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Non-Equivalent Norms on $C^b(K)$

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Abstract. Let A be a non-zero normed vector space and let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. Also, let φ be a non-zero element of A^* such that $\|\varphi\| \leq 1$. We first define a new norm $\|\cdot\|_\varphi$ on $C^b(K)$, that is a non-complete, non-algebraic norm and also non-equivalent to the norm $\|\cdot\|_\infty$. We next show that for $0 \neq \psi \in A^*$ with $\|\psi\| \leq 1$, the two norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are equivalent if and only if φ and ψ are linearly dependent. Also by applying the norm $\|\cdot\|_\varphi$ and a new product " $\cdot$ " on $C^b(K)$, we present the normed algebra $\left(C^{b\varphi}(K), \|\cdot\|_\varphi\right)$. Finally we investigate some relations between strongly zero-product preserving maps on $C^b(K)$ and $C^{b\varphi}(K)$.

1. Introduction

Let $K = \overline{B_1^{(0)}}$ be the closed unit ball of a non-zero normed vector space A and let φ be a non-zero element of A^* such that $\|\varphi\| \leq 1$. We consider $C^b(K)$ for the space of all complex-valued, bounded and continuous functions on K. It is well-known that $C^b(K)$ is a unital algebra with respect to the pointwise algebraic operations. The function 1_K is the identity of $C^b(K)$. The uniform norm on K is

$$\|f\|_\infty = \sup\left\{|f(x)| \quad \Big| \quad x \in K\right\},$$

for all $f \in C^b(K)$. Clearly $\left(C^b(K), \|\cdot\|_\infty\right)$ is a commutative, unital, Banach algebra. For details concerning the Banach algebra $C^b(K)$, we refer to [1] and [9].

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Let A and B be two normed algebras. Then a linear map $T : A \longrightarrow B$ is said to be zero-product preserving, if $T(a)T(c) = 0$ whenever $ac = 0, \quad a, c \in A$. Also T is said to be strongly zero-product preserving, if for any two sequences $\{a_n\}_n, \{c_n\}_n$ in A, $T(a_n)T(c_n) \longrightarrow 0$ whenever $a_n c_n \longrightarrow 0$. Many of the basic properties concerning strongly zero-product preserving maps are investigated in [3–6].

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on A. It is obvious that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, if and only if, for each sequence $\{a_n\}_n \subseteq A$,

$$\|a_n\|_1 \longrightarrow 0 \quad \Leftrightarrow \quad \|a_n\|_2 \longrightarrow 0.$$

On the space $C^b(K)$ we define the product

$$(f \cdot g)(x) = f(x)\varphi(x)g(x), \quad x \in K,$$

for all $f, g \in C^b(K)$. Obviously $\big(C^b(K), \cdot\big)$ is an algebra that we denote it by $C^{b\varphi}(K)$. In [7] it is shown that $\big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ is a non-unital, commutative Banach algebra. Some basic properties such as, idempotent, nilpotent, zero divisor elements and also bounded approximate identities of $C^{b\varphi}(K)$ are investigated in [7]. Also some relations between character spaces of $C^{b\varphi}(K)$ and $C^b(K)$ are characterized in [7].

Let A be a Banach algebra. In [2] R. A. Kamyabi-Gol and M. Janfada defined a new product " $\cdot$ " on A by $a \cdot c = a\varepsilon c$ for all $a, c \in A$, where ε is a fixed element of the closed unit ball $\overline{B_1^{(0)}}$ of A. The pair $(A, \cdot)$ is a Banach algebra which is denoted by A_ε. Some properties such as, Arens regularity, amenability of A_ε and also derivations on A_ε are investigated in [2]. Also biflatness, biprojectivity, $\varphi-$amenability and $\varphi-$contractibility of A_ε are investigated in [8].

