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Almost Multi-Cubic Mappings and a Fixed Point Application

Nasrin Ebrahimi Hoseinzadeh¹, Abasalt Bodaghi^{2*} and Mohammad Reza Mardanbeigi³

ABSTRACT. The aim of this paper is to introduce n -variables mappings which are cubic in each variable and to apply a fixed point theorem for the Hyers-Ulam stability of such mapping in non-Archimedean normed spaces. Moreover, a few corollaries corresponding to some known stability and hyperstability outcomes are presented.

1. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [16] for the Cauchy difference. Later, the result of Hyers was significantly generalized by Aoki [1], Th. M. Rassias [24] (stability incorporated with sum of powers of norms), Găvruta [14] (stability controlled by a general control function) and J. M. Rassias [23] (stability including mixed product-sum of powers of norms).

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [12] that a mapping $f : V^n \rightarrow W$ is called multi-additive if it is additive (satisfies the Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [20] and many other sources. In addition, f is said to be multi-quadratic if it is quadratic (satisfies the quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$) in each variable [13].

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In [30], Zhao et al. proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if it satisfies the equality

$$(1.1) \quad \sum_{t \in \{-1, 1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1, j_2, \dots, j_n \in \{1, 2\}} f(x_{1j_1}, x_{2j_2}, \dots, x_{nj_n})$$

where $x_j = (x_{1j}, x_{2j}, \dots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. In [12] and [13], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [30]). The Jensen type of multi-quadratic mappings and their characterization can be found in [25].

The cubic functional equation has been introduced by J. M. Rassias in [22] as follows:

$$(1.2) \quad f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) = 6f(y).$$

He obtained the general solutions of (1.2) and studied the Hyers-Ulam stability problem for these cubic functional equation. The following alternative cubic functional equation

$$(1.3) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$

has been presented by Jun and Kim in [17]. They found out the general solutions and established the Hyers-Ulam stability for the functional equation (1.3). Furthermore, they considered the cubic functional equation

$$(1.4) \quad f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y)$$

which somewhat different from (1.3) and proved the Hyers-Ulam stability problem for it in Banach spaces setting [18]. Next, the second author in [5] showed that the functional equation

$$(1.5) \quad \begin{aligned} f(rx + sy) + f(rx - sy) \\ = rs^2 [f(x + y) + f(x - y)] + 2r(r^2 - s^2) f(x) \end{aligned}$$

can be a generalization of the equations (1.3) and (1.4) where r, s are integer numbers with $r \pm s \neq 0$; for other forms of the cubic functional equations and their stabilities on the various Banach spaces refer to [3], [4], [6], [7] and [29]. Recently, in [9], the second author and Shojaei introduced the multi-cubic mappings (unified as a equation) and studied the Hyers-Ulam stability for multi-cubic mappings on normed spaces by a fixed point theorem and moreover proved that a multi-cubic functional equation can be hyperstable; see also [21] for more forms of multi-cubic mappings and their stabilities on normed spaces. Besides, for the characterization and stability of multi-quartic mappings refer to [8].

In this paper, by using the functional equation (1.5), we define new multi-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-cubic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for multi-cubic functional equations by applying the fixed point method in non-Archimedean normed spaces which is introduced in [10]; for more applications of this approach for the stability of multi-Cauchy-Jensen and multi-additive-quadratic mappings see [2]. In addition, for the stability of multi-Jensen and multi-additive mappings in non-Archimedean spaces refer to [27] and [28], respectively.

2. CHARACTERIZATION OF MULTI-CUBIC MAPPINGS

Throughout this paper, \mathbb{N} stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $n \in \mathbb{N}$. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $q = (q_1, \dots, q_n) \in \{-1, 1\}^n$ and $x = (x_1, \dots, x_n) \in V^n$ we write $lx := (lx_1, \dots, lx_n)$ and $qx := (q_1x_1, \dots, q_nx_n)$, where lx stands, as usual, for the scalar product of an element l on x in the vector space V .

