Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 3 (2020), 117-130
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2019.111716.644

On the Spaces of λ_r-almost Convergent and λ_r-almost Bounded Sequences

Sinan Ercan

Abstract. The aim of the present work is to introduce the concept of λ_r-almost convergence of sequences. We define the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$ of λ_r-almost convergent and λ_r-almost null sequences. We investigate some inclusion relations concerning those spaces with examples and we determine the β- and γ-duals of the space $f(\lambda_r)$. Finally, we give the characterization of some matrix classes.

1. Preliminaries and Background

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called sequence space. We write ℓ_∞, c, c_0 for the classical sequence spaces of all bounded, convergent, null, respectively. Throughout this paper, we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \to \mathbb{C}, \qquad \tau_k(x) = x_k$$

is a continuous for all $k \in \mathbb{N}$, X is called a $BK-$space. The sequence spaces ℓ_∞, c and c_0 are $BK-$spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbb{N}$.

A continuous linear functional ϕ on ℓ_∞ is called a Banach limit if

(i) $\phi(x) \geq 0$ for $x = (x_k)$ and $x_k \geq 0$ for every k,
(ii) $\phi\left(x_{\sigma(k)}\right) = \phi(x_k)$, where σ is shift operator which is defined on w by $\sigma(k) = k + 1$ and

2010 *Mathematics Subject Classification.* 46A45, 40C05.

Key words and phrases. Almost convergence, Matrix domain, β-, γ-duals, Matrix transformations.

Received: 21 July 2019, Accepted: 21 October 2019.

.

(iii) $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$.

A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α and denoted by $f - \lim x = \alpha$. Lorentz [17] introduced that $f - \lim x_k = \alpha$ uniformly in n if and only if

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} \text{ uniformly in } n.$$

We denote the sets of all almost convergent sequences f by

$$f = \left\{ x = (x_k) \in w : \lim_{m\to\infty} t_{mn}(x) = \alpha \text{ uniformly in } n \right\},$$

where

$$t_{mn}(x) = \sum_{k=0}^{m} \frac{1}{m+1} x_{k+n}, \quad t_{-1,n} = 0.$$

It is well known that $c \subset f \subset \ell_\infty$ strictly hold. Since these inclusions, norms $\|.\|_f$ and $\|.\|_{\ell_\infty}$ of the spaces f and ℓ_∞ are equivalent, so the sets f and f_0 are BK-spaces with the norm $\|x\|_f = \sup_{m,n} |t_{mn}(x)|$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} = 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then $x = U(Vx) = V(Ux)$ holds for all $x \in w$.

If A is an infinite matrix with complex entries a_{nk} for $n, k \in \mathbb{N}$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Any sequence in the n^{th} row of A is indicated by A_n, that is $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. If $x = (x_k) \in w$ then we define the A-transform of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k, \tag{1.1}$$

provided the series (1.1) converges for $n \in \mathbb{N}$. $x = (x_k)$ is called $A-$summable to $a \in \mathbb{C}$ if Ax converges to a which is called $A-$limit of x. If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix mapping from X into Y and denote it by $A : X \to Y$. By $(X : Y)$ we mean the class of all infinite matrices such that $A : X \to Y$.

For an arbitrary sequence space X, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}, \tag{1.2}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$.

Constructing a new sequence space X_A generated by the limitation matrix A from a sequence space X is the expansion or the contraction of the original space X. Using domain of a triangle matrix to construct a new sequence spaces was studied by many authors. (for instance [1]-[15])

2. On the Concept of λ_r-summability

Let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a set which consists of strictly increasing sequence of positive numbers tending to ∞, that is $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. Throughout this paper, we assume that $r \geq 1$ is an integer. We define the infinite matrix $\Lambda^r = (\lambda^r_{nk})$ by

$$\lambda^r_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-r}}{\lambda_n}; & 0 \leq k \leq n \text{ and } r|\, n-k, \\ 0; & \text{otherwise} \end{cases}$$

for $n, k \in \mathbb{N}$. It is clear that the matrix Λ^r is a triangle, that is $\lambda_{nn} \neq 0$ and $\lambda_{nk} = 0$ for $k > n$, $n = 0, 1, 2, \ldots$.

