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About Subspace-Frequently Hypercyclic Operators

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Abstract. In this paper, we introduce subspace-frequently hypercyclic operators. We show that these operators are subspace-hypercyclic and there are subspace-hypercyclic operators that are not subspace-frequently hypercyclic. There is a criterion like to subspace-hypercyclicity criterion that implies subspace-frequent hypercyclicity and if an operator T satisfies this criterion, then $T \oplus T$ is subspace-frequently hypercyclic. Additionally, operators on finite spaces can not be subspace-frequently hypercyclic.

1. Introduction and preliminaries

Let X be an F-space. By an F-space, we mean a topological space, whose topology is induced by a complete and invariant metric d. Let $B(X)$ be the set of linear and continuous operators on X. We say that $T \in B(X)$ is hypercyclic if there exists a vector $x \in X$ such that $\operatorname{orb}(T,x) = \{x, Tx, T^2x, \dots\}$ be dense in X. In this case, we say that x is a hypercyclic vector for T. The notion of hypercyclicity is related to Invariant Subset Problem and it is studied for more than twenty years. The concept of frequently hypercyclicity was introduced by F. Bayart and S. Grivaux in 2006 [2]. By definition, if x is a hypercyclic vector for T, then $\operatorname{orb}(T,x)$ meets any non-empty open set $U \subseteq X$. Now if there exists a vector $x \in X$ such that $\operatorname{orb}(T,x)$ meets any non-empty open set $U \subseteq X$, in the sense of positive lower density, then we say that T is frequently hypercyclic.

Definition 1.1 ([2])**.** Let $T \in B(X)$. We say that $x \in X$ is a frequently hypercyclic vector for T if for any non-empty open set $U \subseteq X$, the set

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$\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. If T has a frequently hypercyclic vector, we say that T is frequently hyperclic.

Recall that if $A \subseteq \mathbb{N}$, then we define the lower density of A as follows:

$$\underline{\text{dens}}(A) = \liminf_{N \to +\infty} \frac{\text{card}\{n \leq N : n \in A\}}{N}. \tag{1.1}$$

The set of frequently hypercyclic vectors for T is denoted by $FHC(T)$. It is shown in [14] that $FHC(T)$ is of first category in X. Bayart and Grivaux [2], proved that Birkhoff's translation operators on $H(\mathbb{C})$ and λB on l^p, where B is the backward shift, λ is a scalar with $|\lambda| > 1$ and $1 < p \leq +\infty$, are frequently hypercyclic. By Definition 1.1, it is clear that a frequently hypercyclic operator is hypercyclic. Bonilla and Grosse-Erdmann [4], have stated a frequent hypercyclicity criterion as follows. First, recall that we say that the series $\sum_{k=1}^{\infty} x_k$ in an F-space X is unconditionally convergent if for any $\varepsilon > 0$, one can find $N \geq 1$ such that $\left\| \sum_{k \in F} x_k \right\| < \varepsilon$, for any finite subset F of natural numbers with $F \cap \{1, 2, \ldots, N\} = \phi$.

Theorem 1.2 ([4] Frequent Hypercyclicity Criterion). *Let $T \in B(X)$ be an operator on a separable F-space X. Suppose that there is a dense subset X_0 of X and there is a mapping $S : X_0 \to X_0$ such that:*

(i) $\sum_{n=1}^{\infty} T^n x$ *converges unconditionally, for all $x \in X_0$;*
(ii) $\sum_{n=1}^{\infty} S^n x$ *converges unconditionally, for all $x \in X_0$;*
(iii) $TSx = x$ *for all $x \in X_0$;*

Then T is frequently hypercyclic.

Also, we have the following interesting theorem:

Theorem 1.3 ([2]). *Let $T \in B(X)$. If T is a frequently hypercyclic operator, then T^n is frequently hypercyclic for any $n \geq 1$. Moreover, T and T^n have the same frequently hypercyclic vectors.*

One can see [5–7] for more information about frequently hypercyclic operators.