From now on, let V and W be vector spaces over the rationals, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \dots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. Let $x_1, x_2 \in V^n$ and $T \in \mathbb{N}_0$ with $0 \leq T \leq n$. Put $\mathcal{M} = \{\mathfrak{N}_n = (N_1, N_2, \dots, N_n) \mid N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, where $j \in \{1, \dots, n\}$. Consider

$$\mathcal{M}_T^n := \{\mathfrak{N}_n = (N_1, N_2, \dots, N_n) \in \mathcal{M} \mid \text{Card}\{N_j : N_j = x_{1j}\} = T\}.$$

We say the mapping $f : V^n \rightarrow W$ is n -multi-cubic or multi-cubic if f is cubic in each variable (see equation (1.5)). For the multi-cubic mappings, we use the following notations:

$$(2.1) \quad \begin{aligned} f(\mathcal{M}_T^n) &:= \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n), \\ f(\mathcal{M}_T^n, z) &:= \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n, z), \quad (z \in V). \end{aligned}$$

Let r be the fixed integer in (1.5) such that $r \neq \pm 1, 0$. We say the mapping $f : V^n \rightarrow W$ satisfies the m -power condition in the j th variable if

$$f(z_1, \dots, z_{j-1}, rz_j, z_{j+1}, \dots, z_n) = r^m f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$$

for all $z_1, \dots, z_n \in V^n$.

Remark 2.1 ([21]). It is easily verified that if f is a multi-cubic mapping, then it satisfies 3-power condition in each of variable. Note that the converse is not true. Here, by means of an example we show that

3-power condition in all variables for a mapping f does not imply being multi-cubic. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in \mathcal{A} (not necessarily unit). Define the mapping $h : \mathcal{A}^n \rightarrow \mathcal{A}$ by $h(a_1, \dots, a_n) = \prod_{j=1}^n \|a_j\|^3 a_0$ for $(a_1, \dots, a_n) \in \mathcal{A}^n$. It is easy to check that the mapping h satisfies 3-power condition in all variables but not multi-cubic even for $n = 1$, which means that h does not satisfy in equation (1.5).

In what follows, $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$.

Theorem 2.2. *Suppose that the mapping $f : V^n \rightarrow W$ is multi-cubic. Then, f satisfies the equation*

$$(2.2) \quad \sum_{q \in \{-1, 1\}^n} f(rx_1 + qsx_2) = \sum_{k=0}^n (rs^2)^{n-k} [2r(r^2 - s^2)]^k f(\mathcal{M}_k^n)$$

where r, s are integer numbers with $r \pm s \neq 0$. The converse is true provided that f has 3-power condition in all variables.

Proof. (Necessity) We prove that f satisfies the equation (2.2) by induction on n . For $n = 1$, it is trivial that f satisfies the equation (1.5). If (2.2) is valid for some positive integer $n > 1$, then

$$\begin{aligned} & \sum_{q \in \{-1, 1\}^{n+1}} f(rx_1^{n+1} + qsx_2^{n+1}) \\ &= rs^2 \sum_{q \in \{-1, 1\}^n} f(rx_1^n + qsx_2^n, x_{1n+1} + x_{2n+1}) \\ & \quad + rs^2 \sum_{q \in \{-1, 1\}^n} f(rx_1^n + qsx_2^n, x_{1n+1} - x_{2n+1}) \\ & \quad + 2r(r^2 - s^2) \sum_{q \in \{-1, 1\}^n} f(rx_1^n + qsx_2^n, x_{1n+1}) \\ &= rs^2 \sum_{k=0}^n \sum_{q \in \{-1, 1\}} (rs^2)^{n-k} [2r(r^2 - s^2)]^k f(\mathcal{M}_k^n, x_{1n+1} + qx_{2n+1}) \\ & \quad + 2r(r^2 - s^2) \sum_{k=0}^n (rs^2)^{n-k} [2r(r^2 - s^2)]^k f(\mathcal{M}_k^n, x_{1n+1}) \\ &= \sum_{k=0}^{n+1} (rs^2)^{n+1-k} [2r(r^2 - s^2)]^k f(\mathcal{M}_k^{n+1}). \end{aligned}$$

This means that (2.2) holds for $n + 1$.

