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Abstract. In this paper, we define a new concept of factorization for a bounded bilinear mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, depended on a natural number n and a cardinal number κ; which is called n-factorization property of level κ. Then we study the relation between n-factorization property of level κ for $\mathcal{X}^*$ with respect to f and automatically boundedness and w^*-w^*-continuity and also strong Arens irregularity. These results may help us to prove some previous problems related to strong Arens irregularity more easier than old. These include some results proved by Neufang in [20] and [22]. Some applications to certain bilinear mappings on convolution algebras, on a locally compact group, are also included. Finally, some solutions related to the Ghahramani-Lau conjecture is raised.

1. Introduction

The factorization property is one of the important properties of some algebraic structures, such as algebras, modules, and in general bilinear mappings, which has a key role in other properties of the algebraic structures. For examples it is a useful property for studying the amenability of groups [16, 23] and so group algebras [17], topological centers [4, 18] and in particular Arens regularity [9, 11–13] and strong Arens irregularity [20] of some module actions and Banach algebras, automatic boundedness [22] and automatic w^*-w^*-continuity [8, 20, 22] of some module homomorphisms.

Cohen in [5], proved a factorization theorem for Banach algebras and then Hewitt extends it to the modules [14]. Another version of this

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theorem which involves 'power-factorization', due in [1, 2, 10], see also [6, Theorem 2.9.24].

The special case of left (right) factorization property of the dual of a Banach algebra via that Banach algebra and its relation with topological centers is studied in [18]. A general factorization theorem depending on a cardinal number κ was applied in [22] and [19, 20] for solving Hofmeier-Wittstock's conjecture [15] concerning the automatic boundedness and Ghahramani-Lau's conjecture [7], respectively. When $\kappa = 1$, then κ-factorization property of the dual of a Banach algebra A is indeed the usual right factorization of A^* via A^{**}.

In this paper, we define a new concept of factorization for a bounded bilinear mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, depended on a natural number n and a cardinal number κ; which is called n-factorization property of level κ. Then we study the relation between n-factorization property of level κ for $\mathcal{X}^*$ with respect to f and automatically boundedness and w^*-w^*-continuity and also strong Arens irregularity. These results may help us to prove some previous problems related to strong Arens irregularity more easier than old. These include some results proved by Neufang in [20] and [22]. Some applications to certain bilinear mappings on convolution algebras, on a locally compact group, are also included.

Before prosesing, let us to recall some preliminaries. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be normed spaces and $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear mapping. Following [3], the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \to \mathcal{Y}^*$ of f is defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle\,, \quad (x \in \mathcal{X}, y \in \mathcal{Y}, z^* \in \mathcal{Z}^*).$$

Similarly we can define f^{**} and f^{***} by $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$, respectively. The bounded bilinear map f^{***} is the unique extension of f for which the maps

$$\cdot \mapsto f^{***}(\cdot, y^{**}), \qquad \cdot \mapsto f^{***}(x, \cdot), \quad (x \in \mathcal{X}, y^{**} \in \mathcal{Y}^{**}),$$

are w^*-w^*-continuous. That is they are continuous when we consider the domain and codomain of these linear maps with w^*-topology. Similarly, f^{t***t} is the unique extension of f for which the maps

$$\cdot \mapsto f^{t***t}(x^{**}, \cdot), \qquad \cdot \mapsto f^{t***t}(\cdot, y), \quad (y \in \mathcal{Y}, x^{**} \in \mathcal{X}^{**}),$$

are w^*-w^*-continuous, where $f^t : \mathcal{Y} \times \mathcal{X} \to \mathcal{Z}$ is the flip map of f defined by $f^t(y, x) = f(x, y)$.

The topological centers $Z_\ell(f)$ and $Z_r(f)$ of f are defined by

$$Z_\ell(f) = \{x^{**} \in \mathcal{X}^{**};\ \begin{array}{ccc} \mathcal{Y}^{**} & \longrightarrow & \mathcal{Z}^{**} \\ y^{**} & \longrightarrow & f^{***}(x^{**}, y^{**}) \end{array} \ \text{is } w^*\text{-}w^*\text{-continuous,}\}$$

and

$$Z_r(f) = \{y^{**} \in \mathcal{Y}^{**};\ \begin{array}{ccc} \mathcal{X}^{**} & \longrightarrow & \mathcal{Z}^{**} \\ x^{**} & \longrightarrow & f^{t***t}(x^{**}, y^{**}) \end{array} \ \text{is } w^*\text{-}w^*\text{-continuous}\}.$$

We say f is (Arens) regular if $f^{***} = f^{t***t}$, or equivalently $Z_\ell(f) = \mathcal{X}^{**}$ which is equivalent to $Z_r(f) = \mathcal{Y}^{**}$. The mapping f is said to be left (respectively, right) strongly Arens irregular if $Z_\ell(f) = \mathcal{X}$ (respectively, $Z_r(f) = \mathcal{Y}$).

