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Continuous k-Frames and their Dual in Hilbert Spaces

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Abstract. The notion of k-frames was recently introduced by Găvruţa in Hilbert spaces to study atomic systems with respect to a bounded linear operator. A continuous frame is a family of vectors in a Hilbert space which allows reproductions of arbitrary elements by continuous super positions. In this manuscript, we construct a continuous k-frame, so called ck-frame along with an atomic system for this version of frames. Also we introduce a new method for obtaining the dual of a ck-frame and prove some new results about it.

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1. Introduction

Frames were first introduced in the context of non-harmonic Fourier series [13]. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of [11]. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly and several new applications have been developed. For example, besides traditional applications as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission [9], and to design high-rate constellation with full diversity in multiple-antenna code design [18]. In [4–6] some more applications have been developed.

In quantum mechanics, specifically in the theory of coherent states [1, 2, 20], this notion was generalized to a family of vectors indexed

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by a locally compact space endowed with a positive Radon measure. They have been introduced originally by Ali, Gazeau and Antoine [1, 2] and also, independently, by Kaiser [19]. Since then, several papers dealt with various aspects of the concept, see for instance [15, 16] or [21]. The continuous wavelet transformation and short time Fourier transformation are two well known examples of continuous frames.

Traditionally, frames were studied for the whole space or for the closed subspaces. Găvruţa in [17] gave another generalization of frames namely K-frames, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space. In general, range is not a closed subspace. K-frames allow us in a stable way, to reconstruct elements from the range of a linear and bounded operator in a Hilbert space.

In this paper, by combining the notion of continuous frame and K-frame, we introduce the notion of continuous k-frames and we investigate some of their properties.

The structure of this article is as follows: in Section 1, we review some basic properties of frame theory in Hilbert spaces. In Section 2, cK-frames and some fundamental properties about them are discussed. Finally, in Section 3 we introduce a new method for obtaining the dual of a cK-frame and we prove some new results about it.

Throughout the paper, H and H_0, are Hilbert spaces, $(H_0)_1$ is the closed unit ball in H_0, (X, μ) is a σ-finite measure space, $\mathcal{L}(H_0, H)$ is the set of all linear mappings of H_0 to H and $\mathcal{B}(H_0, H)$ is the Banach algebra of all bounded linear mappings. Instead of $\mathcal{B}(H, H)$, we simply write $\mathcal{B}(H)$. First, we introduce a result which present a replacement requirement of an inner product and an integral in a measure space.

Definition 1.1. A functions $f : X \to H$ is called Bochner measurable if there exists a sequence of simple function $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$, μ-almost everywhere. If all of f_n are integrable and

$$\lim_{n\to\infty} \int_X \|f_n(x) - f(x)\| \, d\mu(x) = 0,$$

then, we call f is Bochner integrable.

Lemma 1.2 ([23])**.** *Let $f : X \to H$ be a Bochner integrable function. Then for each $h \in H$ we have*

$$\int_X \langle f(x), h \rangle d\mu(x) = \left\langle \int_X f(x) \, d\mu(x), h \right\rangle.$$

Next, we need the following result in operator theory in next section.

Lemma 1.3 ([12])**.** *Suppose, H, H_1, H_2 are Hilbert spaces, $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$,then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1) \subset \mathcal{R}(L_2)$,
(ii) *There exists* $\lambda \geq 0$, *such that* $L_1 L_1^* \leq \lambda L_2 L_2^*$,
(iii) *There exists* $X \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 X$.

In next parts, we aim to review notations of k-frames, k-atoms and continuous frames with the operators of c-frames.

Definition 1.4. Let $k \in \mathcal{B}(H_0, H)$, and $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ where $\mathbb{I} \subseteq \mathbb{Z}$. We say that the sequence $\{f_i\}_{i\in\mathbb{I}}$ is a k-frame for H with respect to H_0, if there exists constants $A, B > 0$ such that

$$A \|k^* h\|^2 \leq \sum_{i\in\mathbb{I}} |\langle h, f_i\rangle|^2 \leq B\|h\|^2, \quad h \in H. \tag{1.1}$$

If $k = id_H$, then we get the discrete frame for H and when only the right hand of the inequality (1.1) holds, we call $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence with the bound B.

Definition 1.5. Let $H_0 \subseteq H$ and $k \in \mathcal{B}(H_0, H)$. A Bessel sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ is called a family of local k-atoms for H_0 if there exists a sequence $\{c_i\}_{i\in\mathbb{I}}$ of linear functionals on H_0 such that the following conditions are satisfied:

(i) There exists $\alpha > 0$ such that for each $f \in H_0$
$$\sum_{i\in\mathbb{I}} |c_i(f)|^2 \leq \alpha \|f\|^2,$$
(ii) for each $f \in H_0$,
$$kf = \sum_{i\in\mathbb{I}} c_i(f) f_i.$$

In this case, we say that the pair $\{f_i, c_i\}$ provides a k-atomic decomposition for H_0. If k is the identity mapping, then we say that $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ is a family of local atoms for H_0.

