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Almost Multi-Cubic Mappings and a Fixed Point Application

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Abstract. The aim of this paper is to introduce n-variables mappings which are cubic in each variable and to apply a fixed point theorem for the Hyers-Ulam stability of such mapping in non-Archimedean normed spaces. Moreover, a few corollaries corresponding to some known stability and hyperstability outcomes are presented.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [16] for the Cauchy difference. Later, the result of Hyers was significantly generalized by Aoki [1], Th. M. Rassias [24] (stability incorporated with sum of powers of norms), Găvruta [14] (stability controlled by a general control function) and J. M. Rassias [23] (stability including mixed product-sum of powers of norms).

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [12] that a mapping $f : V^n \longrightarrow W$ is called multi-additive if it is additive (satisfies the Cauchy's functional equation $A(x+y) = A(x)+A(y)$) in each variable. Some facts on such mappings can be found in [20] and many other sources. In addition, f is said to be multi-quadratic if it is quadratic (satisfies the quadratic functional equation $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$) in each variable [13].

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In [30], Zhao et al. proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if it satisfies the equality

$$\sum_{t\in\{-1,1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1,j_2,\dots,j_n\in\{1,2\}} f(x_{1j_1}, x_{2j_2}, \dots, x_{nj_n}) \tag{1.1}$$

where $x_j = (x_{1j}, x_{2j}, \dots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. In [12] and [13], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [30]). The Jensen type of multi-quadratic mappings and their characterization can be found in [25].

The cubic functional equation has been introduced by J. M. Rassias in [22] as follows:

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) = 6f(y). \tag{1.2}$$

He obtained the general solutions of (1.2) and studied the Hyers-Ulam stability problem for these cubic functional equation. The following alternative cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.3}$$

has been presented by Jun and Kim in [17]. They found out the general solutions and established the Hyers-Ulam stability for the functional equation (1.3). Furthermore, they considered the cubic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) \tag{1.4}$$

which somewhat different from (1.3) and proved the Hyers-Ulam stability problem for it in Banach spaces setting [18]. Next, the second author in [5] showed that the functional equation

$$\begin{aligned} f(rx + sy) &+ f(rx - sy) \\ &= rs^2 [f(x + y) + f(x - y)] + 2r\left(r^2 - s^2\right) f(x) \end{aligned} \tag{1.5}$$

can be a generalization of the equations (1.3) and (1.4) where r, s are integer numbers with $r \pm s \neq 0$; for other forms of the cubic functional equations and their stabilities on the various Banach spaces refer to [3], [4], [6], [7] and [29]. Recently, in [9], the second author and Shojaee introduced the multi-cubic mappings (unified as a equation) and studied the Hyers-Ulam stability for multi-cubic mappings on normed spaces by a fixed point theorem and moreover proved that a multi-cubic functional equation can be hyperstable; see also [21] for more forms of multi-cubic mappings and their stabilities on normed spaces. Besides, for the characterization and stability of multi-quartic mappings refer to [8].

In this paper, by using the functional equation (1.5), we define new multi-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-cubic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for multi-cubic functional equations by applying the fixed point method in non-Archimedean normed spaces which is introduced in [10]; for more applications of this approach for the stability of multi-Cauchy-Jensen and multi-additive-quadratic mappings see [2]. In addition, for the stability of multi-Jensen and multi-additive mappings in non-Archimedean spaces refer to [27] and [28], respectively.

2. Characterization of Multi-Cubic Mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty), n \in \mathbb{N}$. For any $l \in \mathbb{N}_0, n \in \mathbb{N}$, $q = (q_1, \ldots, q_n) \in \{-1, 1\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$ we write $lx := (lx_1, \ldots, lx_n)$ and $qx := (q_1x_1, \ldots, q_nx_n)$, where lx stands, as usual, for the scaler product of an element l on x in the vector space V.