For the multiplication $\pi : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ of a Banach algebra A, π^{***} and π^{t***t} are called the first and second Arens products of $\mathcal{A}^{**}$, respectively. $\mathcal{A}$ is called strongly Arens irregular if and only if π is strongly Arens irregular.

2. Results

We start with the following new definition.

Definition 2.1. Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a bounded bilinear map. A linear map $T : \mathcal{X} \to \mathcal{X}$ is called a $\mathcal{Y}$-morphism for f if

$$T\left(f(x,y)\right) = f\left(T(x), y\right), \quad (x \in \mathcal{X}, y \in \mathcal{Y}).$$

Example 2.2. (i) If $\mathcal{X}$ is a right $\mathcal{A}$-module, then every right $\mathcal{A}$-module morphism T on $\mathcal{X}$ (i.e. $T(xa) = T(x)a$ for all $x \in \mathcal{X}, a \in A$) is an $\mathcal{A}$-morphism for the right module action $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$.

(ii) Let $\mathcal{X}, \mathcal{Y}$ be normed spaces and fix $\lambda \in \mathcal{Y}^*$. Then every linear map from $\mathcal{X}$ into itself is a $\mathcal{Y}$-morphism for $f_\lambda : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, where $f_\lambda(x,y) = \langle \lambda, y\rangle\, x$. Further, every linear map from $\mathcal{X}^*$ into itself is a $\mathcal{Y}^{**}$-morphism for f_λ^{**t} as well as a $\mathcal{Y}$-morphism for f_λ^{t*}.

Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a bilinear mapping. We are interested to investigate the conditions that under which a $\mathcal{Y}^{**}$-morphism $T : \mathcal{X}^* \to \mathcal{X}^*$ for f^{**t} is automatically bounded and w^*-w^*-continuous. When f is the product of a normed algebra $\mathcal{A}$, then each $\mathcal{A}^{**}$-morphism $T : \mathcal{A}^* \to \mathcal{A}^*$ for f^{**t} is an $\mathcal{A}^{**}$-morphism and some results of this paper change to the results in [20]. First we present the following theorem about w^*-w^*-continuity.

Theorem 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and suppose that $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a left strongly Arens irregular bounded bilinear map. Then*

(i) *Every bounded $\mathcal{Y}^{**}$-morphism from $\mathcal{X}^*$ into itself, for f^{**t}, is automatically w^*-w^*-continuous.*

(ii) *A bounded $\mathcal{Y}$-morphism $T : \mathcal{X}^* \to \mathcal{X}^*$ for f^{t*} is w^*-w^*-continuous if T^* is a $\mathcal{Y}$-morphism for f.*

Proof. (i) Let (ψ_α) be a net in $\mathcal{Y}^{**}$ which is w^*-convergent to $\psi \in \mathcal{Y}^{**}$ and let $h \in \mathcal{X}^*$. Then

$$\lim_\alpha \langle f^{***}(T^*(x), \psi_\alpha), h\rangle = \lim_\alpha \langle T(f^{**}(\psi_\alpha, h)), x\rangle$$

$$\begin{aligned}
&= \lim_\alpha \langle f^{**}(\psi_\alpha, T(h)), x\rangle \\
&= \lim_\alpha \langle \psi_\alpha, f^{*}(T(h), x)\rangle \\
&= \langle \psi, f^{*}(T(h), x)\rangle \\
&= \langle f^{**}(\psi, T(h)), x\rangle \\
&= \langle T(f^{**}(\psi, h)), x\rangle \\
&= \langle f^{***}(T^*(x), \psi), h\rangle .
\end{aligned}$$

It follows that $T^*(x) \in Z_\ell(f)$ and by the left strong irregularity we get $T^*(x) \in \mathcal{X}$.