We note that if we choose $r = 1$ the Λ^r matrix reduces to the matrix Λ which is defined in [12]. Also, if $r = 1$ for the sequence $\lambda_k = k + r$ the Λ^r matrix is coincide with the matrix of Cesàro means given in [13] and [14].

Now, let $x = (x_n) \in w$ and $n \geq 1$. Then, we obtain that

$$\begin{aligned} x_n - \Lambda^r_n(x) &= \frac{1}{\lambda_n} \sum_{\substack{i=0 \\ r|n-i}}^{n} (\lambda_i - \lambda_{i-r})(x_n - x_i) \\ &= \frac{1}{\lambda_n} \sum_{\substack{i=0 \\ r|n-i}}^{n} (\lambda_i - \lambda_{i-r}) \sum_{\substack{k=i+r \\ r|n-k}}^{n} (x_k - x_{k-r}) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^{n} (x_k - x_{k-r}) \sum_{\substack{i=0 \\ r|n-i}}^{k-r} (\lambda_i - \lambda_{i-r}) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^{n} \lambda_{k-r}(x_k - x_{k-r}). \end{aligned}$$

Hence we have that

$$x_n - \Lambda^r_n(x) = S^r_n(x), \tag{2.1}$$

for $n \in \mathbb{N}$. Here the sequence $S^r(x) = (S^r_n(x))_{n=0}^{\infty}$ is defined by

$$S^r_0(x) = 0 \quad \text{and} \quad S^r_n(x) = \frac{1}{\lambda_n} \sum_{\substack{k=r \\ r|n-k}}^{n} \lambda_{k-r}(x_k - x_{k-r}). \tag{2.2}$$

We have the following result from (2.1) and Lemma 2.2.

Theorem 2.1. *For a sequence* $x = (x_k) \in w$ *with* $a \in \mathbb{C}$*, let* $f - \lim_{n\to\infty} x_n = a$*. Then,* $f - \lim_{n\to\infty} \Lambda_n^r(x) = a$ *holds if and only if* $S^r(x) \in f_0$*.*

Proof. Firstly, we assume that $f - \lim_{n\to\infty} x_n = f - \lim_{n\to\infty} \Lambda_n^r(x) = a$. We have that the equality

$$\frac{1}{m+1}\sum_{k=0}^{m}\left[x_{n+k} - \Lambda_{n+k}^r(x)\right] = \frac{1}{m+1}\sum_{k=0}^{m} S_{n+k}^r(x), \tag{2.3}$$

holds for all $m, n \in \mathbb{N}$. Since (2.3) tends to zero as $m \to \infty$ uniformly in n, we have

$$\lim_{n\to\infty}\frac{1}{m+1}\sum_{k=0}^{m} S_{n+k}^r(x) = 0 \text{ uniformly in } n.$$

This means that $S^r(x) \in f_0$.

Conversely, assume that $S^r(x) \in f_0$ and let $f - \lim_{n\to\infty} x_n = a$. We have that

$$\lim_{n\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\left[x_{n+k} - \Lambda_{n+k}^r(x)\right] = 0,$$

from (2.3) as $m \to \infty$. Consequently, the desired result

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m} x_{n+k} &= \lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\Lambda_{n+k}^r(x)\\ &= a\end{aligned}$$

is obtained. □

3. The Spaces of λ_r-almost Convergent and λ_r-almost null Sequences

In this section, we introduce the following spaces as the sets of all λ_r-almost convergent sequences and λ_r-almost null sequences, respectively, that is

$$f(\lambda_r) = \left\{x \in w : \lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\Lambda_n^r(x) = l \text{ uniformly in } n\right\},$$

$$f_0(\lambda_r) = \left\{x \in w : \lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\Lambda_n^r(x) = 0 \text{ uniformly in } n\right\}.$$

Using the notation of (1.2) we write again these spaces given above as the matrix domains of the triangle Λ^r in the spaces f and f_0, respectively, such that

$$f(\lambda_r) = (f)_{\Lambda^r}, \qquad f_0(\lambda_r) = (f_0)_{\Lambda^r}.$$

Now we define the sequence $y = (y_n)$ which is connected with the sequence $x = (x_k)$ by the λ_r-transform, i.e.,

$$y_n = \Lambda_n^r(x) = \frac{1}{\lambda_n} \sum_{\substack{k=0 \\ r|n-k}}^{n} (\lambda_k - \lambda_{k-r}) x_k, \tag{3.1}$$

for all $n \in \mathbb{N}$.