From now on, let V and W be vector spaces over the rationals, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. Let $x_1, x_2 \in V^n$ and $T \in \mathbb{N}_0$ with $0 \leq T \leq n$. Put $\mathcal{M} = \{\mathfrak{N}_n = (N_1, N_2, \ldots, N_n) \,|\, N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, where $j \in \{1, \ldots, n\}$. Consider

$$\mathcal{M}_T^n := \{\mathfrak{N}_n = (N_1, N_2, \ldots, N_n) \in \mathcal{M} |\ \mathrm{Card}\{N_j :\ N_j = x_{1j}\} = T\}.$$

We say the mapping $f : V^n \longrightarrow W$ is n-multi-cubic or multi-cubic if f is cubic in each variable (see equation (1.5)). For the multi-cubic mappings, we use the following notations:

$$f\left(\mathcal{M}_T^n\right) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f\left(\mathfrak{N}_n\right), \tag{2.1}$$

$$f\left(\mathcal{M}_T^n, z\right) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f\left(\mathfrak{N}_n, z\right), \quad (z \in V).$$

Let r be the fixed integer in (1.5) such that $r \neq \pm 1, 0$. We say the mapping $f : V^n \longrightarrow W$ satisfies the m-power condition in the jth variable if

$$f\left(z_1, \ldots, z_{j-1}, rz_j, z_{j+1}, \ldots, z_n\right) = r^m f\left(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n\right)$$

for all $z_1, \ldots, z_n \in V^n$.

Remark 2.1 ([21])**.** It is easily verified that if f is a multi-cubic mapping, then it satisfies 3-power condition in each of variable. Note that the converse is not true. Here, by means of an example we show that

3-power condition in all variables for a mapping f does not imply being multi-cubic. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathcal{A}$ (not necessarily unit). Define the mapping $h : \mathcal{A}^n \longrightarrow \mathcal{A}$ by $h(a_1, \ldots, a_n) = \prod_{j=1}^{n} \|a_j\|^3 a_0$ for $(a_1, \ldots, a_n) \in \mathcal{A}^n$. It is easy to check that the mapping h satisfies 3-power condition in all variables but not multi-cubic even for $n = 1$, which means that is h does not satisfy in equation (1.5).

In what follows, $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$.

Theorem 2.2. *Suppose that the mapping $f : V^n \longrightarrow W$ is multi-cubic. Then, f satisfies the equation*

$$\sum_{q\in\{-1,1\}^n} f(rx_1 + qsx_2) = \sum_{k=0}^{n} \left(rs^2\right)^{n-k} \left[2r\left(r^2 - s^2\right)\right]^k f\left(\mathcal{M}_k^n\right) \tag{2.2}$$

where r, s are integer numbers with $r \pm s \neq 0$. The converse is true provided that f has 3-power condition in all variables.

Proof. (Necessity) We prove that f satisfies the equation (2.2) by induction on n. For $n = 1$, it is trivial that f satisfies the equation (1.5). If (2.2) is valid for some positive integer $n > 1$, then

$$\begin{aligned}
&\sum_{q\in\{-1,1\}^{n+1}} f\left(rx_1^{n+1} + qsx_2^{n+1}\right)\\
&= rs^2 \sum_{q\in\{-1,1\}^n} f\left(rx_1^n + qsx_2^n, x_{1n+1} + x_{2n+1}\right)\\
&\quad + rs^2 \sum_{q\in\{-1,1\}^n} f\left(rx_1^n + qsx_2^n, x_{1n+1} - x_{2n+1}\right)\\
&\quad + 2r\left(r^2 - s^2\right) \sum_{q\in\{-1,1\}^n} f\left(rx_1^n + qsx_2^n, x_{1n+1}\right)\\
&= rs^2 \sum_{k=0}^{n} \sum_{q\in\{-1,1\}} \left(rs^2\right)^{n-k} \left[2r\left(r^2 - s^2\right)\right]^k f\left(\mathcal{M}_k^n, x_{1n+1} + qx_{2n+1}\right)\\
&\quad + 2r\left(r^2 - s^2\right) \sum_{k=0}^{n} \left(rs^2\right)^{n-k} \left[2r(r^2 - s^2)\right]^k f\left(\mathcal{M}_k^n, x_{1n+1}\right)\\
&= \sum_{k=0}^{n+1} \left(rs^2\right)^{n+1-k} \left[2r\left(r^2 - s^2\right)\right]^k f\left(\mathcal{M}_k^{n+1}\right).
\end{aligned}$$

This means that (2.2) holds for $n + 1$.