B.F. Madore and A.R. Martinez-Avendano [9], introduced the concept of subspace-hypercyclicity and subspace-transitivity. For a closed and non-empty subspace M of X, we say that T is M-hypercyclic if there exists $x \in X$ such that $\text{orb}(T, x) \cap M$ is dense in M. Also, we say that T is M-transitive if for any two non-empty open sets U and V of M, there exists $n \geq 0$ such that $T^{-n}(U) \cap V$ contains a relatively open non-empty subset of M. Bamerni, Kadets and Kilicman [1], showed that any hypercyclic operator is subspace-hypercyclic with respect to a closed and non-trivial subspace.

Theorem 1.4 ([1]). *Let $T \in B(X)$ be a hypercyclic operator. Then there exists a non-trivial and closed subspace M of X such that T is M-hypercyclic.*

One can see [8, 10, 12] for interesting subjects related to hypercyclicity and subspace-hypercyclicity.

Now we state our main definition.

Definition 1.5. Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. We say that T is subspace-frequently hypercyclic with respect to M if there exists $x \in X$ such that for any non-empty open subset U of M, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density.

In this case, we say that x is a subspace-frequently hypercyclic vector for T with respect to M. We show that the set of all subspace-frequently hypercyclic vectors for T with respect to M by $FHC_M(T)$.

By Definition 1.5, we can deduce that every frequently hypercyclic operator is subspace-frequently hypercyclic; since it is sufficient to consider $M := X$.

Example 1.6. Let $T \in B(X)$ be a frequently hypercyclic operator and let x be a frequently hypercyclic vector for T. Suppose that I is the identity operator on X, that is $Ix = x$ for any $x \in X$. If we consider $M := X \oplus \{0\}$, then $T \oplus I : X \oplus X \to X \oplus X$ is subspace-frequently hypercyclic with respect to $M \oplus \{0\}$ with subspace-frequently hypercyclic vector $x \oplus \{0\}$. Since any open and non-empty subset of M, is of the form $U \oplus \{0\}$, where U is an open and non-empty subset of X and we have:

$$\mathrm{card}\{n \in \mathbb{N} : T^n x \in U\} = \mathrm{card}\{n \in \mathbb{N} : (T \oplus I)^n (x \oplus \{0\}) \in U \oplus \{0\}\}.$$

For example, as it is said the operator $2B$ is frequently hypercyclic on l^p for any $1 \leq p < \infty$. So $2B \oplus I : l^p \oplus l^p \to l^p \oplus l^p$ is subspace-frequently hypercyclic with respect to $M := l^p \oplus \{0\}$, for any $1 \leq p < \infty$.

In Section 2, we show that subspace-frequently hypercyclic operators are subspace-hypercyclic. There is a sufficient condition for subspace-frequent hypercyclicity of weighted shifts and by this, we conclude that there is a subspace-hypercyclic operator that is not subspace-frequently hypercyclic. Additionally, operators on finite spaces can not be subspace-frequently hypercyclic.

In Section 3, we state a subspace-frequent hypercyclicity criterion, like to the subspace-hypercyclicity criterion stated by Madore and Martinez-Avendano in [9]. We show that if an operator T satisfies this criterion, then $T \oplus T$ is subspace-frequently hypercyclic.

In Section 4, we present some open questions about subspace-frequently hypercyclic operators.

2. Subspace-Frequently Hypercyclicity and Subspace-Hypercyclicity

First, we show that subspace-frequently hypercyclic operators are subspace-hypercyclic. Also, we give an example of a subspace-hypercyclic operator that is not subspace-frequently hypercyclic.

Theorem 2.1. *Let $T \in B(X)$ be a subspace-frequently hypercyclic operator with respect to a closed and non-empty subspace M of X. Then T is subspace-hypercyclic with respect to M.*

Proof. Let x be a subspace-frequently hypercyclic vector for T with respect to M. Let $U \subseteq M$ be a non-empty and open set. By definition $\underline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\}$ is positive. As a result, there exists $n > 0$ such that $T^n x \in U$. Consequently x is a subspace-hypercyclic vector for T with respect to M and hence T is M-hypercyclic. □

Remark 2.2. If we denote the set of subspace-hypercyclic vectors for T with respect to M with $HC(T, M)$, then by Theorem 2.1, we can conclude that $FHC_M(T) \subseteq HC(T, M)$.