Theorem 3.1. *The sequence spaces $f(\lambda_r)$ and $f_0(\lambda_r)$ are BK-spaces with the same norm given by*

$$\|x\|_{f(\lambda_r)} = \|\Lambda^r(x)\|_f = \sup_{n,m\in\mathbb{N}} |t_{mn}(\Lambda^r(x))|, \tag{3.2}$$

where

$$t_{mn}(\Lambda^r(x)) = \frac{1}{m+1}\sum_{j=0}^{m} \Lambda_{n+j}^r(x) = \frac{1}{m+1}\sum_{j=0}^{m} \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k,$$

for all $m, n \in \mathbb{N}$.

Proof. It is well known that f and f_0 are BK-spaces with the norm $\|.\|_\infty$. Also the matrix Λ^r is a triangle matrix. Hence $f(\lambda_r)$ and $f_0(\lambda_r)$ are BK-spaces endowed with the norm $\|.\|_{f(\lambda_r)}$ from Theorem 4.3.2 given in [16]. □

Theorem 3.2. *The sequence spaces $f(\lambda_r)$, $f_0(\lambda_r)$ are norm isomorphic to the spaces f and f_0, respectively, that is $f(\lambda_r) \cong f$ and $f_0(\lambda_r) \cong f_0$.*

Proof. To prove $f(\lambda_r) \cong f$ we need show the existence of a linear bijection between the spaces $f(\lambda_r)$ and f which preserves the norm. Let define T as (3.1), from $f(\lambda_r)$ to f by $x \to y = Tx = \Lambda^r(x)$. The linearity of T is clear. Also $x = \theta$ whenever $Tx = T\theta$ and hence T is injective.

Now, let $y = (y_k) \in f$ and $x = (x_k)$ defined by

$$x_k = \begin{cases} -\frac{\lambda_j}{\lambda_k - \lambda_{k-r}}, & j = k-r, \\ \frac{\lambda_j}{\lambda_k - \lambda_{k-r}}, & j = k, \\ 0, & \text{otherwise.} \end{cases}$$

for all $k \in \mathbb{N}$. Then we have

$$\sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k = \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k y_k - \lambda_{k-r} y_{k-r}}{\lambda_{n+j}}$$
$$= y_{n+j}$$

which gives

$$\frac{1}{m+1} \sum_{j=0}^{m} \sum_{\substack{k=0 \\ r|n+j-k}}^{n+j} \frac{\lambda_k - \lambda_{k-r}}{\lambda_{n+j}} x_k = \frac{1}{m+1} \sum_{j=0}^{m} y_{n+j}.$$

Hence,

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} \Lambda^r_{n+j}(x) = \lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} y_{n+j}$$
$$= l \text{ uniformly in } n.$$

This means that $x \in f(\lambda_r)$ and T is surjective. T is also norm preserving from (3.2). The desired result is obtained. □

We note that absolute property does not hold on the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$, that is $\|x\|_{f(\lambda_r)} \neq \||x|\|_{f_0(\lambda_r)}$ for at least one sequence x in each of these spaces, where $|x| = (|x_k|)$. Consequently, these spaces are BK-spaces of non-absolute type. Further, $f(\lambda_r)$ and $f_0(\lambda_r)$ has no Schauder basis from Corollary 3.3 in [11] and Theorem 2.3 in [18].

4. Some Inclusion Relations

Theorem 4.1. *The inclusions $c(\lambda_r) \subset f(\lambda_r) \subset \ell_\infty(\lambda_r)$ strictly hold.*

Proof. Let $x = (x_k)$ be a sequence in $c(\lambda_r)$. Then, $\Lambda^r(x) \in c$ and we know that the inclusion $c \subset f$ holds. Hence, $\Lambda^r(x) \in f$, that is $x \in f(\lambda_r)$. Now to prove strictness of the inclusion we give the following example.