For a normed algebra $(A, \|\cdot\|)$, define $A^\sim$ to be the set of all equivalent classes of Cauchy sequences obtained by the relation $\{a_n\}_n \sim \{b_n\}_n$ if and only if $\lim_{n\longrightarrow\infty} \|a_n - b_n\| = 0$. For $a^\sim = [\{a_n\}_n]$ and $b^\sim = [\{b_n\}_n]$, the operations

$$\begin{aligned} a^\sim + b^\sim &= [\{a_n + b_n\}_n], \\ \lambda a^\sim &= [\{\lambda a_n\}_n], \\ a^\sim b^\sim &= [\{a_n b_n\}_n], \\ \|a^\sim\|_\sim &= \lim_{n\longrightarrow\infty} \|a_n\|, \end{aligned}$$

make $A^\sim$ into a Banach algebra containing a dense subalgebra that is isometric with A. $(A^\sim, \|\cdot\|_\sim)$ is called the completion of A.

In this paper we first define a new norm $\|\cdot\|_\varphi$ on $C^b(K)$, that is a non-complete, non-algebraic norm and also non-equivalent to the norm $\|\cdot\|_\infty$. We next show that for $0 \neq \psi \in A^*$ with $\|\psi\| \leq 1$, the two norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are equivalent if and only if φ and ψ are linearly dependent. Also by applying the norm $\|\cdot\|_\varphi$ and a new product " $\cdot$ " on $C^b(K)$, we present the normed algebra $\left(C^{b\varphi}(K), \|\cdot\|_\varphi\right)$. We finally investigate some relations between strongly zero-product preserving maps on $C^b(K)$ and $C^{b\varphi}(K)$.

2. Non-Equivalent Norms on $C^b(K)$

In this section, let A be a non-zero normed vector space and let φ be a non-zero linear functional on A with $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. We set $\|f\|_\varphi = \|f\varphi\|_\infty$ for all $f \in C^b(K)$. Also let 1_K be the constant function on K such that $1_K(x) = 1$ for all $x \in K$. The following proposition is used repeatedly in the sequel.

Proposition 2.1. *For $f \in C^b(K)$, $f\varphi = 0$ if and only if $f = 0$.*

Proof. Let $f\varphi = 0$. So $f\Big|_{K\setminus\ker\varphi} = 0$. Choose $e \in A$ such that $\varphi(e) = 1$. Since K is convex so, $\frac{1}{n+1}\frac{e}{\|e\|} + \left(1 - \frac{1}{n+1}\right)k_0 \in K \setminus \ker\varphi$ for all $k_0 \in K \cap \ker\varphi$ and for all $n \in \mathbb{N}$. Clearly $\frac{1}{n+1}\frac{e}{\|e\|} + \left(1 - \frac{1}{n+1}\right)k_0 \longrightarrow k_0$ and by continuity of f,

$$0 = f\left(\frac{1}{n+1}\frac{e}{\|e\|} + \left(1 - \frac{1}{n+1}\right)k_0\right) \longrightarrow f(k_0).$$

This shows that $f = 0$ on K. □

Proposition 2.2. $\left(C^b(K), \|\cdot\|_\varphi\right)$ *is a non-complete normed vector space.*

Proof. Let $\|f\|_\varphi = \|f\varphi\|_\infty = 0$. Then $f\varphi = 0$. So by Proposition 2.1 $f = 0$. Clearly $\|\alpha f\|_\varphi = |\alpha|\|f\|_\varphi$ and $\|f + g\|_\varphi \leq \|f\|_\varphi + \|g\|_\varphi$ for all $f, g \in C^b(K)$. We shall show that $\|\cdot\|_\varphi$ is a non-complete norm. To this end, define $f_n : K \longrightarrow \mathbb{C}$ by,

$$f_n(x) = \frac{n\sqrt[3]{|\varphi(x)|}}{n\sqrt[3]{|\varphi(x)|^2} + 1}.$$

So $(f_n\varphi)(x) = f_n(x)\varphi(x) = \frac{n\sqrt[3]{|\varphi(x)|}\varphi(x)}{n\sqrt[3]{|\varphi(x)|^2}+1}$. Hence we can conclude that $f_n\varphi \xrightarrow{\|\cdot\|_\infty} g$ where

$$g(x) = \begin{cases} 0, & x \in K \cap \ker\varphi, \\ \frac{\varphi(x)}{\sqrt[3]{|\varphi(x)|}}, & x \in K \setminus \ker\varphi. \end{cases}$$