Theorem 2.3. *Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. If there exists $p \in \mathbb{N}$ such that T^p is subspace-frequently hypercyclic with respect to M, then T is subspace-frequently hypercyclic with respect to M. Especially T is subspace-hypercyclic with respect to M.*

Proof. Suppose that there exists $p \in \mathbb{N}$ such that T^p is a subspace-frequently hypercyclic operator with respect to M. Let $x \in X$ be a subspace-frequently hypercyclic vector for T^p with respect to M. Let U be a non-empty and open subset of M. So it is clear that:

$$\{n \in \mathbb{N} : (T^p)^n x \in U\} \subseteq \{n \in \mathbb{N} : T^n x \in U\}. \tag{2.1}$$

By hypothesis, $\{n \in \mathbb{N} : (T^p)^n x \in U\}$ has positive lower density. So $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density too. Thus T is a subspace-frequently hypercyclic operator with respect to M. □

Madore and Martinez-Avendano [9], proved that if T is an operator on a finite dimensional Banach space X, then T can not be subspace-hypercyclic for any closed subspace M of X. So by Theorem 2.1, we can deduce immediately the following corollary.

Corollary 2.4. *There is not any subspace-frequently hypercyclic operator on a finite dimensional Banach space X.*

Bayart and Grivaux [2, Example 2.8], stated an equivalent condition for frequently hypercyclicity of B_w. They also showed that under special conditions, B_w is not frequently hypercyclic. We prove that under their conditions, B_w is not subspace-frequently hypercyclic.

Theorem 2.5. *Let B_w be the weighted backward shift on l^p, where $1 \leq p < \infty$, with canonical basis $\{e_i\}_{i\geq 0}$ and with weights $w = \{w_n\}_{n\geq 1}$, where $B_w(e_0) = e_0$ and $B_w(e_n) = e_{n-1}$ for $n \geq 1$. If the series $\sum\limits_{k\geq 1} \frac{1}{(w_1 \cdots w_{n_k})^p}$ is divergent for any sequence $\{n_k\}_{k\geq 1}$, with positive lower density, then B_w is not subspace-frequently hypercyclic for any closed and non-zero subspace M of l^p.*

Proof. Let M be a closed and non-zero subspace of l^p. Suppose on the contrary that x is a subspace-frequently hypercyclic vector of B_w with respect to M. Since M is a subspace of l^p, there exists $e_i \in \{e_n\}_{n\geq 0}$ such that $B(e_i, \frac{1}{2}) \cap M \neq \phi$. Without loss of generality, we can assume that $B(e_0, \frac{1}{2}) \cap M \neq \phi$. Clearly $B(e_0, \frac{1}{2}) \cap M$ is open in the relative topology of M. Let $E = \{n_k\}$ be the set of integers such that ${B_w}^{n_k} x \in B(e_0, \frac{1}{2}) \cap M$. If $n_k \in E$, then we have

$$\|{B_w}^{n_k} x - e_0\| < \frac{1}{2}.$$

Hence $|x_{n_k}|^p > \frac{1}{2(w_1 \cdots w_{n_k})^p}$. On the other hand $\sum |x_{n_k}|^p < \infty$ since $x \in l^p$. Thus $\sum\limits_{k\geq 1} \frac{1}{(w_1 \cdots w_{n_k})^p}$ must be convergent. But this is a contradiction since $E = \{n_k\}$ has positive lower density. □

In the next example, we make a subspace-hypercyclic operator that is not subspace-frequently hypercyclic.

Example 2.6. Let B_w be the weighted backward shift on $l^2(\mathbb{N})$ with weight sequence $w_n = \sqrt{\frac{n+1}{n}}$. Then B_w is a subspace-hypercyclic operator but not a subspace-frequently hypercyclic operator.

As it is proved in [2, Example 2.9], B_w is hypercyclic but not frequently hypercyclic. Now by Theorem 1.4, there exists a closed and non-trivial subspace M of X such that B_w is M-hypercyclic.