Example 4.2. Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} (-1)^k, & if\ r\ is\ even \\ -\frac{\lambda_k+\lambda_{k-r}}{\lambda_k-\lambda_{k-r}}, & if\ r\ is\ odd \end{cases} \tag{4.1}$$

for all $k \in \mathbb{N}$. Then we have

$$\Lambda^r_n(x) = (-1)^n \tag{4.2}$$

for all $n \in \mathbb{N}$. This means that $x \in c(\lambda_r)$. Further, we have

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\Lambda_{n+j}^{r}(x) &= \lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}(-1)^{n+j}\\ &= \lim_{m\to\infty}\frac{(-1)^{n}}{m+1}\left[\frac{1+(-1)^{m}}{2}\right]\\ &= 0.\end{aligned}$$

Hence $x \in f(\lambda_r)$ and the inclusion $c(\lambda_r) \subset f(\lambda_r)$ is strict.

Now we prove the inclusion $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ holds. Let $y = (y_k) \in f(\lambda_r)$. Then, since $\Lambda^r(y) \in f$ and $f \subset \ell_\infty$, we have $\Lambda^r(y) \in \ell_\infty$ and $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ holds. To see strictness of this inclusion consider the sequence z defined in [19] by $y = \Lambda^r(z)$, where

$$y = (0,0,0,\ldots,1,\ldots,1,\ldots,0,\ldots,0,\ldots),$$

and blocks of 0's are increasing by factors of 100 and the blocks of 1 increasing by factors of 10. This sequence is in ℓ_∞ but not in f. Hence, $z \in \ell_\infty(\lambda_r) \backslash f(\lambda_r)$ and the inclusion $f(\lambda_r) \subset \ell_\infty(\lambda_r)$ is strict. □

Theorem 4.3. *The inclusion $f_0(\lambda_r) \subset f(\lambda_r)$ strictly holds.*

Proof. Let $x = (x_k)$ be a sequence in $f_0(\lambda_r)$. Then, we have $\Lambda^r(x) \in f_0$. Since $f_0 \subset f$, $\Lambda^r(x) \in f$ and $x \in f(\lambda_r)$. Consequently, $f_0(\lambda_r) \subset f(\lambda_r)$ holds. Now to see strictness of this inclusion consider $x = (x_k)$ defined by $x_k = 1$ for all $k \in \mathbb{N}$. Obviously, $x \in f(\lambda_r)$ and we have

$$\lim_{n\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\sum_{\substack{k=0\\ r|n+j-k}}^{n+j}\frac{\lambda_k-\lambda_{k-r}}{\lambda_{n+j}}x_k = 1.$$

Hence, $x \notin f_0(\lambda_r)$ and the inclusion $f_0(\lambda_r) \subset f(\lambda_r)$ is strict. □

By taking into account $\lambda \in \Lambda$, we have $\lambda_{k+r}/\lambda_k > 1$ for all $k \in \mathbb{N}$. Hence, there are only two distinct cases of the sequence λ, either $\lim_{k\to\infty}\inf \lambda_{k+r}/\lambda_k = 1$ or $\lim_{k\to\infty}\inf \lambda_{k+r}/\lambda_k > 1$. Clearly, we obtain the following result:

Lemma 4.4. (i) $\lim_{k\to\infty}\inf \lambda_{k+r}/\lambda_k = 1$ *if and only if* $\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-r}}\right) \notin \ell_\infty$.

(ii) $\lim_{k\to\infty}\inf \lambda_{k+r}/\lambda_k > 1$ *if and only if* $\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-r}}\right) \in \ell_\infty$.