(Sufficiency) Assume that f satisfies equation (2.2). Fix $j \in \{1, \ldots, n\}$. Putting $x_{2k} = 0$ for all $k \in \{1, \ldots, n\} \backslash \{j\}$ in the left side of (2.2) and using the assumption, we get

$$
\begin{aligned}
(2.3)\qquad 2^{n-1} \times r^{3(n-1)}&[f(x_{11},\ldots,x_{1j-1},rx_{1j}+sx_{2j},x_{1j+1},\ldots,x_{1n})\\
&+f(x_{11},\ldots,x_{1j-1},rx_{1j}-sx_{2j},x_{1j+1},\ldots,x_{1n})]\\
&=2^{n-1}[f(rx_{11},\ldots,rx_{1j-1},rx_{1j}+sx_{2j},rx_{1j+1},\ldots,rx_{1n})\\
&+f(rx_{11},\ldots,rx_{1j-1},rx_{1j}-sx_{2j},rx_{1j+1},\ldots,rx_{1n})].
\end{aligned}
$$

Set

$$
\begin{aligned}
f^*(x_{1j},x_{2j}) :&= f(x_{11},\ldots,x_{1j-1},x_{1j}+x_{2j},x_{1j+1},\ldots,x_{1n})\\
&+f(x_{11},\ldots,x_{1j-1},x_{1j}-x_{2j},x_{1j+1},\ldots,x_{1n}).
\end{aligned}
$$

By the above replacements in equation (2.2), relation (2.3) implies that
(2.4)

$$
\begin{aligned}
2^{n-1}r^{3(n-1)}&[f(x_{11},\ldots,x_{1j-1},rx_{1j}+sx_{2j},x_{1j+1},\ldots,x_{1n})\\
&+f(x_{11},\ldots,x_{1j-1},rx_{1j}-sx_{2j},x_{1j+1},\ldots,x_{1n})]\\
&=2^{n-1}\left(rs^2\right)^n f^*(x_{1j},x_{2j})\\
&+\sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}2^{n-k}(rs^2)^{n-k}(2r(r^2-s^2))^k\right]f(x_{11},\ldots,x_{1n})\\
&+\sum_{k=1}^{n-1}\left[\binom{n-1}{k}2^{n-k-1}(rs^2)^{n-k}(2r(r^2-s^2))^k\right]f^*(x_{1j},x_{2j})\\
&+\left(2r\left(r^2-s^2\right)\right)^n f(x_{11},\ldots,x_{1n})\\
&=A_{r,s}f^*(x_{1j},x_{2j})+B_{r,s}f(x_{11},\ldots,x_{1n}),
\end{aligned}
$$

where

$$
A_{r,s}=2^{n-1}\left(rs^2\right)^n+\sum_{k=1}^{n-1}\binom{n-1}{k-1}2^{n-k-1}\left(rs^2\right)^{n-k}\left(2r\left(r^2-s^2\right)\right)^k
$$

and

$$
B_{r,s}=\left(2r\left(r^2-s^2\right)\right)^n+\sum_{k=1}^{n-1}\binom{n-1}{k}2^{n-k}\left(rs^2\right)^{n-k}\left(2r\left(r^2-s^2\right)\right)^k.
$$

On the other hand, we have
(2.5)

$$
\begin{aligned}
A_{r,s}&=rs^2\left[(2rs^2)^{n-1}+\sum_{k=1}^{n-1}\binom{n-1}{k}\left(2rs^2\right)^{n-k-1}\left(2r\left(r^2-s^2\right)\right)^k\right]\\
&=rs^2\sum_{k=0}^{n-1}\binom{n-1}{k}\left(2rs^2\right)^{n-k-1}\left(2r\left(r^2-s^2\right)\right)^k
\end{aligned}
$$