We claim that B_w is not subspace-frequently hypercyclic with respect to M or any other non-empty and closed subspace of X. For proving our claim first, note that $w_1 w_2 \cdots w_n = \sqrt{n+1}$ and hence

$$(w_1 w_2 \cdots w_n)^2 = n + 1.$$

Let $E = \{n_k\}$ be a set of positive lower density. As it is said in [3, Example 2.9], since E is a set of positive lower density, there exists a positive number δ such that for any $N \in \mathbb{N}$, there exists $N_1 \geq N$ such that $\text{card}\{N \leq n+1 \leq N_1; n \in E\} \geq \delta N_1$. Hence:

$$\sum_{\substack{N \leq n+1 \leq N_1 \\ n \in E}} \frac{1}{(\sqrt{n+1})^2} = \sum_{\substack{N \leq n+1 \leq N_1 \\ n \in E}} \frac{1}{n+1} \geq \delta N_1 \times \frac{1}{N_1} \geq \delta.$$

So $\sum\limits_{n \in E} \frac{1}{(\sqrt{n+1})^2} = \sum\limits_{n \in E} \frac{1}{n+1}$ diverges and the proof completes by using Theorem 2.5.

Motivated by Example 2.6, one can show the following corollary.

Corollary 2.7. *There exists a subspace-hypercyclic operator that is not subspace-frequently hypercyclic.*

3. Subspace-Frequent Hypercyclicity Criterion

In this section, we state a subspace-frequent hypercyclicity criterion that has similar format to subspace-hypercyclicity criterion that Madore and Martinez-Avendano stated in [9]. For this purpose, first we state to the following definition and lemmas.

The lower density of a strictly increasing sequence $\{n_k\}$ of positive integers is defined as the lower density of the corresponding subset of $\mathbb{N}$, that means:

$$\underline{\text{dens}}(n_k) = \liminf_{N \to +\infty} \frac{\text{card}\{k \in \mathbb{N} : n_k \leq N\}}{N}.$$

Lemma 3.1 ([4])**.** *Let $\{n_k\}$ be a strictly increasing sequence of positive integers. Then*

$$\underline{\text{dens}}(n_k) = \liminf_{k \to +\infty} \frac{k}{n_k}.$$

Lemma 3.2 ([4])**.** *If $\{n_k\}$ and $\{k_l\}$ are strictly increasing sequences of positive lower density, then so is $\{n_{k_l}\}$.*

Theorem 3.3 (Subspace-frequent Hypercyclicity Criterion)**.** *Let $T \in B(X)$ and M be a closed and non-empty subspace of X. Assume that Y and Z are dense subsets of M. Suppose that there exists an increasing sequence $\{n_k\}$ with positive lower density such that:*

i) *For any $y \in Y$, $T^{n_k} y \to 0$;*
ii) *For any $z \in Z$, there exists a sequence $\{x_k\} \subseteq M$ such that $x_k \to 0$ and $T^{n_k} x_k \to z$;*
iii) *$T^{n_k}(M) \subseteq M$ for any k.*

Then T is subspace-frequently hypercyclic with respect to M. Especially T is subspace-hypercyclic with respect to M.

Proof. Let U and V be non-empty open subsets of M. By hypothesis Y and Z are dense in M. So there exist $u \in Y \cap U$ and $v \in Z \cap V$. Since U is open, there exists $\varepsilon > 0$ such that $B(u, \varepsilon) \cap M \subseteq U$ and $B(v, \varepsilon) \cap M \subseteq V$. We have $u \in Y$. So $T^{n_k}u \to 0$ and therefore there exists a positive integerN_1 such that:

$$\text{for any} \quad k \geq N_1; \ \|T^{n_k}u\| < \frac{\varepsilon}{2}. \tag{3.1}$$

Also, we have $v \in Z$. Consequently there exists $\{x_k\} \subseteq M$ such that $x_k \to 0$ and $T^{n_k}x_k \to v$. Therefore there exists a positive integer N_2 such that:

$$\text{for any } k \geq N_2; \quad \|x_k\| < \varepsilon \quad \textit{and} \quad \|T^{n_k}x_k - v\| < \frac{\varepsilon}{2}. \tag{3.2}$$