Definition 1.6. Let $f : X \to H$ be a weakly measurable (i.e. for all $h \in H$, the mapping $x \to \langle f(x), h\rangle$ is measurable). We define the mapping $\int_X \cdot f d\mu : L^2(X) \to H$ as follows:

$$\left\langle \int_X gf\, d\mu, h \right\rangle := \int_X g(x) \langle f(x), h\rangle\, d\mu, \quad h \in H.$$

It is clear that, the vector valued integral $\int_X gfd\mu$ exists in H if for each $h \in H$, $\int_X g(x)\langle f(x), h\rangle\, d\mu$ exists.

Lemma 1.7 ([25])**.** *Let $f : X \to H$ be a weakly measurable. For each $g \in L^2(X)$, the value of $\int_X gfd\mu$ exists in H if and only if for each $h \in H$, $\langle f, h\rangle \in L^2(X)$.*

Definition 1.8. Let $f : X \to H$ be weakly measurable. Then f is called a c-frame for H if there exists $0 < A \leq B < \infty$ such that for all $h \in H$,

$$A\|h\|^2 \leq \int_X |\langle f(x), h\rangle|^2 d\mu \leq B\|h\|^2.$$

The constants A and B are called c-frame bounds. If A, B can be chosen so that $A = B$, we call this c-frame a *c*-tight frame, and if $A = B = 1$ it is called a c-Parseval frame. If we only have the upper bound, we call f a c-Bessel mapping for H. The representation space employed in this setting is

$$L^2(X, H) = \{\varphi : X \to H | \varphi \text{ is measurable and } \|\varphi\|_2 < \infty\},$$

where $\|\varphi\|_2 = \left(\int_X \|\varphi(x)\|^2 \, d\mu\right)^{\frac{1}{2}}$.

For each $f, g \in L^2(X, H)$, the mapping $x \to \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X, H)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{L^2} = \int_X \langle f(x), g(x)\rangle \, d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$. Suppose that f is a c-Bessel mapping, then the synthesis and analysis operators are defined by

$$T_f : L^2(X) \longrightarrow H,$$
$$\langle T_f(g), h\rangle = \int_X g(x) \langle f(x), h\rangle \, d\mu(x),$$

and

$$T_f^* : H \longrightarrow L^2(X),$$
$$T_f^* h = \langle h, f\rangle.$$

For the synthesis operator, by the notation of vector valued integrals, we can write

$$T_f(g) = \int_X gf \, d\mu, \qquad g \in L^2(X).$$

Therefore, the frame operator $S_f := T_f T_f^*$ is given by,

$$S_f h = \int_X \langle h, f\rangle f \, d\mu, \quad \text{for any } h \in H,$$

Now, when f is a c-frame for H with the frame bounds A and B, we get

$$AId_H \leq S_f \leq BId_H.$$

Hence, S_f is a positive, self-adjoint and invertible operator. The next result will be used in the next section.

Lemma 1.9 ([14]). *Let $f : X \to H$ be a c-Bessel mapping for H, and $u \in \mathcal{B}(H, H_0)$. Then $uf : X \to H_0$ is a c-Bessel mapping for H_0 with*

$$uT_f = T_{uf}.$$

2. Continuous k-frames and ck-atoms

In this section, we introduce ck-atoms and continuous k-frames and show that these are equivalent. This work is a generalization of the discrete case, which was presented in [17]. For this, we need the following result.

Proposition 2.1. *Let $f : X \to H$ be weakly measurable. Then f is a c-Bessel mapping for H if and only if $\int_X gfd\mu$ exists in H for each $g \in L^2(X)$.*

Proof. Suppose that $\int_X gfd\mu$ exists for each $g \in L^2(X)$. By Lemma 1.7, $\langle f, h\rangle \in L^2(X)$ for each $h \in H$. We have

$$\begin{aligned}\left|\int_X gfd\mu\right| &= \sup_{\|t\|=1}\left|\left\langle \int_X gfd\mu, t\right\rangle\right| \\ &= \sup_{\|t\|=1}\left|\int_X g(x)\left\langle f(x), t\right\rangle d\mu\right| \\ &\leq \|g\|_2 \sup_{\|t\|=1}\|\langle f, t\rangle\|_2 .\end{aligned}$$

Since, for every $x \in X$,

$$\begin{aligned}\sup_{\|t\|=1} |\left\langle f(x), t\right\rangle| &\leq \|f(x)\| \\ &< \infty,\end{aligned}$$

by Banach-Steinhaus theorem (see [10], page 407), $\sup_{\|t\|=1}\|\langle f, t\rangle\| < \infty$. Hence

$$\begin{aligned}\left\|\int_X \cdot fd\mu\right\| &\leq \sup_{\|t\|=1}\|\langle f, t\rangle\|_2 \\ &< \infty.\end{aligned}$$