Theorem 4.5. (i) *The inclusions $f_0 \subset f_0(\lambda_r)$ and $f \subset f(\lambda_r)$ strictly hold.*

(ii) *The equalities $f_0 = f_0(\lambda_r)$ and $f = f(\lambda_r)$ hold if and only if $\Lambda^r(x) \in f_0$ for every x in the spaces $f(\lambda_r)$ and $f_0(\lambda_r)$, respectively.*

Proof. (i) Let $x = (x_k) \in c$. We know that $c \subset f$ and Λ^r is regular, hence $x \in f$, $\Lambda^r(x) \in c$. Therefore, we obtain that $x \in f(\lambda_r)$ and the inclusion $f \subset f(\lambda_r)$ holds. To see strictness of this inclusion let define a sequence $x = (x_k)$ by (4.1) and suppose that $\lim_{k\to\infty} \inf \frac{\lambda_{k+r}}{\lambda_k} = 1$. Since $x \notin \ell_\infty$, we obtain $x \notin f$. But $\Lambda^r(x) \in f$ and $x \in f(\lambda_r)$. This completes the proof. Similarly, one can prove that the inclusion $f_0 \subset f_0(\lambda_r)$ strictly holds.

(ii) If we assume that $x \in f(\lambda_r)$, we have $S^r(x) \in f_0$. Hence,

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} S(x)_{n+k} = 0.$$

Then, we have

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} [x_{n+k} - \Lambda^r_{n+k}(x)] = 0$$

from (2.2). This means that

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{n+k} = \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} \Lambda^r_{n+k}(x)$$
$$= l \text{ uniformly in } n.$$

Hence, $f(\lambda_r) \subset f$. By combining the inclusion $f \subset f(\lambda_r)$ the equality $f(\lambda_r) = f$ is obtained.

Conversely, assume that the equality $f = f(\lambda_r)$ holds. By (2.2), we have $S^r(x) \in f_0$. Following similar way, the results which are concerning to $f_0(\lambda_r)$ will be obtained.

□

Theorem 4.6. *Neither of the spaces ℓ_∞ and $f(\lambda_r)$ includes the other.*

Proof. Consider the sequences defined by $\lambda_k = k$ and $x_k = 1/r$. Then, since $\Lambda^r(x) = e \in f$, $x \in f(\lambda_r)$. It is obvious that $x \in \ell_\infty \cap f(\lambda_r)$. Now, consider the sequence x given by (4.1) and suppose that $\lim_{k\to\infty} \inf \frac{\lambda_{k+r}}{\lambda_k} = 1$. Then, since $\Lambda^r(x) = (-1)^n \in f$, $x \in f(\lambda_r)$ but $x \notin \ell_\infty$. Further, let take $\lambda_k = k$ and define another sequence

$$y = (0, \ldots, 0, 1/r, \ldots, 1/r, 0, \ldots, 0, 1/r, \ldots, 1/r, 0, \ldots, 0, \ldots),$$

where the block's of 0's are increasing by factors of 100 and the blocks of $1/r$ are increasing by factors of 10. Then,

$$\Lambda^r(y) = (0, \ldots 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, \ldots) \notin f$$

and $y \notin f(\lambda_r)$, where the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10, but $y \in \ell_\infty$. This means that $y \in \ell_\infty \backslash f(\lambda_r)$. Hence the spaces ℓ_∞ and $f(\lambda_r)$ overlap, but neither of them include each other. □

5. The $\beta-$ and $\gamma-$duals of the set $f(\lambda_r)$ and Some Certain Matrix Classes

In this section, we determine the β-, γ-duals of the space $f(\lambda_r)$. The β-, γ- duals of a sequence space μ are defined as followings;

$$\mu^\beta = \{x = (x_k) \in w : xy = (x_k y_k) \in cs \; for \; all \; y = (y_k) \in \mu\}$$

$$\mu^\gamma = \{x = (x_k) \in w : xy = (x_k y_k) \in bs \; for \; all \; y = (y_k) \in \mu\}.$$

Then, we characterize some matrix transformations between $f(\lambda_r)$ and classical sequence spaces. Now we begin the following lemmas which will be used in the proof of our results.