Consider $N := \max\{N_1, N_2\}$. So for any $k \geq N$, we have (3.1) and (3.2). Also, $x_k + u \in U$ since:

$$\begin{aligned} \|T^{n_k}(x_k + u) - v\| &= \|T^{n_k}(x_k) + T^{n_k}(u) - v\| \\ &\leq \|T^{n_k}(x_k) - v\| + \|T^{n_k}(u)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore $T^{n_k}(U) \cap V \neq \phi$ for any $k \geq N$. Hence:

$$\{n_k\}_{k \geq N} \subseteq \{n \in N : T^n(U) \cap V \neq \phi\}.$$

But $\{n_k\}_{k \geq N}$ is of positive lower density by Lemma 3.2. Thus

$$\{n \in N : T^n(U) \cap V \neq \phi\},$$

is of positive lower density and hence T is subspace-frequently hypercyclic with respect to M. □

Corollary 3.4. *Let $T \in B(X)$ and M be a closed and non-empty subspace of X. Suppose that T satisfies the subspace-frequent hypercyclicity criterion with respect to M. Then $T \oplus T$ is subspace-frequently hypercyclic. Especially $T \oplus T$ is subspace-hypercyclic.*

Proof. We prove that $T \oplus T$ satisfies the subspace-frequent hypercyclicity criterion with respect to $M \oplus M$. By hypothesis T satisfies in the subspace-frequent hypercyclicity criterion with respect to M. So there exist Y and Z, dense subsets of M and an increasing sequence $\{n_k\}$ with positive lower density such that satisfies all three conditions of subspace-frequent hypercyclicity criterion.

Let U_1, U_2, V_1 and V_2 be non-empty open subsets of M. By hypothesis there exist $u_1 \in Y \cap U_1$, $u_2 \in Y \cap U_2$, $v_1 \in Z \cap V_1$ and $v_2 \in Z \cap V_2$. Thus there exists $\varepsilon > 0$ such that $B(u_i, \varepsilon) \cap M \subseteq U$ and $B(v_i, \varepsilon) \cap M \subseteq V$ for $i = 1, 2$. It is not hard to find a positive integer N_1 such that:

$$\text{for any} \quad k \geq N_1; \ \|T^{n_k}(u_1) + T^{n_k}(u_2)\| < \frac{\varepsilon}{2}. \tag{3.3}$$

Also, we have $v_1, v_2 \in Z$. So there exists $\{x_k\} \subseteq M$ and $\{x'_k\} \subseteq M$ such that $x_k \to 0$, $x'_k \to 0$, $T^{n_k}x_k \to v_1$ and $T^{n_k}x'_k \to v_2$. Consequently there exists a positive integer N_2 such that for any $k \geq N_2$, $\|x_k + x'_k\| < \varepsilon$ and $\|T^{n_k}(x_k + x'_k) - (v_1 + v_2)\| < \frac{\varepsilon}{2}$.

Consider $N := \max\{N_1, N_2\}$. It is not hard to see that for any $k \geq N$, we have

$$x_k + u_1 \oplus x'_k + u_2 \in U_1 \oplus U_2$$

and

$$T^{n_k}(x_k + u_1) \oplus T^{n_k}(x'_k + u_2) \in V_1 \oplus V_2$$

The remainder of the proof is similar to that of Theorem 3.3. □

Example 3.5. Consider $T := \lambda B$, where λ is a scalar with $|\lambda| > 1$ and B is the backward shift on l^2. Suppose that

$$M := \{\{a_n\} \in l^2 : a_{2k} = 0, \text{ for all } k \in \mathbb{N}\}.$$

Madore and Martinez-Avendano [9], showed that T satisfies $(i), (ii)$ and (iii), if we consider $n_k := 2k$. Also, $\{n_k\}$ has positive lower density, since

$$\underline{\text{dens}}(n_k) = \liminf_{N \to +\infty} \frac{k}{n_k} = \liminf_{N \to +\infty} \frac{k}{2k} = \frac{1}{2}.$$

So, $\lambda B \oplus \lambda B$ is subspace-transitive with respect to $M \oplus M$.

4. Some Questions

We proved that if T^n is subspace-frequently hypercyclic with respect to M for some natural number n, then T is subspace-frequently hypercyclic with respect to M. Now this question arises:

Question 1.: Let $T \in B(X)$ be a subspace-frequently hypercyclic operator with respect to a closed and non-empty subspace M of X. Can we conclude that T^n is subspace-frequently hypercyclic with respect to M for any $n \in \mathbb{N}$?