The above inequality implies that $\int_X \cdot fd\mu$ is bounded and $\sup\limits_{\|t\|=1}\|\langle f, t\rangle\|_2$ is an upper bounded for $\int_X \cdot fd\mu$. Now the adjoint of $\int_X .fd\mu$ is calculated as follow:
For each $h \in H$ and $g \in L^2(X)$, we have

$$\left\langle g, \left(\int_X \cdot fd\mu\right)^* (h)\right\rangle = \left\langle \int_X gfd\mu, h\right\rangle$$

$$= \int_X g(x) \langle f(x), h \rangle \, d\mu$$
$$= \langle g, \langle h, f \rangle \rangle .$$

Thus, for each $h \in H$,

$$\left(\int_X \cdot f d\mu \right)^* (h) = \langle h, f \rangle . \tag{2.1}$$

Therefore,

$$\begin{aligned} \int_X | \langle h, f(x) \rangle |^2 d\mu &= \left\| \left(\int_X \cdot f d\mu \right)^* (h) \right\|^2 \\ &\leq \left\| \left(\int_X \cdot f d\mu \right)^* \right\|^2 \|h\|^2 \\ &= \left\| \int_X \cdot f d\mu \right\|^2 \|h\|^2 \\ &\leq \left(\sup_{\|t\|=1} \|\langle f, t \rangle\|_2^2 \right) \|h\|^2 . \end{aligned}$$

Hence, f is a c-Bessel mapping for H. Now, if f is a c-Bessel mapping for H, then for each $h \in H$, we have $\langle h, f \rangle \in L^2(X)$. Consequently, by Lemma 1.7, $\int_X .gf d\mu$ exists for each $g \in L^2(X)$. □

Definition 2.2. Let $H_0 \subseteq H$ and $f : X \to H$ be weakly measurable and $k \in \mathcal{B}(H_0, H)$. Then f is called a local ck-atoms for H_0 if the following conditions are satisfied:

- (i) For each $g \in L^2(X)$, the vector valued integral $\int_X gf d\mu$ exists in H.
- (ii) There exist $a > 0$ and $\ell : X \to \mathcal{L}(H_0, \mathbb{C})$ such that $\ell(\cdot)(h) \in L^2(X)$ for each $h \in H_0$ and also

$$\|\ell(\cdot)(h)\|_2 \leq a \|h\| ,$$
$$kh = \int_X \ell(\cdot)(h) f d\mu.$$

Now, in this case, when k is the identity function on H_0, we call f as a local *c-atoms* for H_0.

Definition 2.3. Let $k \in \mathcal{B}(H_0, H)$ and $f : X \to H$ be weakly measurable. Then f is called a ck-frame with respect to H_0, if there exist constants $0 < A \leq B < \infty$ such that for each $h \in H$,

$$A \|k^* h\|^2 \leq \int_X | \langle h, f(x) \rangle |^2 d\mu \leq B \|h\|^2 .$$

Theorem 2.4. *Let $H_0 \subseteq H$ and $k \in \mathcal{B}(H_0, H)$. If $f : X \to H$ is weakly measurable, then the following assertions are equivalent:*

(i) *f is a local ck-atoms for H_0.*

(ii) *f is a ck-frame for H with respect to H_0.*

(iii) *f is a c-Bessel mapping for H, and there exists $g \in \mathcal{B}(H_0, L^2(X))$ such that*

$$kh = \int_X g(h) f d\mu, \quad h \in H_0.$$

Proof. (i)⇒(ii). By the hypothesis and Proposition 2.1, f is a c-Bessel mapping for H. For each $h \in H$ we have

$$\begin{aligned} \|k^* h\| &= \sup_{\|t\|=1} |\langle k^*(h), t\rangle| \\ &= \sup_{\|t\|=1} |\langle h, k(t)\rangle|. \end{aligned}$$

Now by (i) there exist $c > 0$ and $\ell : X \to \mathcal{L}(H_0, \mathbb{C})$ such that for every $h \in H_0$, $\ell(\cdot)(h) \in L^2(X)$, and also

$$\begin{aligned} \|\ell(\cdot)(h)\|_2 &\leq c \|h\|, \\ kh &= \int_X \ell(\cdot)(h) f d\mu. \end{aligned}$$