Lemma 5.1 ([21]). *$A = (a_{nk}) \in (f : \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{5.1}$$

Lemma 5.2 ([21]). *$A = (a_{nk}) \in (f : c)$ if and only if (5.1) holds and there are α, $\alpha_k \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \tag{5.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{5.3}$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0. \tag{5.4}$$

Lemma 5.3 ([22]). *$A \in (\ell_\infty : f)$ if and only if (5.1) holds and*

$$f - \lim a_{nk} = \alpha_k \; exists \; for \; each \; fixed \; k, \tag{5.5}$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0 \; uniformly \; in \; n. \tag{5.6}$$

Lemma 5.4 ([23]). *$A \in (c : f)$ if and only if (5.1), (5.5) hold and*

$$f - \lim \sum_k a_{nk} = \alpha. \tag{5.7}$$

Lemma 5.5 ([22]). *$A \in (f : f)$ if and only if (5.1), (5.5), (5.7) hold and*

$$\lim_{m\to\infty} \sum_k |\Delta[a(n,k,m) - \alpha_k]| = 0 \; uniformly \; in \; n. \tag{5.8}$$

Theorem 5.6. *The $\gamma-$dual of the space $f(\lambda_r)$ is the set $e_1 \cap e_2$, where*

$$e_1 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \left| \overline{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \right) \lambda_k \right| < \infty \right\},$$
$$e_2 = \left\{ a = (a_k) \in w : \left(\frac{a_n}{\lambda_n - \lambda_{n-r}} \lambda_n \right) \in \ell_\infty \right\}.$$

Proof.

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left[-\frac{\lambda_{k-r}}{\lambda_k - \lambda_{k-r}} y_{k-r} + \frac{\lambda_k}{\lambda_k - \lambda_{k-r}} y_k \right] \\ &= \sum_{k=0}^{n-1} \overline{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \right) \lambda_k y_k + \frac{a_n}{\lambda_n - \lambda_{n-r}} \lambda_n y_n \\ &= T_n(y) \end{aligned}$$

for all $n \in \mathbb{N}$. $T = (t_{nk})$ is the matrix defined by

$$t_{nk} = \begin{cases} \bar{\Delta}_r \left(\frac{a_k}{\lambda_k - \lambda_{k-r}} \% \right) \lambda_k, & \text{if } 0 < k < n-1, \\ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_n, & \text{if } k = n, \\ 0, & \text{if } k > n. \end{cases} \tag{5.9}$$

for all $k, n \in \mathbb{N}$. We deduce from $T_n(y)$ that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in f(\lambda_r)$ if and only if $Ty \in \ell_\infty$ whenever $y = (y_k) \in f$, where $T = (t_{nk})$ is defined by (5.9). Therefore, we obtain from Lemma 5.1 that $(f(\lambda_r))^\gamma = e_1 \cap e_2$. □

Theorem 5.7. *Define the sets e_3 and e_4 by*

$$e_3 = \left\{ a = (a_k) \in w : \left\{ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_n \right\} \in c \right\},$$
$$e_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k |\Delta(t_{nk} - \alpha_k)| = 0 \right\},$$

then $\{f(\lambda_r)\}^\beta = e_3 \cap e_4$.

Proof. Take any $a = (a_k) \in w$. It is eaisly seen from $T_n(y)$ that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(\lambda_r)$ if and only if $Ty \in c$ whenever $y = (y_k) \in f$. It is clear that the columns of the matrix T in lie c where $T = (t_{nk})$ defined in (5.9). We have the consequence by Lemma 5.2 that $\{f(\lambda_r)\}^\beta = e_3 \cap e_4$. □

Theorem 5.8. *Let assume that $A = (a_{nk})$ and $B = (b_{nk})$ are the infinite matrices which are connected with relation*

$$\hat{a}_{nk} = b_{nk} \tag{5.10}$$

$$= \overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k$$

and μ is any given sequence space. Then, $A \in (f(\lambda_r) : \mu)$ if and only if $B \in (f : \mu)$ and

$$\left\{ \frac{\lambda_n}{\lambda_n - \lambda_{n-r}} a_{nk} \right\}_{k \in \mathbb{N}} \in c_0. \tag{5.11}$$