Now, let (h_α) be a net in $\mathcal{X}^*$ which is w^*-convergent to zero. Then

$$\langle T(h_\alpha), x\rangle = \langle h_\alpha, T^*(x)\rangle \to 0,$$

and this says that T is w^*-w^*-continuous.

(ii) Let $\varphi \in \mathcal{X}^{**}$ and $\psi \in \mathcal{Y}^{**}$ and consider the bounded nets (x_α) and (y_β) which are w^*-convergent to φ and ψ, respectively. Then we have

$$\begin{aligned}
T^*\left(f^{***}(\varphi, \psi)\right) &= w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta T^*\left(f(x_\alpha, y_\beta)\right) \\
&= w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta f\left(T^*(x_\alpha), y_\beta\right) \\
&= f^{***}\left(T^*(\varphi), \psi\right).
\end{aligned}$$

Therefore, T^* is a $\mathcal{Y}^{**}$-morphism for f^{***}. Further, for each $h \in \mathcal{X}^*$ we have

$$\begin{aligned}
\left\langle f^{t***t}\left(T^*(x), \psi\right), h\right\rangle &= \left\langle \psi, f^{t**}\left(T^*(x), h\right)\right\rangle \\
&= \lim_\beta \left\langle T^*(x), f^{t*}(h, y_\beta)\right\rangle \\
&= \lim_\beta \left\langle x, f^{t*}\left(T(h), y_\beta\right)\right\rangle \\
&= \lim_\beta \langle T(h), f(x, y_\beta)\rangle \\
&= \langle f^{***}(x, \psi), T(h)\rangle \\
&= \langle f^{***}\left(T^*(x), \psi\right), h\rangle .
\end{aligned}$$

Similar to the proof of part (i) we conclude that $T^*(x) \in \mathcal{X}$ and T is w^*-w^*-continuous. □

The following definition, which is motivatied by [20, Definition 2.1], introduces a certain factorization property for dual space of a Banach space.

Definition 2.4. Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear mapping on normed spaces and let $n \in \mathbb{N}$ and κ be a cardinal number. We say that $\mathcal{X}^*$ has an n-factorization property of level κ with respect to f (property $F^n_\kappa(f)$, for short) if for every family $\{h_\alpha\}_{\alpha \in I} \subseteq Ball\left(\mathcal{X}^*\right)$ with

$card(I)=\kappa$, there exist the families $\left\{\psi_{\alpha}^{i}\right\}$ in $Ball\left(\mathcal{Y}^{**}\right)$ and functionals ν^{i} in $\mathcal{Z}^{*}$, $1\leq i\leq n$, such that for every α,

$$\sum_{i=1}^{n} f^{**}\left(\psi_{\alpha}^{i},\nu^{i}\right)=h_{\alpha}.$$

Example 2.5. Let $f:\begin{bmatrix}\mathbb{C} & \mathbb{C}\\ 0 & \mathbb{C}\end{bmatrix}\times\begin{bmatrix}\mathbb{C}\\ \mathbb{C}\end{bmatrix}\to\begin{bmatrix}\mathbb{C}\\ \mathbb{C}\end{bmatrix}$ be defined by $f\left(\begin{bmatrix}a & b\\ 0 & c\end{bmatrix},\begin{bmatrix}t\\ s\end{bmatrix}\right)=\begin{bmatrix}at+bs\\ cs\end{bmatrix}$. Then for each cardinal number κ, $\begin{bmatrix}\mathbb{C} & \mathbb{C}\\ 0 & \mathbb{C}\end{bmatrix}^{*}$ has the property $F_{\kappa}^{2}(f)$. But it doesn't have the property $F_{\kappa}^{1}(f)$. Since it is easy to verify that $f^{**}\left(\begin{bmatrix}t\\ s\end{bmatrix},\begin{bmatrix}k\\ l\end{bmatrix}\right)=\begin{bmatrix}tk & sk\\ 0 & sl\end{bmatrix}$ and so for each $\left\{\begin{bmatrix}a_{\alpha} & b_{\alpha}\\ 0 & c_{\alpha}\end{bmatrix}\right\}$ of cardinality κ with $|a_{\alpha}|+|b_{\alpha}|+|c_{\alpha}|\leq 1$, for each α, we have

$$\begin{bmatrix}a_{\alpha} & b_{\alpha}\\ 0 & c_{\alpha}\end{bmatrix}=f^{**}\left(\begin{bmatrix}a_{\alpha}\\ b_{\alpha}\end{bmatrix},\begin{bmatrix}1\\ 1\end{bmatrix}\right)+f^{**}\left(\begin{bmatrix}0\\ c_{\alpha}-b_{\alpha}\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix}\right).$$