It follows that

$$\begin{aligned} \lim_{m,n\longrightarrow\infty} \|f_n - f_m\|_\varphi &= \lim_{m,n\longrightarrow\infty} \|f_n\varphi - f_m\varphi\|_\infty \\ &= 0. \end{aligned}$$

So $\{f_n\}_n$ is a Cauchy sequence in $\big(C^b(K), \|\cdot\|_\varphi\big)$. We shall show that there is no function $h \in C^b(K)$ such that, $f_n \xrightarrow{\|\cdot\|_\varphi} h$. On the contrary, if $f_n \xrightarrow{\|\cdot\|_\varphi} h$ for some $h \in C^b(K)$ then

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|f_n\varphi - h\varphi\|_\infty &= \lim_{n\longrightarrow\infty} \|f_n - h\|_\varphi \\ &= 0.\end{aligned}$$

Hence $g = h\varphi$. So $h(x) = \frac{g(x)}{\varphi(x)} = \frac{\frac{\varphi(x)}{\sqrt[3]{|\varphi(x)|}}}{\varphi(x)} = \frac{1}{\sqrt[3]{|\varphi(x)|}}$ for all $x \in K \setminus \ker\varphi$. This shows that, h is not a bounded and continuous function on K, that is a contradiction. So $\big(C^b(K), \|\cdot\|_\varphi\big)$ is not complete. □

Corollary 2.3. *$\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are not equivalent norms.*

Proof. Since by Proposition 2.2 $\big(C^b(K), \|\cdot\|_\varphi\big)$ is not complete, so $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are not equivalent norms. □

In the following example we present a sequence $\{f_n\}_n$ in $C^b(K)$ such that, $\|f_n\|_\varphi \to 0$, whereas $\|f_n\|_\infty \nrightarrow 0$.

Example 2.4. Define $f_n : K \longrightarrow \mathbb{C}$ by $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$.
Clearly $f_n(0) = 1 \nrightarrow 0$. So $\|f_n\|_\infty \nrightarrow 0$. But

$$\begin{aligned}|f_n(x)\varphi(x)| &= f_n(x)|\varphi(x)| \\ &= \frac{|\varphi(x)| - |\varphi(x)|^2}{1 + n|\varphi(x)|} \\ &\leq \frac{1}{n},\end{aligned}$$

for all $x \in K$. So $\|f_n\|_\varphi = \|f_n\varphi\|_\infty \to 0$.

In the following proposition, we shall show that for two non-zero linear functionals $\varphi, \psi \in A^*$ such that $\|\varphi\| \leq 1, \|\psi\| \leq 1$, $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are non-equivalent norms whenever φ and ψ are linearly independent.

Proposition 2.5. *The norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are equivalent if and only if φ and ψ are linearly dependent.*

Proof. Let $\psi = \lambda\varphi$ for some $0 \neq \lambda \in \mathbb{C}$. So,

$$\begin{aligned}\|f\|_\psi &= \|f\psi\|_\infty \\ &= \|\lambda f\varphi\|_\infty \\ &= |\lambda|\|f\varphi\|_\infty \\ &= |\lambda|\|f\|_\varphi.\end{aligned}$$

This shows that $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are equivalent. For the converse, let $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ be equivalent norms, and on the contrary, let φ and ψ be linearly independent. So $\ker\varphi \nsubseteq \ker\psi$. Hence there exists an element $x_0 \in \ker\varphi$ such that $\psi(x_0) \neq 0$. Define $f_n : K \longrightarrow \mathbb{C}$ by $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$ for all $x \in K$. By Example 2.4 we have, $\|f_n\|_\varphi = \|f_n\varphi\|_\infty \to 0$, whereas

$$\begin{aligned}\|f_n\|_\psi &= \|f_n\psi\|_\infty \\ &\geq \left|(f_n\psi)\left(\frac{x_0}{\|x_0\|}\right)\right| \\ &= \frac{|\psi(x_0)|}{\|x_0\|}.\end{aligned}$$