Proof. Firstly, keep in mind that the sequences f and $f(\lambda_r)$ are norm isomorphic. Then, we assume that $A \in (f(\lambda_r) : \mu)$ and take any $y = (y_k) \in f$. Then, $B\Lambda^r$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(\lambda_r)\}^{\beta}$ which yields that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence By exists for each $y \in f$ and thus letting $m \to \infty$ in the equality

$$\begin{aligned}\sum_{k=0}^{m} b_{nk} y_k &= \sum_{k=0}^{m} \left[\overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k \right] \cdot \left(\sum_{\substack{i=0 \\ r|k-i}}^{k} \frac{(\lambda_k - \lambda_{k-r})}{\lambda_k} x_i \right) \\ &= \sum_{k=0}^{m} a_{nk} x_k,\end{aligned}$$

for all $n, m \in \mathbb{N}$. We have by (5.10) that $By = Ax$ which gives the result $B \in (f : \mu)$.

Conversely, suppose that (5.11) holds for every fixed $k \in \mathbb{N}$ and $B \in (f : \mu)$. Let take any $x = (x_k) \in f(\lambda_r)$. Then Ax exists. Further, we obtain

$$\begin{aligned}\sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m-1} \overline{\Delta}_r \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-r}} \right) \lambda_k y_k + \frac{a_{nm}}{\lambda_m - \lambda_{m-r}} \lambda_m y_m \\ &= \sum_{k=0}^{m} b_{nk} y_k\end{aligned}$$

for all $n, m \in \mathbb{N}$, as $m \to \infty$, that $Ax = By$ and this shows that $A \in (f(\lambda_r) : \mu)$. $\square$

Theorem 5.9. *Let μ be any sequence space and assume that $A = (a_{nk})$ and $E = (e_{nk})$ are the infinite matrices which are connected by the relation*

$$e_{nk} = \sum_{\substack{j=0 \\ r|n-j}}^{n} \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} a_{jk},$$

for all $n, k \in \mathbb{N}$. Then $D \in (\mu : f(\lambda_r))$ if and only if $E \in (\mu : f(\lambda_r))$.

Proof. Let $x = (x_k) \in \mu$ and consider the following equality

$$\sum_{\substack{j=0 \\ r|n-j}}^{n} \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} \sum_{k=0}^{j} a_{jk} x_k = \sum_{k=0}^{j} e_{nk} x_k,$$

for $n \in \mathbb{N}$. Further, by letting $j \to \infty$,

$$\sum_{\substack{j=0 \\ r|n-j}}^{n} \frac{\lambda_j - \lambda_{j-r}}{\lambda_n} \sum_{k=0}^{\infty} a_{jk} x_k = \sum_{k=0}^{\infty} e_{nk} x_k,$$

for $n \in \mathbb{N}$. Then, we have $\{\Lambda^r (Ax)\}_n = (Ex)_n$ for all $n \in \mathbb{N}$. Since $Ax \in f(\lambda_r)$, $Ex \in f$ whenever $x \in \mu$. This completes the proof. □

Corollary 5.10. *The following statements hold:*

(i) $A = (a_{nk}) \in (f(\lambda_r) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(\lambda_r)\}^\beta$ *and (5.1) holds with* $\hat{a}_{nk}$ *instead of* a_{nk}.

(ii) $A = (a_{nk}) \in (f(\lambda_r) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(\lambda_r)\}^\beta$ *and (5.1), (5.2), (5.3), (5.4) hold with* $\hat{a}_{nk}$ *instead of* a_{nk}.

(iii) $A = (a_{nk}) \in (\ell_\infty : f(\lambda_r))$ *if and only if (5.1), (5.5) and (5.6) hold with* e_{nk} *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in (f : f(\lambda_r))$ *if and only if (5.1), (5.5), (5.7) and (5.8) hold with* e_{nk} *instead of* a_{nk}.

(v) $A = (a_{nk}) \in (c : f(\lambda_r))$ *if and only if (5.1), (5.5), (5.7) hold with* e_{nk} *instead of* a_{nk}.

Acknowledgment. The author would like to thank the referees for their valuable comments and suggestions which led to a number of improvements in this paper.