(Sufficiency) Assume that f satisfies equation (2.2). Fix $j \in \{1, \dots, n\}$. Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\} \setminus \{j\}$ in the left side of (2.2) and

using the assumption, we get

$$(2.3) \quad \begin{aligned} & 2^{n-1} \times r^{3(n-1)} [f(x_{11}, \dots, x_{1j-1}, rx_{1j} + sx_{2j}, x_{1j+1}, \dots, x_{1n}) \\ & \quad + f(x_{11}, \dots, x_{1j-1}, rx_{1j} - sx_{2j}, x_{1j+1}, \dots, x_{1n})] \\ & = 2^{n-1} [f(rx_{11}, \dots, rx_{1j-1}, rx_{1j} + sx_{2j}, rx_{1j+1}, \dots, rx_{1n}) \\ & \quad + f(rx_{11}, \dots, rx_{1j-1}, rx_{1j} - sx_{2j}, rx_{1j+1}, \dots, rx_{1n})]. \end{aligned}$$

Set

$$f^*(x_{1j}, x_{2j}) := f(x_{11}, \dots, x_{1j-1}, x_{1j} + x_{2j}, x_{1j+1}, \dots, x_{1n}) \\ + f(x_{11}, \dots, x_{1j-1}, x_{1j} - x_{2j}, x_{1j+1}, \dots, x_{1n}).$$

By the above replacements in equation (2.2), relation (2.3) implies that

(2.4)

$$\begin{aligned} & 2^{n-1} r^{3(n-1)} [f(x_{11}, \dots, x_{1j-1}, rx_{1j} + sx_{2j}, x_{1j+1}, \dots, x_{1n}) \\ & \quad + f(x_{11}, \dots, x_{1j-1}, rx_{1j} - sx_{2j}, x_{1j+1}, \dots, x_{1n})] \\ & = 2^{n-1} (rs^2)^n f^*(x_{1j}, x_{2j}) \\ & \quad + \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} 2^{n-k} (rs^2)^{n-k} (2r(r^2 - s^2))^k \right] f(x_{11}, \dots, x_{1n}) \\ & \quad + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} 2^{n-k-1} (rs^2)^{n-k} (2r(r^2 - s^2))^k \right] f^*(x_{1j}, x_{2j}) \\ & \quad + (2r(r^2 - s^2))^n f(x_{11}, \dots, x_{1n}) \\ & = A_{r,s} f^*(x_{1j}, x_{2j}) + B_{r,s} f(x_{11}, \dots, x_{1n}), \end{aligned}$$

where

$$A_{r,s} = 2^{n-1} (rs^2)^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{n-k-1} (rs^2)^{n-k} (2r(r^2 - s^2))^k$$

and

$$B_{r,s} = (2r(r^2 - s^2))^n + \sum_{k=1}^{n-1} \binom{n-1}{k} 2^{n-k} (rs^2)^{n-k} (2r(r^2 - s^2))^k.$$

On the other hand, we have

(2.5)

$$\begin{aligned} A_{r,s} & = rs^2 \left[(2rs^2)^{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k} (2rs^2)^{n-k-1} (2r(r^2 - s^2))^k \right] \\ & = rs^2 \sum_{k=0}^{n-1} \binom{n-1}{k} (2rs^2)^{n-k-1} (2r(r^2 - s^2))^k \end{aligned}$$

$$\begin{aligned}
&= rs^2 (2rs^2 + 2r^3 - 2rs^2)^{n-1} \\
&= 2^{n-1} r^{3(n-1)} r s^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
(2.6) \quad B_{r,s} &= (2r(r^2 - s^2))^n \\
&\quad + \sum_{k=0}^{n-2} \binom{n-1}{k} (2rs^2)^{n-k-1} (2r(r^2 - s^2))^{k+1} \\
&= 2r(r^2 - s^2) [(2r(r^2 - s^2))^{n-1} \\
&\quad + \sum_{k=0}^{n-2} \binom{n-1}{k} (2rs^2)^{n-k-1} (2r(r^2 - s^2))^k] \\
&= 2r(r^2 - s^2) \sum_{k=0}^{n-1} \binom{n-1}{k-1} (2rs^2)^{n-k-1} (2r(r^2 - s^2))^k \\
&= 2r(r^2 - s^2) (2rs^2 + 2r^3 - 2rs^2)^{n-1} \\
&= 2^{n-1} r^{3(n-1)} 2r(r^2 - s^2).
\end{aligned}$$