Thus $\|f_n\|_\psi \nrightarrow 0$. This shows that $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are non-equivalent norms, that is a contradiction. □

Remark 2.6. Since K is connected and $|\varphi| : K \longrightarrow \mathbb{C}$ is continuous, so $|\varphi|(K) := \left\{|\varphi(x)| \quad \middle| \quad x \in K\right\}$ is connected in $\mathbb{R}$. Thus, $|\varphi|(K) = [0, a)$ or $|\varphi|(K) = [0, a]$ for some $a > 0$. It follows that,

$$\begin{aligned}\|\varphi\| &= \|\varphi\|_\infty \\ &= \sup\{|\varphi(x)| \quad | \quad x \in K\} \\ &= a.\end{aligned}$$

So, $|\varphi|(K) = [0, \|\varphi\|_\infty)$ or $|\varphi|(K) = [0, \|\varphi\|_\infty]$.

Theorem 2.7. *The norm $\|\cdot\|_\varphi$ is not an algebraic norm on $C^b(K)$.*

Proof. Define $f_n : K \longrightarrow \mathbb{C}$ and $g_n : K \longrightarrow \mathbb{C}$ by $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$ and $g_n(x) = \frac{n}{1+n|\varphi(x)|}$ for all $x \in K$. So,

$$\begin{aligned}|(f_n\varphi)(x)| &= \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}|\varphi(x)| \\ &= \frac{|\varphi(x)| - |\varphi(x)|^2}{1+n|\varphi(x)|},\end{aligned}$$

$$\begin{aligned}|(g_n\varphi)(x)| &= \frac{n}{1+n|\varphi(x)|}|\varphi(x)| \\ &= \frac{n|\varphi(x)|}{1+n|\varphi(x)|},\end{aligned}$$

and

$$\begin{aligned}|(f_n g_n\varphi)(x)| &= \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}\frac{n}{1+n|\varphi(x)|}|\varphi(x)| \\ &= \frac{n|\varphi(x)| - n|\varphi(x)|^2}{(1+n|\varphi(x)|)^2},\end{aligned}$$

for all $x \in K$. By Example 2.4 we have $\|f_n\varphi\|_\infty \longrightarrow 0$.
Set $z = |\varphi(x)|$ for $x \in K$. So, by Remark 2.6 we have,

$$\|g_n\varphi\|_\infty = \sup\left\{\frac{nz}{1+nz} \quad \Bigg| \quad z \in |\varphi|(K)\right\},$$

and

$$\|f_n g_n\varphi\|_\infty = \sup\left\{\frac{nz-nz^2}{(1+nz)^2} \quad \Bigg| \quad z \in |\varphi|(K)\right\}.$$

It follows that,

$$\|g_n\varphi\|_\infty = \frac{n\|\varphi\|_\infty}{1+n\|\varphi\|_\infty}, \quad n \in \mathbb{N},$$
$$\|f_n g_n\varphi\|_\infty = \frac{n^2+n}{4n^2+8n+4}, \quad n > \frac{1}{\|\varphi\|_\infty} - 2.$$

Indeed, let $G_n(z) = \frac{nz}{1+nz}$ and $H_n(z) = \frac{nz-nz^2}{(1+nz)^2}, z \in |\varphi|(K)$.

Clearly ${G_n}'(z) = \frac{n}{(1+nz)^2}$. So G_n is increasing on $|\varphi|(K)$ and consequently,

$$\begin{aligned}\|g_n\varphi\|_\infty &= \|G_n\|_\infty \\ &= \lim_{z\longrightarrow\|\varphi\|_\infty} G_n(z) \\ &= \frac{n\|\varphi\|_\infty}{1+n\|\varphi\|_\infty}.\end{aligned}$$