References

1. A. Sönmez, *Almost convergence and triple band matrix*, Math. Comput. Modelling, 57 (2013), pp. 2393-2402.
2. M. Candan, *Almost convergence and double sequential band matrix*, Acta. Math. Sci., 34 (2014), pp. 354-366.
3. M. Kirişci, *Almost convergence and generalized weighted mean II*, J. Inequal. Appl., 1 (2014), pp. 1-13.
4. M. Kirişci, *Almost convergence and generalized weighted mean*, In: AIP Conference Proceedings, AIP (2012), pp. 191-194.
5. M. Şengönül and K. Kayaduman, *On the Riesz almost convergent sequences space*, Abstr. Appl. Anal., 2012 (2012), Article ID 691694, 18 pages.

6. A. Karaisa and F. Özger, *Almost difference sequence space derived by using a generalized weighted mean*, J. Comput. Anal. Appl., 19 (2015), pp. 27-38.
7. F. Başar and R. Çolak, *Almost-conservative matrix transformations*, Turkish J. Math., 13 (1989), pp. 91-100.
8. M. Kirişci, *On the spaces of Euler almost null and Euler almost convergent sequences*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat, 62 (2013), pp. 1-16.
9. Qamaruddin, S.A. Mohuiddine, *Almost convergence and some matrix transformations*, Filomat, 21 (2007), pp. 261-266.
10. M. Candan and K. Kayaduman, *Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core*, Brithish J. Math. Comput. Sci., 7 (2015), pp. 150-167.
11. F. Başar and M. Kirişçi, *Almost convergence and generalized difference matrix*, Comput. Math. Appl., 61 (2011), pp. 602-611.
12. M. Mursaleen and A. K. Noman, *On the spaces of $\lambda-$convergent sequences and bounded sequences*, Thai J. Math, 8 (2010), pp. 311-329.
13. P.N. Ng and P.Y. Lee, *Cesàro sequence spaces of non-absolute type*, Comment. Math. Prace Mat., 20 (1978), pp. 429-433.
14. M. Şengönül and F. Başar, *Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c*, Soochow J. Math., 31 (2005), pp. 107-119.
15. M. Yeşilkayagil and F. Başar, *Space of A_λ-almost null and A_λ-almost convergent sequences*, J. Egypt. Math. Soc., 23 (2015), pp. 119-126.
16. A. Wilansky, *Summability Through Functional Analysis*, in: North-Holland Mathematics Studies, Elsevier Science Publishers, Amsterdam, New York, 1984.
17. G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., 80 (1948), pp. 167-190.
18. A. M. Jarrah and E. Malkowsky, *BK spaces, bases and linear operators*, Ren. Circ. Mat. Palermo II, 52 (1990), pp. 177-191.
19. H.I. Miller and C. Orhan, *On almost convergent and statistically convergent subsequences*, Acta Math. Hungar., 93 (2001), pp. 135-151.
20. G.M. Petersen, *Regular Matrix Transformations*, McGraw-Hill, New York-Toronto-Sydney, 1970.
21. J.A. Sıddıqi, *Infinite matrices summing every almost periodic sequences*, Pac. J. Math., 39 (1971), pp. 235-251.
22. J.P. Duran, *Infinite matrices and almost convergence*, Math. Z., 128 (1972), pp. 75-83.

23. J.P. King, *Almost summable sequences*, Proc. Am. Math. Soc., 17 (1966), pp. 1219-1225.
24. F. Başar and İ. Solak, *Almost-coercive matrix transformations*, Rend. Mat. Appl., 11 (1991), pp. 249-256.
25. F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul, 2012.
26. S. Nanda, *Infinite matrices and almost convergence*, J. Indian Math. Soc., 40 (1976), pp. 173-184.
27. P. Kórus, *On Λ^r-strong convergence of numerical sequences and Fourier series*, J. Class. Anal., 9 (2016), pp. 89-98.
28. S. Ercan, *On λ_r-Convergence and λ_r-Boundedness*, Journal of Advanced Physics, 7 (2018), pp. 123-129.

Department of Mathematics, Faculty of Science, Firat University, 23119, Elazig, Turkey.

E-mail address: sinanercan45gmail.com