But if it has the property $F_{\kappa}^{1}(f)$, then

$$\begin{aligned}\begin{bmatrix}a_{\alpha} & b_{\alpha}\\ 0 & c_{\alpha}\end{bmatrix}&=f^{**}\left(\begin{bmatrix}k_{\alpha}\\ l_{\alpha}\end{bmatrix},\begin{bmatrix}t\\ s\end{bmatrix}\right)\\ &=\begin{bmatrix}k_{\alpha}t & l_{\alpha}t\\ 0 & l_{\alpha}s\end{bmatrix}.\end{aligned}$$

That is $\frac{b_{\alpha}}{c_{\alpha}}=\frac{t}{s}$ is constant, which is impossible for many examples.

Remark 2.6. (i) If A is a Banach algebra and π is the multiplication of A, then the 1-factorization property of level κ with respect to π (π^{t}) for A^{*}, is the left (right) A^{**}-factorization property of level κ, as introduced in [20].

(ii) If $m>n$ and $\mathcal{X}^{*}$ has the property $F_{\kappa}^{n}(f)$, then it has the property $F_{\kappa}^{m}(f)$. Since for every family $\{h_{\alpha}\}_{\alpha\in I}\subseteq Ball(\mathcal{X}^{*})$ with $card(I)=\kappa$, there exist the families $\{\psi_{\alpha}^{i}\}$ in $Ball(\mathcal{Y}^{**})$ and functionals ν^{i} in $\mathcal{Z}^{*}$, $1\leq i\leq n$, such that for every α,

$$\begin{aligned}h_{\alpha}&=\sum_{i=1}^{n} f^{**}(\psi_{\alpha}^{i},\nu^{i})\\ &=\sum_{i=1}^{n} f^{**}(\psi_{\alpha}^{i},\nu^{i})+\sum_{i=n+1}^{m} f^{**}(0,0).\end{aligned}$$

Also if $\kappa > \kappa'$ and $\mathcal{X}^*$ has the property $F_\kappa^n(f)$, then it has the property $F_{\kappa'}^n(f)$. Since for every family $\{h_\alpha\}_{\alpha\in J} \subseteq Ball(\mathcal{X}^*)$ with $card(J) = \kappa'$, we can consider a family $\{h_\beta\}_{\beta\in I} \subseteq Ball(\mathcal{X}^*)$ with $card(I) = \kappa$ such that $\{h_\alpha\}$ is a subnet of $\{h_\beta\}$ which factors.

Theorem 2.7. *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a bounded bilinear mapping on normed spaces and $\mathcal{X}^*$ has the property $F_\kappa^m(f)$, for some $m \in \mathbb{N}$ and $\kappa \geq \aleph_0$. Then every $\mathcal{Y}^{**}$-morphism $T : \mathcal{X}^* \to \mathcal{X}^*$ for f^{***t} is automatically bounded.*

Proof. If T is not bounded, then for each $n \in \mathbb{N}$ there is an element $h_n \in Ball(\mathcal{X}^*)$ such that $\|T(h_n)\| \geq n$. On the other hand, since $\mathcal{X}^*$ has the property $F_\kappa^m(f)$ there exist the families $\{\psi_n^i\} \in Ball(\mathcal{Y}^{**})$ and $h^i \in \mathcal{X}^*$ such that for each $n \in \mathbb{N}$, $h_n = \sum_{i=1}^m f^{**}(\psi_n^i, h^i)$. It follows that, for each $n \in \mathbb{N}$,

$$\begin{aligned} n &\leq \|T(h_n)\| \\ &\leq \left\| T\left(\sum_{i=1}^m f^{**}\left(\psi_n^i, h^i\right)\right)\right\| \\ &= \left\| \sum_{i=1}^m f^{**}\left(\psi_n^i, T(h^i)\right)\right\| \\ &\leq \sum_{i=1}^m \|f^{**}\| \left\|T(h^i)\right\|, \end{aligned}$$

which is a contradiction. □

We also quote the following definition from [21].