So for each $h \in H$,

$$\begin{aligned} \|k^*(h)\|^2 &= \sup_{\|t\|=1} \left| \left\langle h, \int_X \ell(\cdot)(t) f d\mu \right\rangle \right|^2 \\ &= \sup_{\|t\|=1} \left| \int_X \ell(x)(t) \langle h, f(x)\rangle d\mu \right|^2 \\ &\leq \sup_{\|t\|=1} \|\ell(\cdot)(t)\|_2^2 \left(\int_X |\langle h, f(x)\rangle|^2 d\mu \right) \\ &\leq \sup_{\|t\|=1} c^2 \|t\|^2 \int_X |\langle h, f(x)\rangle|^2 d\mu \\ &= c^2 \int_X |\langle h, f(x)\rangle|^2 d\mu. \end{aligned}$$

(ii)⇒(iii). Since f is a c-Bessel mapping for H, $T_f : L^2(X) \to H$ is a bounded linear operator. By (ii), for each $h \in H$

$$A \|k^*(h)\|^2 \leq \left\|T_f^*(h)\right\|^2.$$

Now for each $h \in H$, we have

$$\begin{aligned} A\langle kk^*(h), h\rangle &= A\|k^*(h)\|^2 \\ &\leq \left\|T_f^*(h)\right\|^2 \\ &= \left\langle T_f T_f^*(h), h\right\rangle. \end{aligned}$$

thus

$$kk^* \leq \frac{1}{A} T_f T_f^*.$$

Finally, by Lemma 1.3, there exists a bounded linear operator $M : H_0 \longrightarrow L^2(X)$ such that $k = T_f M$. So for each $h \in H_0$

$$\begin{aligned} kh &= T_f(M(h)) \\ &= \int_X M(h) f d\mu. \end{aligned}$$

(iii)$\Rightarrow$ (i). Since f is a c-Bessel mapping for H, $\int_X gfd\mu$ for each $g \in L^2(X)$, by Lemma 2.1 exists. By (iii), there exists $g \in \mathcal{B}(H_0, L^2(X))$ such that

$$kh = \int_X g(h) f d\mu, \quad h \in H_0.$$

Now we define

$$\ell : X \to L(H_0, \mathbb{C}), \qquad \ell(\cdot)(h) := g(h)(\cdot), \quad h \in H_0,$$

so we have

$$k(h) = \int_X \ell(\cdot)(h) f d\mu, \quad h \in H_0,$$

also

$$\begin{aligned} \|\ell(\cdot)(h)\|_2 &= \|g(h)(\cdot)\|_2 \\ &\leq \|g\| \, \|h\|. \end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $k \in B(H_0, H)$. Suppose that $f : X \to H$ is weakly measurable. Then f is a ck-frame for H with respect to H_0 if and only if the mapping*

$$L_f : L^2(X) \longrightarrow H, \qquad L_f(g) = \int_X g f d\mu,$$

is a well-defined bounded linear operator with $\mathcal{R}(k) \subset \mathcal{R}(L_f)$.

Proof. First we assume that f is a ck-frame for H with respect to H_0. Then, by the definition, f is a c-Bessel mapping for and we have

$$\begin{aligned} A\|k^*(h)\|^2 &\leq \int_X |\langle h, f(x)\rangle|^2 d\mu \\ &= \left\|T_f^*(h)\right\|^2, \end{aligned}$$

thus

$$Akk^* \leq T_f T_f^*,$$

and Theorem 1.3 implies that

$$\mathcal{R}(k) \subset \mathcal{R}(T_f).$$

Since $L_f = T_f$, L_f is a bounded linear operator. Now, let

$$L_f : L^2(X) \longrightarrow H$$
$$L_f(g) = \int_X g f d\mu,$$

be a well-defined bounded linear operator of $L^2(X)$ into H with $\mathcal{R}(k) \subset \mathcal{R}(L_f)$. By Proposition 2.1, f is a c-Bessel mapping for H. So it is sufficient to show that it has a lower ck-frame bound. Since k and L_f are bounded linear operators and $\mathcal{R}(k) \subset \mathcal{R}(L_f)$, by Lemma 1.3, there exists $A > 0$ such that $Akk^* \leq L_f L_f^*$. Now for each $h \in H$,

$$\langle Akk^*(h), h\rangle \leq \left\langle L_f L_f^*(h), h\right\rangle,$$

consequently by (2.1) we have

$$\begin{aligned} A\left\|k^*(h)\right\|^2 &\leq \left\|L_f^*(h)\right\|^2 \\ &= \int_X |\langle h, f(x)\rangle|^2 d\mu, \end{aligned}$$

and the proof is complete. □

Generally, in ck-frames, as in k-frames, the frame operator is not invertible. However, we have the following:

Theorem 2.6. *Let $k \in \mathcal{B}(H_0, H)$, and $f : X \to H$ be a ck-frame for H with respect to H_0, with bounds A,B. If k has a closed range, then S_f is invertible on $\mathcal{R}(k)$, and for each $h \in \mathcal{R}(k)$*

$$B^{-1}\left\|h\right\|^2 \leq \left\langle (S_f|_{\mathcal{R}(k)})^{-1}h, h\right\rangle \leq A^{-1}\|k^\dagger\|^2\|h\|^2.$$