Obviously the only root of the equation ${H_n}'(z) = \frac{(-n^2-2n)z+n}{(1+nz)^3} = 0$ is $z = \frac{1}{n+2}$. Thus if $n > \frac{1}{\|\varphi\|_\infty} - 2$, or equivalently, $\frac{1}{n+2} < \|\varphi\|_\infty$, then H_n is increasing on $\left[0, \frac{1}{n+2}\right]$ and decreasing on $\left[\frac{1}{n+2}, \|\varphi\|_\infty\right)$. Therefore,

$$\begin{aligned}\|f_n g_n\varphi\|_\infty &= \|H_n\|_\infty \\ &= H_n\left(\frac{1}{n+2}\right) \\ &= \frac{n^2+n}{4n^2+8n+4}.\end{aligned}$$

We claim that there is no $\alpha \in \mathbb{R}^+$ such that $\|fg\|_\varphi \le \alpha\|f\|_\varphi\|g\|_\varphi$ for all $f, g \in C^b(K)$. To obtain a contradiction, let there exists $\alpha \in \mathbb{R}^+$ such that $\|fg\|_\varphi \le \alpha\|f\|_\varphi\|g\|_\varphi$ for all $f, g \in C^b(K)$. So $\|f_n g_n\|_\varphi \le \alpha\|f_n\|_\varphi\|g_n\|_\varphi$ for all $n \in \mathbb{N}$. It follows that $\|f_n g_n\varphi\|_\infty \le \alpha\|f_n\varphi\|_\infty\|g_n\varphi\|_\infty$ for all $n \in \mathbb{N}$. Hence if $n > \frac{1}{\|\varphi\|_\infty} - 2$ we have,

$$\frac{n^2+n}{4n^2+8n+4} \le \alpha\|f_n\varphi\|_\infty \frac{n\|\varphi\|_\infty}{1+n\|\varphi\|_\infty}. \tag{2.1}$$

Letting $n \longrightarrow \infty$ in (2.1) we obtain, $\frac{1}{4} \leq \alpha \times 0 \times 1 = 0$, that is a contradiction. □

Remark 2.8. Clearly $\|\cdot\|_\varphi$ is an algebraic norm on $C^{b\varphi}(K)$. Indeed,

$$\begin{aligned}\|f \cdot g\|_\varphi &= \|f\varphi g\|_\varphi \\ &= \|f\varphi g\varphi\|_\infty \\ &\leq \|f\varphi\|_\infty \|g\varphi\|_\infty \\ &= \|f\|_\varphi \|g\|_\varphi.\end{aligned}$$

Since $\big(C^b(K), \|\cdot\|_\varphi\big)$ is a non-complete normed vector space, so $\big(C^{b\varphi}(K), \|\cdot\|_\varphi\big)$ is a non-complete normed algebra.

Let $C^{b\varphi}(K)^\sim$ be the completion of $C^{b\varphi}(K)$. Then $\big(C^{b\varphi}(K)^\sim, \|\cdot\|_{\varphi_\sim}\big)$ is a Banach algebra and $\overline{C^{b\varphi}(K)}^{\|\cdot\|_{\varphi_\sim}} = C^{b\varphi}(K)^\sim$.

In the following proposition we characterize the norm $\|\cdot\|_{\varphi_\sim}$.

Proposition 2.9. *Let $[\{f_n\}_n] \in C^{b\varphi}(K)^\sim$. Then $\|[\{f_n\}_n]\|_{\varphi_\sim} = \|g\|_\infty$ for some $g \in C^b(K)$.*

Proof. Let $[\{f_n\}_n] \in C^{b\varphi}(K)^\sim$. Since $\{f_n\}_n$ is Cauchy in $\big(C^{b\varphi}(K), \|\cdot\|_\varphi\big)$, so

$$\begin{aligned}0 &= \lim_{m,n\longrightarrow\infty} \|f_m - f_n\|_\varphi \\ &= \lim_{m,n\longrightarrow\infty} \|f_m\varphi - f_n\varphi\|_\infty.\end{aligned}$$