Definition 2.8. Let $\mathcal{X}$ be a Banach space and $\kappa \geq \aleph_0$ be a cardinal number.

(i) A functional $\Phi \in \mathcal{X}^{**}$ is called w^*-κ-continuous if for all nets $(x_\alpha)_{\alpha\in I} \subseteq Ball(\mathcal{X}^*)$ of cardinality $\aleph_0 \leq |I| \leq \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have $\langle \Phi, x_\alpha\rangle \to 0$.

(ii) We say that $\mathcal{X}$ has the Mazur property of level κ (property M_κ, for short) if every w^*-κ-continuous functional in $\mathcal{X}^{**}$ actually is w^*-continuous, that is, an element of $\mathcal{X}$.

The following result investigates the strong Arens irregularity of $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ under certain conditions.

Theorem 2.9. *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear map on normed spaces.*

(i) *If $\mathcal{X}$ has the property M_κ and $\mathcal{X}^*$ has the property $F^n_\kappa(f)$, for some $n \in \mathbb{N}$ and $\kappa \geq \aleph_0$, then f is left strongly Arens irregular.*
(ii) *If $\mathcal{Y}$ has the property M_κ and $\mathcal{Y}^*$ has the property $F^n_\kappa(f^t)$, for some $n \in \mathbb{N}$ and $\kappa \geq \aleph_0$, then f is right strongly Arens irregular.*

Proof. (i) Let $\phi \in Z_\ell(f)$ and (h_α) be a net of cardinality κ in $Ball(\mathcal{X}^*)$ such that $h_\alpha \xrightarrow{w^*} 0$, since $\mathcal{X}$ has the property M_κ, it suffices to show that $\langle \phi, h_\alpha \rangle$ is convergent to zero.
Since $\mathcal{X}^*$ has the property $F^n_\kappa(f)$, there exist the nets $\left(\psi^i_\alpha\right)$ in $Ball\left(\mathcal{Y}^{**}\right)$ and ν_i in $\mathcal{Z}^*$ such that for each α, $\sum\limits_{i=1}^{n} f^{**}\left(\psi^i_\alpha, \nu_i\right) = h_\alpha$.

Let $\left(\langle \phi, h_{\alpha_\beta} \rangle\right)$ be a convergent subnet. Since $nBall\left(\mathcal{Y}^{**}\right)$ is w^*-compact and $\left\{\sum\limits_{i=1}^{n} \psi^i_{\alpha_\beta}\right\} \subseteq nBall\left(\mathcal{Y}^{**}\right)$ there is a subnet $\left\{\sum\limits_{i=1}^{n} \psi^i_{\alpha_{\beta_\gamma}}\right\}_\gamma$, w^*-converging to an element $\psi \in \mathcal{Y}^{**}$. On the other hand, since $Ball\left(\mathcal{Y}^{**}\right)$ is w^*-compact, by a similar method as above there are subnets

$$\left\{\psi^1_{\alpha_{\beta_{\gamma_{\lambda_1}}}}\right\}_{\lambda_1}, \left\{\psi^2_{\alpha_{\beta_{\gamma_{\lambda_1 \lambda_2}}}}\right\}_{\lambda_2}, \ldots, \left\{\psi^n_{\alpha_{\beta_{\gamma_{\lambda_1 \cdots \lambda_n}}}}\right\}_{\lambda_n},$$

which are w^*-convergent to $\psi_1, \psi_2, \ldots, \psi_n$, respectively. Now
$\left\{\sum\limits_{i=2}^{n} \psi^i_{\alpha_{\beta_{\gamma_{\lambda_1}}}}\right\}_{\lambda_1}$ is w^*-convergent to $\psi - \psi_1$,
$\left\{\sum\limits_{i=3}^{n} \psi^i_{\alpha_{\beta_{\gamma_{\lambda_1 \lambda_2}}}}\right\}_{\lambda_2}$ is w^*-convergent to $\psi - \psi_1 - \psi_2$,
and by continuing this argument, $\left\{\psi^i_{\alpha_{\beta_{\gamma_{\lambda_1 \cdots \lambda_n}}}}\right\}_{\lambda_n}$ is w^*-convergent to $\psi_n = \psi - \sum\limits_{i=1}^{n-1} \psi_i$. So $\psi = \sum\limits_{i=1}^{n} \psi_i$ and after passing to subnets we can assume $\psi^i_{\alpha_{\beta_\gamma}} \xrightarrow{w^*} \psi_i$ for each $i = 1, \ldots, n$.