Proof. For each $h \in H$

$$\begin{aligned} A\|k^*h\|^2 &\leq \int_X |\langle h, f\rangle|^2 \, d\mu \\ &= \langle S_f(h), h\rangle \\ &\leq B\|h\|^2, \end{aligned}$$

hence

$$Akk^* \leq S_f \leq BI.$$

Since $kk^\dagger|_{\mathcal{R}(k)} = I_{\mathcal{R}(k)}$, for each $h \in \mathcal{R}(k)$, then

$$\|h\| = \|I^*_{\mathcal{R}(k)}h\| = \|(k^\dagger|_{\mathcal{R}(k)})^*k^*h\| \leq \|k^\dagger\|.\|k^*h\|.$$

Therefore, for each $h \in \mathcal{R}(k)$,

$$A\|k^{\dagger}\|^{-2}\|h\|^{2} \leq \langle S_f(h), h\rangle \leq B\,\|h\|^{2}.$$

hence, S_f is invertibale on $\mathcal{R}(k)$, and for each $h \in \mathcal{R}(k)$,

$$\begin{aligned} B^{-1}\,\|h\|^{2} &\leq \left\langle \left(S_f|_{\mathcal{R}(k)}\right)^{-1}(h), h\right\rangle \\ &\leq A^{-1}\|k^{\dagger}\|^{2}\|h\|^{2}. \end{aligned}$$

□

Corollary 2.7. *Let $k \in \mathcal{B}(H_0, H)$, and $f : X \to H$ be a ck-frame for H with respect to H_0, with bounds A,B. If k has a closed range with $\mathcal{R}(k) \subset \mathcal{R}(f)$, then f is a c-frame for $\mathcal{R}(k)$ with bounds $A\|k^{\dagger}\|^{-2}$ and B, respectively.*

3. Continuous k-dual

In this section, we introduce the dual of ck-frames and prove some theorems about them. Throughout this section, the orthogonal projection of H onto a closed subspace $V \subseteq H$.

Theorem 3.1. *Let $k \in \mathcal{B}(H_0, H)$, and let $f : X \to H$ be a c-Bessel mapping for H, and $g : X \to H_0$ be a c-Bessel mapping for H_0. Then the following assertions are equivalent:*

(i) *For each $h_0 \in H_0$, $kh_0 = T_f(\langle h_0, g\rangle)$.*
(ii) *For each $h \in H$, $k^*h = T_g(\langle h, f\rangle)$.*
(iii) *For each $h \in H, h_0 \in H_0$,*

$$\langle kh_0, h\rangle = \int_X \langle h_0, g(x)\rangle \langle f(x), h\rangle \, d\mu.$$

(iv) *For each $h \in H, h_0 \in H_0$,*

$$\langle k^*h, h_0\rangle = \int_X \langle h, f(x)\rangle \langle g(x), h_0\rangle \, d\mu.$$

(v) *For any orthonormal bases $\{\gamma_j\}_{j\in J}$ for H_0, and $\{e_i\}_{i\in I}$ for H,*

$$\langle k^*e_i, \gamma_j\rangle = \int_X \langle e_i, f(x)\rangle \langle g(x), \gamma_j\rangle \, d\mu, \quad i \in I \; j \in J.$$

Proof. (i)⇒(ii). If $h \in H$ and $h_0 \in H_0$, then

$$\begin{aligned} \langle h_0, k^*h\rangle &= \langle T_f(\langle h_0, g\rangle), h\rangle \\ &= \int_X \langle h_0, g(x)\rangle \langle f(x), h\rangle \, d\mu \\ &= \overline{\int_X \langle g(x), h_0\rangle \langle h, f(x)\rangle \, d\mu} \end{aligned}$$

$$= \overline{\langle T_g(\langle h, f\rangle), h_0\rangle}$$
$$= \langle h_0, T_g(\langle h, f\rangle)\rangle .$$

Hence (ii) is proved.
The part (ii)⇒(i) follows similarly.
(iv)⇒(v), (ii)⇔(iii), and (iii)⇔(iv) are evident.
(v)⇒(iv). Let $h \in H, h_0 \in H_0$. Then