$$= rs^2\left(2rs^2+2r^3-2rs^2\right)^{n-1}$$
$$= 2^{n-1}r^{3(n-1)}rs^2.$$

In addition,

$$B_{r,s} = \left(2r\left(r^2-s^2\right)\right)^n + \sum_{k=0}^{n-2}\binom{n-1}{k}\left(2rs^2\right)^{n-k-1}\left(2r\left(r^2-s^2\right)\right)^{k+1}$$
$$= 2r\left(r^2-s^2\right)\left[\left(2r\left(r^2-s^2\right)\right)^{n-1} + \sum_{k=0}^{n-2}\binom{n-1}{k}(2rs^2)^{n-k-1}\left(2r\left(r^2-s^2\right)\right)^{k}\right]$$
$$= 2r\left(r^2-s^2\right)\sum_{k=0}^{n-1}\binom{n-1}{k-1}\left(2rs^2\right)^{n-k-1}\left(2r\left(r^2-s^2\right)\right)^{k}$$
$$= 2r\left(r^2-s^2\right)\left(2rs^2+2r^3-2rs^2\right)^{n-1}$$
$$= 2^{n-1}r^{3(n-1)}2r\left(r^2-s^2\right). \tag{2.6}$$

It follows from relations (2.4), (2.6) and (2.5) that

$$f\left(x_{11},\dots,x_{1j-1},rx_{1j}+sx_{2j},x_{1j+1},\dots,x_{1n}\right) + f\left(x_{11},\dots,x_{1j-1},rx_{1j}-sx_{2j},x_{1j+1},\dots,x_{1n}\right)$$
$$= rs^2f^*\left(x_{1j},x_{2j}\right)+2r\left(r^2-s^2\right)f\left(x_{11},\dots,x_{1n}\right).$$

This means that f is cubic in the jth variable. Since j is arbitrary, we obtain the desired result. $\square$

3. Stability Results for (2.2)

We firstly express some basic facts concerning non-Archimedean spaces and some preliminary results. Let us recall that a metric d on a nonempty set X is said to be non-Archimedean (or an ultrametric) provided

$$d(x,z)\le \max\{d(x,y),d(y,z)\}$$

for $x,y,z\in X$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|a|=0$ if and only if $a=0$, $|ab|=|a||b|$, and $|a+b|\le\max\{|a|,|b|\}$ for all $a,b\in\mathbb{K}$. Clearly, $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:\mathcal{X}\longrightarrow\mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|ax\|=|a|\|x\|, \quad (x\in\mathcal{X}, a\in\mathbb{K})$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|,\|y\|\}, \quad (x,y\in\mathcal{X}).$$

Then, $(\mathcal{X},\|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n-x_m\| \leq \max\{\|x_{j+1}-x_j\|; m\leq j\leq n-1\}, \quad (n\geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent. If $(\mathcal{X},\|\cdot\|)$ is a non-Archimedean normed space, then it is easily verified that the function $d_{\mathcal{X}}:\mathcal{X}\times\mathcal{X}\longrightarrow\mathbb{R}_+$, given by $d_{\mathcal{X}}(x,y):=\|x-y\|$, is a non-Archimedean metric on $\mathcal{X}$ that is invariant (i.e., $d_{\mathcal{X}}(x+z,y+z)=d_{\mathcal{X}}(x,y)$ for $x,y,z\in\mathcal{X}$). Hence, non-Archimedean normed spaces are also special cases of metric spaces with invariant metrics.

The most important examples of non-Archimedean normed spaces are the p-adic numbers, which have gained the interest of physicists because of their connections with some problems coming from quantum physics, p-adic strings and superstrings [19]. Indeed, Hensel [15] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x,y>0$, there exists an integer n such that $x<ny$.

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1+\cdots+1}^{n-\text{times}}=0$.

Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. In this section, we prove the generalized Hyers-Ulam stability of equation (2.2) in non-Archimedean spaces. The proof is based on a fixed point result that can be derived from [10, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j\in\mathbb{N}$, $g_1,\ldots,g_j:E\longrightarrow E$ and $L_1,\ldots,L_j:E\longrightarrow\mathbb{R}_+$,

(H2) $\mathcal{T}:Y^E\longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\| \leq \max_{i\in\{1,\ldots,j\}} L_i(x)\,\|\lambda(g_i(x))-\mu(g_i(x))\|,$$

for all $\lambda,\mu\in Y^E, x\in E$,

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i\in\{1,\dots,j\}} L_i(x)\delta(g_i(x)), \quad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [10, Theorem 1]. This result plays a key tool to obtain our goal in this paper.

