^{\alpha-2} h'(z) g^\gamma(z)}{z^{\alpha-1} h'(z) g^\gamma(z)} \right\} \\
&= (\alpha-1) \operatorname{Re} \left(\frac{1}{b} \right) + \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z h''(z)}{h'(z)} \right) \right\} + \gamma \operatorname{Re} \left\{ \frac{1}{b} \left(\frac{z g'(z)}{g(z)} \right) \right\}.
\end{aligned}$$

By this hypothesis and (2.1), we obtain

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z(I_g^{\gamma, \alpha} h)''(z)}{(I_g^{\gamma, \alpha} h)'(z)} \right) \right\} > 0.$$

Therefore, $I_g^{\gamma, \alpha} h \in K_b$ for all $h \in K_b$.

(ii) We can prove this part in a similar manner as the proof of part (i). □

Lemma 2.2. *Let $\alpha > 0$, $0 \neq b \in \mathbb{C}$ and $0 < \gamma < \alpha \operatorname{Re}(\frac{1}{b})$. If $g \in K_b$, then $J_g^{\gamma, \alpha}$ is an operator from S_b^* to K_b .*

Proof. Let $h \in S_b^*$, then for $z \in \mathbb{D}$,

$$\begin{aligned}
& \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(J_g^{\gamma, \alpha} h)''(z)}{(J_g^{\gamma, \alpha} h)'(z)} \right\} \\
&= \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(z^{\alpha-1} h'(z) (g'(z))^\gamma + \gamma z^{\alpha-1} h(z) g''(z) (g'(z))^{\alpha-1})}{z^{\alpha-1} h(z) (g'(z))^\gamma} \right\} \\
&\quad + \operatorname{Re} \left\{ \frac{1}{b} \frac{z(\alpha-1) z^{\alpha-2} h(z) (g'(z))^\gamma}{z^{\alpha-1} h(z) (g'(z))^\gamma} \right\} \\
&= \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z h'(z)}{h(z)} \right) + \frac{\gamma}{b} \left(\frac{z g''(z)}{g'(z)} \right) + \frac{\alpha-1}{b} \right\} \\
&= \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z h'(z)}{h(z)} - 1 \right) \right\} + \gamma \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z g''(z)}{g'(z)} \right) \right\} + \alpha \operatorname{Re} \left(\frac{1}{b} \right) - \gamma.
\end{aligned}$$

By this hypothesis, we obtain $J_g^{\gamma, \alpha} h \in K_b$. □

Theorem 2.3. *Let $\gamma > 0$, $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha + \beta\gamma < 2$. Also, let $g \in P(\beta, 1)$ and $h \in K(\alpha)$. Then, the image $(I_g^{\gamma, \alpha} h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z| \rightarrow 1} (1 - |z|) \left| \frac{z h''(z)}{h'(z)} + \frac{\gamma z g'(z)}{g(z)} + \alpha + 1 \right| < 1.$$

Proof. By using Part (2) of Lemma 2.1, we have $I_g^{\gamma, \alpha} h \in K$. By replacing $g = I_g^{\gamma, \alpha} h$ in (1.1), then there exists $\psi \in H(\mathbb{D})$ such that

$$\frac{(I_g^{\gamma, \alpha} h)''(z)}{(I_g^{\gamma, \alpha} h)'(z)} = \frac{2\psi(z)}{1 - z\psi(z)}.$$

Therefore

$$\begin{aligned} \psi(z) &= \frac{\frac{(I_g^{\gamma, \alpha} h)''(z)}{(I_g^{\gamma, \alpha} h)'(z)}}{2 + \frac{z(I_g^{\gamma, \alpha} h)''(z)}{(I_g^{\gamma, \alpha} h)'(z)}} \\ &= \frac{\frac{h''(z)}{h'(z)} + \frac{\gamma g'(z)}{g(z)} + \frac{\alpha-1}{z}}{1 + \alpha + \frac{zh''(z)}{h'(z)} + \frac{\gamma zg'(z)}{g(z)}}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} (2.2) \quad \frac{1 - |z|}{|1 - z\psi(z)|} &= \frac{1 - |z|}{\frac{2}{\left| \frac{zh''(z)}{h'(z)} + \frac{\gamma zg'(z)}{g(z)} + \alpha + 1 \right|}} \\ &= \frac{1}{2}(1 - |z|) \left| 1 + \alpha + \frac{zh''(z)}{h'(z)} + \frac{\gamma zg'(z)}{g(z)} \right|. \end{aligned}$$

By Theorem 2 in [10], we can conclude that the image $(I_g^{\gamma, \alpha} h)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z| \rightarrow 1} \frac{1 - |z|}{|1 - z\psi(z)|} < \frac{1}{2},$$

and so by (2.2), the proof is complete. \square

Similarly, by using Lemma 2.2, the following theorem is achieved:

Theorem 2.4. *Let $0 < \gamma < \alpha$, $g \in K$ and $h \in S^*$. Then, the image $(J_g^{\gamma, \alpha} h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z| \rightarrow 1} (1 - |z|) \left| \frac{zh'(z)}{h(z)} + \frac{\gamma zg''(z)}{g'(z)} + \alpha + 1 \right| < 1.$$

By using part 2 of Lemma 2.1, we can obtain the below result:

Corollary 2.5. *Let $\gamma > 0$, $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha + \beta\gamma < 2$. If $g \in P(\beta, 1)$ and $h \in K(\alpha)$, then $\|S_{I_g^{\gamma, \alpha} h}\| \leq 2$.*

And also, by Theorem 2.3, the below conclusion is gained:

Corollary 2.6. *Let $\gamma > 0$, $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, where $1 \leq 2\alpha + \beta\gamma < 2$. If*

$$\limsup_{|z| \rightarrow 1} (1 - |z|) \left| \frac{zh''(z)}{h'(z)} + \frac{\gamma zg'(z)}{g(z)} + \alpha + 1 \right| < 1,$$

then $\|S_{I_g^{\gamma, \alpha} h}\| < 2$.

Finally, a similar corollary to those above is also true for the operator $J_f^{\gamma, \alpha} h$.

3. INTEGRAL OPERATORS ON BESOV SPACES

We use $dA(z)$ to denote the area measure of \mathbb{D} which is normalized, so the area of \mathbb{D} is 1. We have

$$\begin{aligned} dA(z) &= \frac{1}{\pi} dx dy \\ &= \frac{r}{\pi} dr d\theta, \end{aligned}$$

where $z = x + iy = re^{i\theta}$ and we set

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

where $\alpha > -1$. It is clear that if α is a real number then

$$\int_{\mathbb{D}} (1 - |z|^2)^\alpha dA(z) < \infty,$$

if and only if $\alpha > -1$.

For $1 < p < \infty$ and $\delta \geq 1$, the Besov space B_δ^p is defined as the set of all $g \in H(\mathbb{D})$ such that

$$\begin{aligned} \|g\|_{B_\delta^p} &:= |g(0)| + \left\{ (p-1) \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |g'(z)|^p dA_\delta(z) \right\}^{\frac{1}{p}} \\ &< \infty. \end{aligned}$$

For simplicity, the space B_1^p will be denoted by B^p . Many authors have studied the properties of the Besov spaces [1, 2]. The space H^∞ consists of bounded analytic functions g in \mathbb{D} where

$$\|g\|_{H^\infty} := \lim_{r \rightarrow 1^-} (\max_{|z| \leq r} |g(z)|) < \infty.$$

In this section, we study two operators $I_g^{\gamma, \alpha}$ and $J_g^{\gamma, \alpha}$ on H^∞ and Besov space B_δ^p .

Theorem 3.1. *Let $1 < p < \infty$ and $\delta \geq 1$. If $g \in B_\delta^p$ then $J_g^{\gamma, \alpha}$ is bounded on H^∞ and $\|J_g^{\gamma, \alpha}\|_{B_\delta^p} \leq \|g\|_{B_\delta^p}$ where $\gamma \leq 1$ and $\alpha \geq 1$.*

Proof. Let $\|h\|_{H^\infty} = 1$. Therefore,

$$\begin{aligned} \|J_g^{\gamma, \alpha}\|_{B_\delta^p}^p &= (p-1) \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |h(z)(g'(z))^\gamma z^{\alpha-1}|^p dA_\delta(z) \\ &\leq (p-1) \|h\|_{H^\infty}^p \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |(g'(z))^\gamma|^p dA_\delta(z) \\ &\leq \left(\|g\|_{B_\delta^p} - |g(0)| \right)^p < \infty, \end{aligned}$$

since $\gamma \leq 1$ and $\alpha \geq 1$. □

Theorem 3.2. *Let $1 < p < \infty$, $\delta \geq 1$ and $g \in H^\infty$. Then $I_g^{\gamma, \alpha} \in B_\delta^p$, where $\alpha + \gamma \geq 1$. Moreover, $\|I_g^{\gamma, \alpha}\|_{B_\delta^p} \leq \|I_g^\gamma\|_{H^\infty}$, where $I(z) = z^{\alpha-1}$ ($z \in \mathbb{D}$).*

Proof. Suppose that $g \in H^\infty$. There exists a number $N > 0$ such that $\frac{|g(z)|}{N} < |z|$ ($z \in \mathbb{D}$), thus,

$$|z^{\alpha-1}g^\gamma(z)| \leq N^\gamma, \quad (z \in \mathbb{D}),$$

where $\gamma + \alpha - 1 \geq 0$.