$$\begin{aligned}
\int_X \langle h, f(x)\rangle \langle g(x), h_0\rangle \, d\mu &= \langle \langle h, f\rangle , \langle h_0, g\rangle\rangle_{L^2} \\
&= \left\langle \left\langle h, \sum_i \overline{\langle e_i, f\rangle} \; e_i \right\rangle , \left\langle h_0, \sum_j \overline{\langle \gamma_j, g\rangle} \; \gamma_j \right\rangle \right\rangle_{L^2} \\
&= \sum_{i,j} \left\langle \left\langle h, \overline{\langle e_i, f\rangle} e_i \right\rangle \left\langle h_0, \overline{\langle \gamma_j, g\rangle} \gamma_j \right\rangle \right\rangle_{L^2} \\
&= \sum_{i,j} \langle h, e_i\rangle \langle \gamma_j, h_0\rangle \langle \langle e_i, f\rangle , \langle \gamma_j, g\rangle\rangle_{L_2} \\
&= \sum_{i,j} \langle h, e_i\rangle \langle \gamma_j, h_0\rangle \int_X \langle e_i, f(x)\rangle \langle g(x), \gamma_j\rangle \, d\mu \\
&= \sum_{i,j} \langle h, e_i\rangle \langle \gamma_j, h_0\rangle \langle k^* e_i, \gamma_j\rangle \\
&= \sum_{i,j} \langle h, e_i\rangle \langle e_i, k\gamma_j\rangle \langle \gamma_j, h_0\rangle \\
&= \sum_j \langle h, k\gamma_j\rangle \langle \gamma_j, h_0\rangle \\
&= \sum_j \langle k^* h, \gamma_j\rangle \langle \gamma_j, h_0\rangle \\
&= \langle k^* h, h_0\rangle .
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $k \in \mathcal{B}(H_0, H)$ and $f : X \to H$ be a c-Bessel mapping for H and $g : X \to H_0$ be a c-Bessel mapping for H_0.*

(i) *The following condition is equivalent to the assertions of Theorem 3.1:*

$$\|kh_0\|^2 = \int_X \langle h_0, g(x)\rangle \langle f(x), kh_0\rangle \, d\mu, \quad h_0 \in H_0.$$

(ii) *The following condition is equivalent to the assertions of Theorem 3.1:*

$$\|k^* h\|^2 = \int_X \langle h, f(x)\rangle \langle g(x), k^* h\rangle \, d\mu, \quad h \in H.$$

Proof. (i) Define $F : H_0 \to H_0$ by

$$F(h_0) := T_g \langle kh_0, f \rangle, \quad h_0 \in H_0.$$

F is clearly linear and bounded, since for each $h_0 \in H_0$,

$$\begin{aligned}
\|F(h_0)\| &= \sup_{k_0 \in (H_o)_1} |\langle F(h_0), k_0 \rangle| \\
&= \sup_{k_0 \in (H_o)_1} \left| \int_X \langle g(x), k_0 \rangle \langle kh_0, f(x) \rangle \, d\mu \right| \\
&\leq \sup_{k_0 \in (H_o)_1} \left(\int_X |\langle k_0, g(x) \rangle|^2 \, d\mu \right)^{1/2} \\
&\quad \times \sup_{\ell_0 \in (H_o)_1} \left(\int_X |\langle \ell_0, f(x) \rangle|^2 \, d\mu \right)^{1/2} \|kh_0\| \\
&\leq \sup_{k_0 \in (H_o)_1} \left(\int_X |\langle k_0, g(x) \rangle|^2 \, d\mu \right)^{1/2} \\
&\quad \times \sup_{\ell_0 \in (H_o)_1} \left(\int_X |\langle \ell_0, f(x) \rangle|^2 \, d\mu \right)^{1/2} \|k\| \, \|h_0\| .
\end{aligned}$$

For each $h_0 \in H_0$, we have

$$\begin{aligned}
\langle h_0, k^* kh_0 \rangle &= \|kh_0\|^2 \\
&= \int_X \langle h_0, g(x) \rangle \langle f(x), kh_0 \rangle \, d\mu \\
&= \overline{\langle T_g \langle kh_0, f \rangle, h_0 \rangle} \\
&= \langle h_0, T_g \langle kh_0, f \rangle \rangle .
\end{aligned}$$

Hence, $k^* kh_0 = T_g \langle kh_0, f \rangle$. The part (ii) follows similarly. $\square$

Now, we can define the ck-dual pair for two c-Bessel mappings as follows.

Definition 3.3. Let $k \in \mathcal{B}(H_0, H)$, and let $f : X \to H$ be a c-Bessel mapping for H, and $g : X \to H_0$ be a c-Bessel mapping for H_0. We say that f, g is a ck-dual pair, if one of the assertions of Theorem 3.1 holds.