Theorem 3.1. *Let hypotheses* (H1)-(H3) *hold and the function* $\theta : E \longrightarrow \mathbb{R}_+$ *and the mapping* $\varphi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \qquad \lim_{l\to\infty} \Lambda^l\theta(x) = 0, \quad (x \in E).$$

Then, for every $x \in E$, *the limit* $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ *and the mapping* $\psi \in Y^E$, *defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l\theta(x), \qquad (x \in E).$$

In the sequel, given the mapping $f : V^n \longrightarrow W$, we delineate the difference operator $\mathfrak{D}_c f : V^n \times V^n \longrightarrow W$ by

$$\mathfrak{D}_c f(x_1, x_2) = \sum_{q\in\{-1,1\}^n} f(rx_1 + qsx_2) - \sum_{k=0}^{n} \left(rs^2\right)^{n-k} \left[2r\left(r^2 - s^2\right)\right]^k f\left(\mathcal{M}_k^n\right)$$

where $f(\mathcal{M}_k^n)$ is defied in (2.1).

Here and subsequently, without loss of generality we assume that r is a fixed positive integer such that $r \geq 2$. We now are ready to indicate the upcoming result which is the main result in this paper.

Theorem 3.2. *Let* $\beta \in \{-1, 1\}$ *be fixed,* V *be a linear space and* W *be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* r. *Suppose that* $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a function satisfying the equality*

$$\lim_{l\to\infty} \left(\frac{1}{|r|^{3n\beta}}\right)^l \varphi(r^{l\beta}x_1, r^{l\beta}x_2) = 0 \tag{3.1}$$

for all $x_1, x_2 \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathfrak{D}_c f(x_1, x_2)\| \leq \varphi(x_1, x_2) \tag{3.2}$$

for all $x_1, x_2 \in V^n$. *Then, there exists a unique multi-cubic mapping* $\mathcal{C} : V^n \longrightarrow W$ *such that*

$$\|f(x) - \mathcal{C}(x)\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^n \times |r|^{3n\frac{\beta+1}{2}}} \left(\frac{1}{|r|^{3n\beta}}\right)^l \varphi\left(r^{l\beta+\frac{\beta-1}{2}}, 0\right) \tag{3.3}$$

for all $x \in V^n$.

Proof. Putting $x = x_1$ and $x_2 = 0$ in (3.2), we have

$$\left\| 2^n f(rx) - \left(\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \left(rs^2\right)^{n-k} \left(2r\left(r^2 - s^2\right)\right)^k \right) f(x) \right\| \leq \varphi(x,0) \tag{3.4}$$

for all $x \in V^n$. By an easy computation, we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \left(rs^2\right)^{n-k} \left(2r\left(r^2 - s^2\right)\right)^k &= \left(2rs^2 + 2r^3 - 2rs^2\right)^n \\ &= \left(2r^3\right)^n. \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\left\| f(rx) - r^{3n} f(x) \right\| \leq \frac{1}{|2|^n} \varphi(x,0) \tag{3.6}$$

for all $x \in V^n$. The inequality (3.4) implies that

$$\|f(x) - \mathcal{T} f(x)\| \leq \theta(x) \tag{3.7}$$

for all $x \in V^n$, where

$$\theta(x) := \frac{1}{|2|^n \times |r|^{3n\frac{\beta+1}{2}}} \varphi\left(r^{\frac{\beta-1}{2}} x, 0\right), \qquad \mathcal{T}\xi(x) := \frac{1}{r^{3n\beta}} \xi\left(r^\beta x\right)$$

for all $\xi \in W^{V^n}$ and $x \in V^n$. Define $\Lambda\eta(x) := \frac{1}{|r|^{3n\beta}} \eta\left(r^\beta x\right)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easy to see that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := r^\beta x$ for all $x \in V^n$ and $L_1(x) = \frac{1}{|r|^{3n\beta}}$. Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\begin{aligned} \|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\| \frac{1}{r^{3n\beta}} \lambda(r^\beta x) - \frac{1}{r^{3n\beta}} \mu(r^\beta x) \right\| \\ &\leq L_1(x) \left\| \lambda(g_1(x)) - \mu(g_1(x)) \right\|. \end{aligned}$$