It follows from relations (2.4), (2.6) and (2.5) that

$$\begin{aligned}
&f(x_{11}, \dots, x_{1j-1}, rx_{1j} + sx_{2j}, x_{1j+1}, \dots, x_{1n}) \\
&\quad + f(x_{11}, \dots, x_{1j-1}, rx_{1j} - sx_{2j}, x_{1j+1}, \dots, x_{1n}) \\
&= rs^2 f^*(x_{1j}, x_{2j}) + 2r(r^2 - s^2) f(x_{11}, \dots, x_{1n}).
\end{aligned}$$

This means that f is cubic in the j th variable. Since j is arbitrary, we obtain the desired result. \square

3. STABILITY RESULTS FOR (2.2)

We firstly express some basic facts concerning non-Archimedean spaces and some preliminary results. Let us recall that a metric d on a nonempty set X is said to be non-Archimedean (or an ultrametric) provided

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for $x, y, z \in X$. By a non-Archimedean field we mean a field \mathbb{K} equipped with a function (valuation) $|\cdot|$ from \mathbb{K} into $[0, \infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$, and $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let \mathcal{X} be a vector space over a scalar field \mathbb{K} with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|ax\| = |a|\|x\|$, $(x \in \mathcal{X}, a \in \mathbb{K})$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max \{\|x\|, \|y\|\}, \quad (x, y \in \mathcal{X}).$$

Then, $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max \{\|x_{j+1} - x_j\|; m \leq j \leq n - 1\}, \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space \mathcal{X} . By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent. If $(\mathcal{X}, \|\cdot\|)$ is a non-Archimedean normed space, then it is easily verified that the function $d_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$, given by $d_{\mathcal{X}}(x, y) := \|x - y\|$, is a non-Archimedean metric on \mathcal{X} that is invariant (i.e., $d_{\mathcal{X}}(x + z, y + z) = d_{\mathcal{X}}(x, y)$ for $x, y, z \in \mathcal{X}$). Hence, non-Archimedean normed spaces are also special cases of metric spaces with invariant metrics.

The most important examples of non-Archimedean normed spaces are the p -adic numbers, which have gained the interest of physicists because of their connections with some problems coming from quantum physics, p -adic strings and superstrings [19]. Indeed, Hensel [15] discovered the p -adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p -adic numbers. A key property of p -adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

We recall that for a field \mathbb{K} with multiplicative identity 1, the characteristic of \mathbb{K} is the smallest positive number n such that $\overbrace{1 + \dots + 1}^{n\text{-times}} = 0$.

Throughout, for two sets A and B , the set of all mappings from A to B is denoted by B^A . In this section, we prove the generalized Hyers-Ulam stability of equation (2.2) in non-Archimedean spaces. The proof is based on a fixed point result that can be derived from [10, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \dots, g_j : E \rightarrow E$ and $L_1, \dots, L_j : E \rightarrow \mathbb{R}_+$,

(H2) $\mathcal{T} : Y^E \rightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{i \in \{1, \dots, j\}} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|,$$

for all $\lambda, \mu \in Y^E, x \in E$,

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i \in \{1, \dots, j\}} L_i(x)\delta(g_i(x)), \quad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [10, Theorem 1]. This result plays a key tool to obtain our goal in this paper.