Therefore, $z^{\alpha-1}g^\gamma(z) \in H^\infty$. Set $I(z) = z^{\alpha-1}$ ($z \in \mathbb{D}$), so there exists a number $c > 0$ such that $\|I_g^\gamma\|_{H^\infty} = c$. Now, for any $\|h\|_{B_\delta^p} = 1$, we have

$$\begin{aligned} \|I_g^{\gamma, \alpha}h\|_{B_\delta^p}^p &= (p-1) \int_{\mathbb{D}} (1-|z|^2)^{p-2} |h'(z)z^{\alpha-1}g^\gamma(z)|^p dA_\delta(z) \\ &\leq c^p(p-1) \int_{\mathbb{D}} (1-|z|^2)^{p-2} |h'(z)|^p dA_\delta(z) \\ &\leq c^p \|h\|_{B_\delta^p}^p \\ &= c^p, \end{aligned}$$

and the proof is complete. \square

Let $\lambda > 0$ and g be a locally univalent function. Also let

$$B(\lambda) = \{g \in H(\mathbb{D}); \left\| \frac{g''}{g'} \right\| \leq 2\lambda\},$$

where

$$\left\| \frac{g''}{g'} \right\| = \sup_{z \in \mathbb{D}} (1-|z|^2) \left| \frac{g''(z)}{g'(z)} \right|,$$

is the norm of the pre-Schwarzian derivative $\frac{g''}{g'}$ of g . Kim and Sugawa [5, 6] investigated the properties of the class $B(\lambda)$.

Theorem 3.3. *Let $1 < p < \infty$, $\delta > 1$ and $\lambda < 1$. Therefore, $B(\lambda) \subseteq B_\delta^p$.*

Proof. Let $|z| = t < 1$ and $g \in B(\lambda)$. Then we have

$$\begin{aligned} \log \left| \frac{g'(z)}{g'(0)} \right| &\leq \left| \log \frac{g'(z)}{g'(0)} \right| \\ &= \left| \int_0^z \frac{g''(w)}{g'(w)} dw \right| \\ &\leq t \int_0^1 \left| \frac{g''(rz)}{g'(rz)} \right| dr \\ &\leq t \int_0^1 \frac{2\lambda}{1-t^2r^2} dr \end{aligned}$$

$$= 2\lambda \log \sqrt{\frac{1+t}{1-t}}.$$

This implies

$$(3.1) \quad |g'(z)| \leq |g'(0)| \left(\frac{1+t}{1-t} \right)^\lambda, \quad (|z| = t < 1),$$

therefore by using relationship (3.1), we can obtain

$$(3.2) \quad \begin{aligned} \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| &\leq |g'(0)| \sup_{0 < t < 1} (1 - t^2) \left(\frac{1+t}{1-t} \right)^\lambda \\ &\leq 2^{1+\lambda} |g'(0)| \sup_{0 < t < 1} (1 - t)^{1-\lambda} \\ &= 2^{1+\lambda} |g'(0)|. \end{aligned}$$

If we set $m = 2^{1+\lambda} |g'(0)|$ and use relationship (3.2), it is deduced that

$$\begin{aligned} \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2} dA_\delta(z) &= (\delta + 1) \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p+\delta-2} dA(z) \\ &\leq (\delta + 1) m^p \int_{\mathbb{D}} (1 - |z|^2)^{\delta-2} dA(z) < \infty, \end{aligned}$$

where $\delta > 1$ and finally it is concluded that $g \in B_\delta^p$. \square

Theorem 3.4. *Assume that $\alpha + \gamma \geq 1$ and $g \in H^\infty$. Then the integral operator $I_g^{\gamma, \alpha}$ is compact from B_δ^p space to B_δ^p space where $1 < p < \infty$ and $\delta \geq 1$.*

Proof. Let $g \in H^\infty$ and (h_n) be a sequence in B_δ^p such that $h_n \rightarrow 0$. For $n = 1, 2, 3, \dots$, we have

$$\begin{aligned} \|I_g^{\gamma, \alpha} h_n\|_{B_\delta^p}^p &= (p-1) \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |h_n'(z) g^\gamma(z) z^{\alpha-1}|^p dA_\delta(z) \\ &\leq \|g\|_{H^\infty}^p \cdot \|h_n\|_{B_\delta^p}^p. \end{aligned}$$

Since for $h_n \rightarrow 0$ on $\overline{\mathbb{D}}$, we have $\|h_n\|_{B_\delta^p} \rightarrow 0$ and by considering $n \rightarrow \infty$ in the last inequality, we obtain that

$$\lim_{n \rightarrow \infty} \|I_g^{\gamma, \alpha} h_n\|_{B_\delta^p} = 0.$$

Therefore, $I_g^{\gamma, \alpha}$ is compact. \square

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