The above relation shows that the hypothesis (H2) is valid. By induction on l, one can check that for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$\begin{aligned} \Lambda^l \theta(x) &:= \left(\frac{1}{|r|^{3n\beta}}\right)^l \theta\left(r^{l\beta} x\right) \\ &= \frac{1}{|2|^n \times |r|^{3n\frac{\beta+1}{2}}} \left(\frac{1}{|r|^{3n\beta}}\right)^l \varphi\left(r^{l\beta + \frac{\beta-1}{2}}, 0\right) \end{aligned} \tag{3.8}$$

for all $x \in V^n$. The relations (3.7) and (3.8) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping

$\mathcal{C} : V^n \longrightarrow W$ such that $\mathcal{C}(x) = \lim_{l\to\infty} \left(\mathcal{T}^l f\right)(x)$ for all $x \in V^n$, and also (3.3) holds. We also can verified by induction on l that

$$\left\|\mathfrak{D}_c \left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \leq \left(\frac{1}{|r|^{3n\beta}}\right)^l \varphi\left(r^{l\beta}x_1, r^{l\beta}x_2\right) \tag{3.9}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.9) and applying (3.1), we arrive at $\mathfrak{D}_c\mathcal{C}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies equation (2.2) and the proof is now completed. □

Remark 3.3. We note that in Theorem 3.2, it is assumed that the non-Archimedean field has the characteristic different from r and so the conditions of Theorem 3.1 are valid because $|r| < |2| < 1$.

The following corollaries are some direct applications of Theorem 3.2 concerning the stability of (2.2).

Corollary 3.4. *Let $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r and $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}_c f(x_1, x_2)\| \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{1}{|2|^n}\delta$$

for all $x \in V^n$.

Proof. We firstly note that $|r| < 1$. Letting $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 3.2, we have $\lim_{l\to\infty}\left(\frac{1}{|r|^{-3n}}\right)^l \delta = 0$. Therefore, one can obtain the desired result. □

Corollary 3.5. *Let $p \in \mathbb{R}$ fulfills $p \neq 3n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from r and $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}_c f(x_1, x_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^p$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \frac{1}{|2|^n \times |r|^{3n}} \sum_{j=1}^{n} \|x_{1j}\|^p & p > 3n \\ \\ \frac{1}{|2|^n \times |r|^{p}} \sum_{j=1}^{n} \|x_{1j}\|^p & p < 3n \end{cases}$$

for all $x = x_1 \in V^n$.

Proof. Putting $\varphi(x_1, x_2) = \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p$, we have $\varphi\left(r^l x_1, r^l x_2\right) = |r|^{lp} \varphi(x_1, x_2)$. It now follows from Theorem 3.2 the first and second inequalities in the cases $\beta = 1$ and $\beta = -1$, respectively. □

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1 \varphi(a_1, \ldots, a_n) = \mathcal{F}_2 \varphi(a_1, \ldots, a_n) \tag{3.10}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d\left(\mathcal{F}_1 \varphi_0(a_1, \ldots, a_n), \mathcal{F}_2 \varphi_0(a_1, \ldots, a_n)\right) \leq \psi(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n \in A$, fulfils (3.10); this definition is introduced in [11]. In other words, a functional equation $\mathcal{F}$ is hyperstable if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions the functional equation (2.2) can be hyperstable as follows.

Corollary 3.6. *Suppose that* $p_{kj} > 0$ *for* $k \in \{1, 2\}$ *and* $j \in \{1, \ldots, n\}$ *fulfill* $\sum_{k=1}^{2} \sum_{j=1}^{n} p_{kj} \neq 3n$. *Let* V *be a normed space and* W *be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* r *and* $|2| < 1$. *If* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathfrak{D}_c f(x_1, x_2)\| \leq \prod_{k=1}^{2} \prod_{j=1}^{n} \|x_{kj}\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, *then* f *is multi-cubic.*

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