Theorem 3.1. *Let hypotheses (H1)-(H3) hold and the function $\theta : E \longrightarrow \mathbb{R}_+$ and the mapping $\varphi : E \longrightarrow Y$ fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \quad \lim_{l \rightarrow \infty} \Lambda^l \theta(x) = 0, \quad (x \in E).$$

Then, for every $x \in E$, the limit $\lim_{l \rightarrow \infty} \mathcal{T}^l \varphi(x) =: \psi(x)$ and the mapping $\psi \in Y^E$, defined in this way, is a fixed point of \mathcal{T} with

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l \in \mathbb{N}_0} \Lambda^l \theta(x), \quad (x \in E).$$

In the sequel, given the mapping $f : V^n \longrightarrow W$, we delineate the difference operator $\mathfrak{D}_c f : V^n \times V^n \longrightarrow W$ by

$$\begin{aligned} \mathfrak{D}_c f(x_1, x_2) &= \sum_{q \in \{-1, 1\}^n} f(rx_1 + qsx_2) \\ &\quad - \sum_{k=0}^n (rs^2)^{n-k} [2r(r^2 - s^2)]^k f(\mathcal{M}_k^n) \end{aligned}$$

where $f(\mathcal{M}_k^n)$ is defined in (2.1).

Here and subsequently, without loss of generality we assume that r is a fixed positive integer such that $r \geq 2$. We now are ready to indicate the upcoming result which is the main result in this paper.

Theorem 3.2. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r . Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a function satisfying the equality*

$$(3.1) \quad \lim_{l \rightarrow \infty} \left(\frac{1}{|r|^{3n\beta}} \right)^l \varphi(r^{l\beta} x_1, r^{l\beta} x_2) = 0$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$(3.2) \quad \|\mathfrak{D}_c f(x_1, x_2)\| \leq \varphi(x_1, x_2)$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$(3.3) \quad \|f(x) - \mathcal{C}(x)\| \leq \sup_{l \in \mathbb{N}_0} \frac{1}{|2|^n \times |r|^{3n\frac{\beta+1}{2}}} \left(\frac{1}{|r|^{3n\beta}} \right)^l \varphi \left(r^{l\beta + \frac{\beta-1}{2}}, 0 \right)$$

for all $x \in V^n$.

Proof. Putting $x = x_1$ and $x_2 = 0$ in (3.2), we have

$$(3.4) \quad \left\| 2^n f(rx) - \left(\sum_{k=0}^n \binom{n}{k} 2^{n-k} (rs^2)^{n-k} (2r(r^2 - s^2))^k \right) f(x) \right\| \leq \varphi(x, 0)$$

for all $x \in V^n$. By an easy computation, we have

$$(3.5) \quad \begin{aligned} \sum_{k=0}^n \binom{n}{k} 2^{n-k} (rs^2)^{n-k} (2r(r^2 - s^2))^k &= (2rs^2 + 2r^3 - 2rs^2)^n \\ &= (2r^3)^n. \end{aligned}$$

It follows from (3.4) and (3.5) that

$$(3.6) \quad \|f(rx) - r^{3n}f(x)\| \leq \frac{1}{|2|^n} \varphi(x, 0)$$

for all $x \in V^n$. The inequality (3.4) implies that

$$(3.7) \quad \|f(x) - \mathcal{T}f(x)\| \leq \theta(x)$$

for all $x \in V^n$, where

$$\theta(x) := \frac{1}{|2|^n \times |r|^{3n \frac{\beta+1}{2}}} \varphi\left(r^{\frac{\beta-1}{2}}x, 0\right), \quad \mathcal{T}\xi(x) := \frac{1}{r^{3n\beta}} \xi(r^\beta x)$$

for all $\xi \in W^{V^n}$ and $x \in V^n$. Define $\Lambda\eta(x) := \frac{1}{|r|^{3n\beta}} \eta(r^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$. It is easy to see that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := r^\beta x$ for all $x \in V^n$ and $L_1(x) = \frac{1}{|r|^{3n\beta}}$. Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\begin{aligned} \|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\| \frac{1}{r^{3n\beta}} \lambda(r^\beta x) - \frac{1}{r^{3n\beta}} \mu(r^\beta x) \right\| \\ &\leq L_1(x) \|\lambda(g_1(x)) - \mu(g_1(x))\|. \end{aligned}$$