Now we have for each $x \in \mathcal{X}$,

$$\begin{aligned}
\left\langle \sum_{i=1}^{n} f^{**}(\psi_i, \nu_i), x \right\rangle &= \sum_{i=1}^{n} \langle \psi_i, f^*(\nu_i, x) \rangle \\
&= \sum_{i=1}^{n} \lim_\gamma \left\langle \psi^i_{\alpha_{\beta_\gamma}}, f^*(\nu_i, x) \right\rangle \\
&= \lim_\gamma \left\langle \sum_{i=1}^{n} f^{**}\left(\psi^i_{\alpha_{\beta_\gamma}}, \nu_i\right), x \right\rangle \\
&= \lim_\gamma \left\langle h_{\alpha_{\beta_\gamma}}, x \right\rangle
\end{aligned}$$

$$= 0.$$

Hence, from $\phi \in Z_\ell(f)$ we get

$$\begin{aligned}
\lim_\beta \langle \phi, h_{\alpha_\beta} \rangle &= \lim_\gamma \left\langle \phi, h_{\alpha_{\beta_\gamma}} \right\rangle \\
&= \lim_\gamma \left\langle \phi, \sum_{i=1}^n f^{**}\left(\psi^i_{\alpha_{\beta_\gamma}}, \nu_i\right) \right\rangle \\
&= \sum_{i=1}^n \lim_\gamma \left\langle f^{***}\left(\phi, \psi^i_{\alpha_{\beta_\gamma}}\right), \nu_i \right\rangle \\
&= \sum_{i=1}^n \langle f^{***}(\phi, \psi_i), \nu_i \rangle \\
&= \left\langle \phi, \sum_{i=1}^n f^{**}(\psi_i, \nu_i) \right\rangle \\
&= 0.
\end{aligned}$$

Therefore, every convergent subnet converges to zero, and so $\langle \phi, h_\alpha \rangle$ is convergent to zero.

(ii) follows from (i) and the fact that $Z_\ell\left(f^t\right) = Z_r(f)$. □

Corollary 2.10. *Let $\kappa \geq \aleph_0$, $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a bilinear mapping. If $\mathcal{X}$ has the property M_κ and $\mathcal{X}^*$ has the property $F^n_\kappa(f)$, for some $n \in \mathbb{N}$, then every left $\mathcal{Y}^{**}$-morphism $T : \mathcal{X}^* \to \mathcal{X}^*$ for f^{***t} is automatically bounded and w^*-w^*-continuous.*

Proof. Apply Theorems 2.7, 2.9 and 2.3. □

3. Aplications to Convolution Algebras

Let G be a locally compact group. The least cardinality of a compact covering of G is called the compact covering number of G and is denoted by $\kappa(G)$. Also the least cardinality of an open basis at the neutral element of G is called the local weight of G and will be denoted by $\chi(G)$. For every infinite locally compact group G, we have $|G| = \kappa(G)2^{\chi(G)}$ [16].

In this section, G is a locally compact non-compact group with a compact covering number $\kappa(G)$ and so $\kappa(G) \geq \aleph_0$. We study the factorization property of level $\kappa(G)$ with respect to certain bilinear mappings for dual of a convolution algebra.

Example 3.1. Define $f : L^1(G) \times M(G) \to M(G)$ by $f(g, \mu) = g * \mu \quad (g \in L^1(G), \mu \in M(G))$. For a family $\{h_i\}_{i \in I} \subseteq Ball(L^\infty(G))$ of cardinality $\kappa(G)$, let φ_i be a Hann Banach extension of h_i to $M(G)$, $\quad (i \in$