Theorem 3.4. *Let $k \in \mathcal{B}(H_0, H)$, and let $f : X \to H$ be c-Bessel mapping for H, and $g : X \to H_0$ be a c-Bessel mapping for H_0. Let f, g be a ck-dual pair. Then f is a ck-frame for H with respect to H_0, and g is a ck*-frame for H_0 with respect to H.*

Proof. For each $h \in H_0$

$$\|kh\|^4 = |\langle T_f(\langle h, g \rangle), kh \rangle|^2$$

$$
\begin{aligned}
&= \left| \int_X \langle h, g(x)\rangle \,.\, \langle f(x), kh\rangle \; d\mu \right|^2 \\
&\le \left(\int_X |\langle h, g(x)\rangle|^2 \; d\mu \right) \left(\int_X |\, \langle f(x), kh\rangle \,|^2 \, d\mu \right) \\
&\le \left(\int_X |\, \langle h, g(x)\rangle \,|^2 \, d\mu \right) B\|kh\|^2,
\end{aligned}
$$

where B is an upper bound for the c-Bessel mapping f. This shows that g is a ck^*-frame for H_0 with respect to H, with the lower bound B^{-1}. Similarly, f is a ck-frame for H. □

Now, we characterize all ck-duals for a ck-frame with the same method of Proposition 1 in [3].

Theorem 3.5. *Let $k \in \mathcal{B}(H_0, H)$ with the closed range and $f : X \to H$ be a Bochner integrable function and ck-frame for H with respect to H_0. Then $k^*(S_f|_{\mathcal{R}(k)})^{-1}\pi_{S_f(\mathcal{R}(k))}f$ is a ck-dual of $\pi_{\mathcal{R}(k)}f$ with bounds B^{-1} and $A^{-1}\|k\|^2\|k^\dagger\|^2$, respectively, where A and B are ck-frame bounds for f.*

Proof. Since

$$
k^* \left(S_f|_{\mathcal{R}(k)}\right)^{-1} \pi_{S_f(\mathcal{R}(k))} \in \mathcal{B}(H, H_0),
$$

then, by Lemma 1.9, $k^*(S_f|_{\mathcal{R}(k)})^{-1}\pi_{S_f(\mathcal{R}(k))}f$ is a c-Bessel mapping for H_0 and moreover by Lemma 1.2 we get $S_f^* = \pi_{\mathcal{R}(k)}S_f|_{S_f(\mathcal{R}(k))}$. Hence, for each $h \in \mathcal{R}(k)$ and $h_0 \in H_0$, we have

$$
\begin{aligned}
\langle h_0, k^*h\rangle &= \left\langle \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1} S_f \right)^* kh_0, h \right\rangle \\
&= \left\langle S_f^* \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1} \right)^* kh_0, h \right\rangle \\
&= \left\langle \pi_{\mathcal{R}(k)} S_f|_{S(\mathcal{R}(k))} \pi_{S_f(\mathcal{R}(k))} \left((S_f|_{\mathcal{R}(k)})^{-1} \right)^* kh_0, h \right\rangle \\
&= \left\langle T_f \left(\left\langle \pi_{S_f(\mathcal{R}(k))} \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1} \right)^* kh_0, f \right\rangle \right), \pi_{\mathcal{R}(k)}h \right\rangle \\
&= \int_X \left\langle \pi_{S_f(\mathcal{R}(k))} = \left((S_f|_{\mathcal{R}(k)})^{-1} \right)^* kh_0, f(x) \right\rangle \left\langle f(x), \pi_{\mathcal{R}(k)}h \right\rangle \, d\mu \\
&= \int_X \left\langle h_0, k^* \left(S_f|_{\mathcal{R}(k)}\right)^{-1} \pi_{S_f(\mathcal{R}(k))} f(x) \right\rangle \left\langle \pi_{\mathcal{R}(k)} f(x), h \right\rangle d\mu.
\end{aligned}
$$

Therefore,

$$
\langle k^*h, h_0\rangle = \int_X \left\langle h, \pi_{\mathcal{R}(k)} f(x) \right\rangle \left\langle k^*(S_f|_{\mathcal{R}(k)})^{-1} \pi_{S_f(\mathcal{R}(k))} f(x), h_0 \right\rangle d\mu.
$$

Now, put $g := k^*(S_f|_{\mathcal{R}(k)})^{-1}\pi_{S_f(\mathcal{R}(k))}f$. By Theorem 3.1, g is a ck-dual for $\pi_{\mathcal{R}(k)}f$ with the lower bound B^{-1}. Furthermore, by Theorem 2.6, for each $h \in \mathcal{R}(k)$,

$$\left\|\left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* h\right\|^2 = \left\langle S_f|_{\mathcal{R}(k)}^{-1}(S_f|_{\mathcal{R}(k)}^{-1})^* h, h\right\rangle \leq A^{-1}\|k^\dagger\|^2\| \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* h\|\|h\|.$$