The above relation shows that the hypothesis (H2) is valid. By induction on l , one can check that for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$(3.8) \quad \begin{aligned} \Lambda^l \theta(x) &:= \left(\frac{1}{|r|^{3n\beta}} \right)^l \theta(r^{l\beta} x) \\ &= \frac{1}{|2|^n \times |r|^{3n \frac{\beta+1}{2}}} \left(\frac{1}{|r|^{3n\beta}} \right)^l \varphi\left(r^{l\beta + \frac{\beta-1}{2}} x, 0\right) \end{aligned}$$

for all $x \in V^n$. The relations (3.7) and (3.8) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping

$\mathcal{C} : V^n \longrightarrow W$ such that $\mathcal{C}(x) = \lim_{l \rightarrow \infty} (\mathcal{T}^l f)(x)$ for all $x \in V^n$, and also (3.3) holds. We also can verified by induction on l that

$$(3.9) \quad \left\| \mathfrak{D}_c \left(\mathcal{T}^l f \right) (x_1, x_2) \right\| \leq \left(\frac{1}{|r|^{3n\beta}} \right)^l \varphi \left(r^{l\beta} x_1, r^{l\beta} x_2 \right)$$

for all $x_1, x_2 \in V^n$. Letting $l \rightarrow \infty$ in (3.9) and applying (3.1), we arrive at $\mathfrak{D}_c \mathcal{C}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies equation (2.2) and the proof is now completed. \square

Remark 3.3. We note that in Theorem 3.2, it is assumed that the non-Archimedean field has the characteristic different from r and so the conditions of Theorem 3.1 are valid because $|r| < |2| < 1$.

The following corollaries are some direct applications of Theorem 3.2 concerning the stability of (2.2).

Corollary 3.4. *Let $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r and $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| \mathfrak{D}_c f (x_1, x_2) \right\| \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{1}{|2|^n} \delta$$

for all $x \in V^n$.

Proof. We firstly note that $|r| < 1$. Letting $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 3.2, we have $\lim_{l \rightarrow \infty} \left(\frac{1}{|r|^{-3n}} \right)^l \delta = 0$. Therefore, one can obtain the desired result. \square

Corollary 3.5. *Let $p \in \mathbb{R}$ fulfills $p \neq 3n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r and $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| \mathfrak{D}_c f (x_1, x_2) \right\| \leq \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|^p$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \frac{1}{|2|^n \times |r|^{3n}} \sum_{j=1}^n \|x_{1j}\|^p & p > 3n \\ \frac{1}{|2|^n \times |r|^p} \sum_{j=1}^n \|x_{1j}\|^p & p < 3n \end{cases}$$

for all $x = x_1 \in V^n$.

Proof. Putting $\varphi(x_1, x_2) = \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|^p$, we have $\varphi(r^l x_1, r^l x_2) = |r|^{lp} \varphi(x_1, x_2)$. It now follows from Theorem 3.2 the first and second inequalities in the cases $\beta = 1$ and $\beta = -1$, respectively. \square

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^A . We say that operator equation

$$(3.10) \quad \mathcal{F}_1 \varphi(a_1, \dots, a_n) = \mathcal{F}_2 \varphi(a_1, \dots, a_n)$$

is ψ -hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1 \varphi_0(a_1, \dots, a_n), \mathcal{F}_2 \varphi_0(a_1, \dots, a_n)) \leq \psi(a_1, \dots, a_n)$$

for all $a_1, \dots, a_n \in A$, fulfils (3.10); this definition is introduced in [11]. In other words, a functional equation \mathcal{F} is hyperstable if any mapping f satisfying the equation \mathcal{F} approximately is a true solution of \mathcal{F} . Under some conditions the functional equation (2.2) can be hyperstable as follows.

Corollary 3.6. *Suppose that $p_{kj} > 0$ for $k \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{k=1}^2 \sum_{j=1}^n p_{kj} \neq 3n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r and $|2| < 1$. If $f : V^n \rightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}_c f(x_1, x_2)\| \leq \prod_{k=1}^2 \prod_{j=1}^n \|x_{kj}\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, then f is multi-cubic.

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