I). Mimic the methods of [20, Theorem 3.2], set

$$h := \sum_{\alpha} \sum_{i} i\left(\delta_{y^{-1}(\alpha,i)}\right) \Box \chi_{K(\alpha,i)} \varphi_i \in M(G)^*, \quad (w^*\text{-limits}),$$

and

$$\psi_j := w^*\text{-}\lim_{\beta \to \mathfrak{F}} i\left(\delta_{y(\beta,j)}\right) \in Ball\left(M(G)^{**}\right), \quad (j \in I),$$

where $\mathfrak{F}$ is an ultrafilter on I which dominates the order filter and $\Box$ is denoted for the second adjoint of the convolution of $M(G)$. Also in the sequel the first adjoint of the convolution of $M(G)$ is denoted by $\Box$. Now, we see in [20, Theorem 3.2] that for each $j \in I$, $\psi_j \Box h = \varphi_j$ and so for every $g \in L^1(G)$ and every $j \in I$,

$$\begin{aligned}
\langle f^{**}(\psi_j, h), g\rangle &= \langle \psi_j, f^*(h,g)\rangle \\
&= \lim_{\beta \to \mathfrak{F}} \left\langle f^*(h,g), i\left(\delta_{y(\beta,j)}\right)\right\rangle \\
&= \lim_{\beta \to \mathfrak{F}} \left\langle h, f\left(g, i\left(\delta_{y(\beta,j)}\right)\right)\right\rangle \\
&= \lim_{\beta \to \mathfrak{F}} \left\langle h, g * i\left(\delta_{y(\beta,j)}\right)\right\rangle \\
&= \lim_{\beta \to \mathfrak{F}} \left\langle h \Box g, i\left(\delta_{y(\beta,j)}\right)\right\rangle \\
&= \langle \psi_j, h \Box g\rangle \\
&= \langle \psi_j \Box h, g\rangle \\
&= \langle \varphi_j, g\rangle \\
&= \langle h_j, g\rangle .
\end{aligned}$$

Therefore, $L^\infty(G)$ has the property $F^1_{\kappa(G)}(f)$. On the other hand, since $L^1(G)$ has the Mazur property of level $\kappa(G).\aleph_0 = \kappa(G)$ [16, Theorem 3.4], Theorem 2.9 and Corollary 2.10 imply that f is left strongly Arens irregular and every $M(G)^{**}$-morphism on $L^\infty(G)$ for f^{**t} is automatically bounded and w^*-w^*-continuous.

For another examples we conclude with some hereditary properties. In this direction, consider normed spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and $\mathcal{W}$ and bounded bilinear mappings $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ and $h : \mathcal{X} \times \mathcal{Y} \to \mathcal{W}$ and a bounded linear mapping $g : \mathcal{Z} \to \mathcal{W}$ such that $gf = h$. Then since for each $\psi \in \mathcal{Y}^{**}$ and $\eta \in \mathcal{W}^*$ we have

$$h^{**}(\psi, \eta) = f^{**}\left(\psi, g^*(\eta)\right),$$

so, if for some positive integer n, $\mathcal{X}^*$ has the property $F^n_\kappa(h)$, then it has the property $F^n_\kappa(f)$. In particular, if $\mathcal{X}^*$ has the property $F^n_\kappa(h)$ and $\mathcal{M}, \mathcal{N}$ are subspaces of $\mathcal{X}, \mathcal{W}$, respectively, and $f : \mathcal{M} \times \mathcal{Y} \to \mathcal{N}$ is a bilinear mapping such that

$$f(m,y)=h(m,y),\quad (m\in\mathcal{M}, y\in\mathcal{Y}),$$

then $\mathcal{M}^*$ has the property $F_\kappa^n(f)$. Since for the identity mapping $i:\mathcal{Y}\to\mathcal{Y}$ and the inclusion mappings $j:\mathcal{M}\to\mathcal{X}$ and $k:\mathcal{N}\to\mathcal{W}$, we have $k\circ f=h\circ(j\times i)$ and it is easy to check that $\mathcal{M}^*$ has the property $F_\kappa^n(h\circ(j\times i))$.

Similar to the above, in the case where g is one-to-one (and so g^* is onto) and $\mathcal{X}^*$ has the property $F_\kappa^n(f)$, we can conclude that it has also the property $F_\kappa^n(gf)$. In particular if $\mathcal{Z}$ is a subspace of $\mathcal{W}$ with inclusion mapping J and $\mathcal{X}^*$ has the property $F_\kappa^n(f)$, then $\mathcal{X}^*$ has the property $F_\kappa^n(Jf)$.

This results give some another examples of dual spaces which have the factorization property with respect to a bounded bilinear mapping. Some of these examples are going in the following. In particular, we find an easier proof for example 3.1.