Therefore, for each $h \in H$

$$\begin{aligned}\int_X |\langle h, g(x)\rangle|^2 \, d\mu &= \int_X \left|\left\langle \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* k(h), f(x)\right\rangle\right|^2 d\mu \\ &= \left\langle S_f|_{\mathcal{R}(k)} \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* k(h), \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* k(h)\right\rangle \\ &= \left\langle \left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* k(h), k(h)\right\rangle \\ &\leq \|k(h)\| \left\|\left(\left(S_f|_{\mathcal{R}(k)}\right)^{-1}\right)^* k(h)\right\| \\ &\leq \|k(h)\| A^{-1}\|k^\dagger\|^2\|k(h)\| \\ &\leq A^{-1}\|k\|^2\|k^\dagger\|^2\|h\|^2,\end{aligned}$$

and the result follows. □

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References

1. S.T. Ali, J.P. Antoine, and J.P. Gazeau, *Continuous frames in Hilbert spaces*, Ann.Phys., 222 (1993), pp. 1-37.
2. S.T. Ali, J.P. Antoine, and J.P. Gazeau, Coherent States, *Wavelets and their Generalizations*, Springer Graduate Texts in Contemporary Physics, 1999.
3. F. Arabyani and A.A. Arefijamal, *Some constructions of k-frames and their duals*, Rocky Mountain., 47(6)(2017), pp. 1749-1764.
4. J. Benedetto, A. Powell, and O. Yilmaz, *Sigma-Delta quantization and finite frames*, IEEE Trans. Inform.Th., 52(2006), pp. 1990-2005.
5. H. Bolcskel , F. Hlawatsch, and H.G Feichyinger, *Frame-Theoretic analysis of oversampled filter bank*, IEEE Trans. Signal Processing.,46(12)(1998), pp. 3256- 3268.
6. E.J. Candes and D.L. Donoho, *New tight frames of curvelets and optimal representation of objects with piecewise C^2 singularities*, Comm. Pure and App. Math.,56 (2004), pp. 216-266.
7. P.G. Casazza and G. Kutyniok, *Frame of subspaces*, Contemp. Math. 345, Amer. Math. Soc., Providence, RI., (2004), pp. 87-113.

8. P.G. Casazza, G. Kutyniok, and S. Li, *Fusion frames and Distributed Processing*, Appl. Comput. Harmon. Anal.,25 (2008), pp. 114-132.
9. P.G. Casazza and J. Kovaĉević, *Equal-norm tight frames with erasures*, Adv. Comput. Math., 18 (2003), pp. 387-430.
10. O. Christensen, *An Introduction to Frames and Riesz Bases* 2nd ed. Birkhäuser,Boston, 2016.
11. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal Expansions*, J. Math. Phys., 27(1986), pp. 1271-1283.
12. R.G. Douglas,*On majorization, factorization and range inclusion of operators on Hilbert spaces*, Proc. Amer. Math. Soc., 17(2) (1966), pp. 413-415.
13. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonik Fourier series*, Trans. Amer. Math. Soc.,72 (1952), pp. 341-366.
14. M.H. Faroughi and E. Osgooei, *C-Frames and C-Bessel Mappings*, Bull. Iranian Math. Soc., 38(1) (2012), pp. 203-222.
15. M. Fornasier and H. Rauhut, *Continuous frames, function spaces, and the discretization problem*, J. Fourier Anal. Appl., 11(3) (2005), pp.245-287.
16. J.P. Gabardo and D. Han, *Frames associated with measurable spaces*, Adv. Comput. Math., 18(2003), pp.127-147.
17. L.Găvruţa, *Frames for operators*, Appl. Comput. Harmon. Anal.,32 (2012), pp. 139-144.
18. B. Hassibi, B. Hochwald, A. Shokrollahi, and W. Sweldens, *Representation theory for high-rate multiple-antenna code design*, IEEE Trans. Inform.Theory., 47 (2001), pp. 2335-2367.
19. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhuser Boston, 2011.
20. M. Mirzaee, M. Rezaei, and M.A. Jafarizadeh, *Quantum tomography with wavelet transform in Banach space on homogeneous space*, Eur. Phys. J. B., 60 (2007), pp. 193-201.
21. A. Rahimi A. Najati, and Y.N. Dehgan, *Continuous frame in Hilbert space*, Methods Func. Anal. Top., 12 (2006), pp. 170-182.
22. A. Rahimi, A. Najati, and M H. Faroughi, *Continuous and discrete frames of subspaces in Hilbert spaces*, Southeast Asian Bull. Math., 32 (2008), pp. 305-324.
23. M. Rahmani, *On some properties of c-frames*, J. Math. Research with Appl., 37(4) (2017), pp. 466-476.
24. W. Rudin, *Functional Analysis*, New York, Tata Mc Graw-Hill Editions, 1973.
25. W. Rudin, *Real and Complex Analysis*, New York, Tata Mc Graw-Hill Editions, 1987.

26. S. Sakai, *C^*-Algebras and W^*-Algebras*, New York, Springer-Verlag, 1998.
27. X. Xiao, Y. Zhu, and L. Găvruţa, *Some Properties of k-frames in Hilbert Spaces*, Results. Math., 63 (2012), pp.1243-1255.

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