As a result of Theorem 1.4, hypercyclic operators are subspace-hypercyclic. So we would like to know the answer to the following question:

Question 2.: If $T \in B(X)$ is frequently hypercyclic, then is there a non-trivial and closed subspace M of X such that T is subspace-frequently hypercyclic with respect to M?

We say that an operator $T \in B(X)$ is chaotic if it is hypercyclic and has a dense set of periodic points in X. Bayart and Grivaux [3], posed this question: Is every chaotic operator is frequently hypercyclic? Menet [11], answered this question and proved that there exists an operator T on l^2 such that T is chaotic but not frequently hypercyclic. Now this question appears:

Question 3.: Is there a subspace-frequently hypercyclic operator that is not chaotic?

Talebi and Moosapoor [15], introduced subspace-chaotic operators. If an operator T is M-transitive and has a dense set of periodic points in M, then we say that T is M-chaotic. Now we are interested in finding the answer to the following question too:

Question 4.: Is there a subspace-frequently hypercyclic operator that is not subspace-chaotic?

Let T be an invertible operator. It is an open problem [2, Question 4.3], that if the frequently hypercyclicity of T, implies that T^{-1} is frequently hypercyclic? Also, we do not know is it true for subspace-frequently hypercyclic operators or not?

Question 5.: Let T be an invertible subspace-frequently hypercyclic operator. Can we deduce that T^{-1} is subspace-frequently hypercyclic?

Shakrin [13], showed that there are some infinite dimensional Banach spaces that do not support frequently hypercyclic operators. Now the following question appears:

Question 6.: Do subspace-frequently hypercyclic operators exist on any separable infinite dimensional Banach space?

References

1. N. Bamerni, V. Kadets and A. Kilicman, *Hypercyclic operators are subspace-hypercyclic*, J. Math. Anal. Appl., 435(2)(2016), pp. 1812-1815.
2. F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc., 358(11) (2006), pp. 5083-5117.
3. F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. London Math. Soc., 94(3) (2007), pp. 181-210.
4. A. Bonilla and K.G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*, Ergod. Theor. Dyn. Syst., 27 (2007), pp. 383-404.
5. S. Grivaux, *Frequently hypercyclic operators with irregularly visiting orbits*, J. Math. Anal. Appl., 462 (2018), pp. 542-553.

6. K.G. Grosse-Erdmann, *Frequently hypercyclic bilateral shifts*, Glasgow. Math. J., 61(2) (2019), pp. 271-286.
7. K.G. Grosse-Erdmann and A. Peris, *Frequently dense orbits*, C. R. Acad. Sci. Paris, Ser. I, 341 (2005), pp. 123-128.
8. K.G. Grosse-Erdmann and A. Peris Manguillot, *Linear chaos*, Springer, 2011.
9. B.F. Madore and R.A. Martinez-Avendano, *Subspace hypercyclicity*, J. Math. Anal. Appl., 373(2) (2011), pp. 502-511.
10. R.A. Martinez-Avendano and O. Zatarain-Vera, *Subspace-hypercyclicity for Toeplitz operators*, J. Math. Anal. Appl., 422(1) (2015), pp. 772-775.
11. Q. Menet, *Linear chaos and frequent hypercyclicity*, Trans. Amer. Math. Soc., 369(7) (2017), pp. 4977-4994.
12. H. Rezaei, *Notes on subspace-hypercyclic operators*, J. Math. Anal. Appl., 397(1) (2013), pp. 428-433.
13. S. Shkarin, *On the spectrum of frequently hypercyclic operators*, Proc. Amer. Math. Soc., 137(1) (2009), pp. 123-134.
14. T.K. Subrahmonian Moothathu, *Two remarks on frequent hypercyclicity*, J. Math. Anal. Appl., 408 (2013), pp. 843-845.
15. S. Talebi and M. Moosapoor, *Subspace-chaotic operators and subspace-weakly mixing operators*, Int. J. of Pure and Applied Math., 78 (2012), pp. 879-885.

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