Example 3.2. (i) Define $T:L^1(G)\times L^1(G)\to L^1(G\times G)$ and $\Theta: L^1(G)\times L^1(G)\to M(G\times G)$ by $\Theta=JT$ and $[T(f,g)](x,y)=f(x)g(y)$, where $f,g\in L^1(G), x,y\in G$ and J is the inclusion mapping from $L^1(G\times G)$ into $M(G\times G)$. Also define $S: L^1(G\times G)\to L^1(G)$ by $S(F)(x)=\int F\left(xy,y^{-1}\right)dy$, for every $F\in L^1(G\times G)$. Then since ST is nothing else than the convolution h on $L^1(G)$ and Theorem 2.1 of [22] implies that $L^\infty(G)$ has the property $F^1_{\kappa(G)}(h)$, we conclude that $L^\infty(G)$ has property $F^1_{\kappa(G)}(T)$ and so thr property $F^1_{\kappa(G)}(\Theta)$. On the other hand, $L^1(G)$ has the Mazur property of level $\kappa(G)$ so T and Θ are left strongly Arens irregular and every $L^1(G)^{**}$-morphism from $L^\infty(G)$ into itself, for T^{**t} or Θ^{**t}, is automatically bounded and w^*-w^*-continuous.

(ii) Let $f:M(G)\times M(G)\to M(G)$ be the convolution on $M(G)$ and $g:L^1(G)\times M(G)\to L^1(G)$ be its restriction, then since $M(G)^*$ has the property $F^1_{\kappa(G)}(f)$ [20, Theorem 3.2], therefore $L^\infty(G)$ has the property $F^1_{\kappa(G)}(g)$. Now since $L^1(G)$ has the property $M_{\kappa(G)}$, therefore g is left strongly Arens irregular and every $M(G)^{**}$- morphism on $L^\infty(G)$, for g^{**t}, is automatically bounded and w^*-w^*- continuous.

Also, if h is the convolution on $L^1(G)$, then since $L^\infty(G)$ has the property $F^1_{\kappa(G)}(h)$ and since $L^1(G)$ has the property $M_{\kappa(G)}$, therefore $L^1(G)$ is strongly Arens irregular and every $L^1(G)^{**}$-module morphism on $L^\infty(G)$ is automatically bounded and w^*-w^*- continuous, which is the results of [20] and [22].

4. A Problem Concerning to the Conjecture by Ghahramani-Lau

In section 1 we saw that the property $F^n_\kappa(f)$ implies the property $F^m_\kappa(f)$, for each $n \leq m$. But as we see in Example 2.5, the converse is not true. Neufang in [20] shown that for every locally compact non-compact group G, $M(G)^*$ has the $M(G)^{**}$-factorization property of level $\kappa(G)$ and then he concludes that $M(G)$ is strongly irregular for some classes of locally compact groups. The results of this paper show that we don't need this type of factorization in general and for studying the strong Arens irregularity of a Banach algebra A it is only required to factor a family with a linear span. Indeed, it is required to study the property $F^n_\kappa(f)$ for some n, instead of $F_\kappa(f)$, where f is the product of A.

Question 4.1. *Does for each non-discrete locally compact group G, $M(G)^*$ have the property $F^n_{|G|}$ with respect to the convolution of $M(G)$, for some natural number n?*

If the answer is positive, then we can conclude the Ghahramani-Lau conjecture. That is for each locally compact group G, $M(G)$ is strongly Arens irregular. Since when G is discrete and compact, it is finite and so $M(G)$ is self-adjoint and so strongly Arens irregular. If G is discrete and non-compact, then $\kappa(G) \geq 2^{\chi(G)}$ and so $M(G)$ is strongly Arens irregular [20, Theorem 3.5]. When G is non-discrete, then $|G| \geq \aleph_0$ and so it has the Mazur property of level $|G|\aleph_0 = |G|$, [20, Proposition 3.4]. So if this conjecture is positive, then Theorem 2.9 implies the Ghahramani-Lau conjecture.

Question 4.2. *Does the dual of the triangular Banach algebra $T = \begin{bmatrix} M(G) & M(G) \\ 0 & L^1(G) \end{bmatrix}$ with the usual matrix operation π have the property $F^n_{|G|}(\pi)$, for some $n \in \mathbb{N}$?*

If it has an affirmative answer, then similar to the method of Example 2.5, we can show that $M(G)^*$ has the property $F^{2n}_{|G|}$ with respect to the convolution and then by the above discussion we conclude the Ghahramani-Lau conjecture.

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