It follows that $\{f_n\varphi\}_n$ is a Cauchy sequence in $\big(C^b(K), \|\cdot\|_\infty\big)$. So there exists $g \in C^b(K)$ such that $f_n\varphi \xrightarrow{\|\cdot\|_\infty} g$. Hence $\|f_n\varphi\|_\infty \longrightarrow \|g\|_\infty$. Thus by definition,

$$\begin{aligned}\|[\{f_n\}_n]\|_{\varphi_\sim} &= \lim_{n\longrightarrow\infty} \|f_n\|_\varphi \\ &= \lim_{n\longrightarrow\infty} \|f_n\varphi\|_\infty \\ &= \|g\|_\infty.\end{aligned}$$

□

3. Strongly Zero-Product Preserving Maps on $C^b(K)$ and $C^{b\varphi}(K)$

In this section we investigate some relations between strongly zero-product preserving maps on $C^b(K)$ and $C^{b\varphi}(K)$.

Proposition 3.1. *Let $T : C^b(K) \longrightarrow C^b(K)$ be a linear map. Then $T : \big(C^b(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^b(K), \|\cdot\|_\infty\big)$ is zero-product preserving if and only if $T : \big(C^{b\varphi}(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ is so.*

Proof. Let $T : \big(C^b(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^b(K), \|\cdot\|_\infty\big)$ be a zero-product preserving map and let $f \cdot g = 0, \quad f, g \in C^{b\varphi}(K)$. So $f\varphi g = 0$ and consequently by Proposition 2.1, $fg = 0$. Therefore $T(f)T(g) = 0$ and so $T(f) \cdot T(g) = T(f)\varphi T(g) = 0$. Thus T is zero-product preserving on $C^{b\varphi}(K)$. Conversely, let $T : \big(C^{b\varphi}(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ be zero-product preserving and let $fg = 0, \quad f, g \in C^b(K)$. So $f \cdot g = 0$. It follows that $T(f)\varphi T(g) = T(f) \cdot T(g) = 0$. So by Proposition 2.1, $T(f)T(g) = 0$. Therefore T is zero-product preserving on $C^b(K)$. □

The following result shows that Proposition 3.1 is not the case when we replace strongly zero-product preserving map instead of zero-product preserving map.

Example 3.2. Define $T : \big(C^b(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^b(K), \|\cdot\|_\infty\big)$ by $T(f) = f(0)\varphi$ for all $f \in C^b(K)$. Clearly T is a linear map. Let $f_n g_n \xrightarrow{\|\cdot\|_\infty} 0$. So $f_n(0)g_n(0) \longrightarrow 0$. It follows that

$$\begin{aligned}\|T(f_n)T(g_n)\|_\infty &= \|f_n(0)g_n(0)\varphi^2\|_\infty \\ &= |f_n(0)g_n(0)|\|\varphi\|_\infty^2 \\ &\longrightarrow 0.\end{aligned}$$

So T is a strongly zero-product preserving map. We shall show that $T : \big(C^{b\varphi}(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ is not strongly zero-product preserving. To this end, let $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$ for all $n \in \mathbb{N}$ and for all $x \in K$. By previous example we have, $f_n \cdot 1_K = f_n\varphi \xrightarrow{\|\cdot\|_\infty} 0$. But

$$\begin{aligned}T(f_n) \cdot T(1_K) &= f_n(0)\varphi^3 \\ &= \varphi^3 \\ &\xrightarrow{\|\cdot\|_\infty} \varphi^3 \\ &\neq 0.\end{aligned}$$

Example 3.3. Define $T : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ by $T(f)(x) = f\left(\frac{e}{\|e\|}\right), x \in K$, where $e \in A$ is an element such that $\varphi(e) = 1$. Then,

$$T : \left(C^{b\varphi}(K), \|\cdot\|_\infty\right) \longrightarrow \left(C^{b\varphi}(K), \|\cdot\|_\infty\right),$$

and

$$T : \left(C^{b\varphi}(K), \|\cdot\|_\varphi\right) \longrightarrow \left(C^{b\varphi}(K), \|\cdot\|_\varphi\right),$$

are both strongly zero-product preserving maps. Indeed, let $f_n \cdot g_n \xrightarrow{\|\cdot\|_\varphi} 0$. So $\|f_n\varphi g_n\varphi\|_\infty = \|f_n \cdot g_n\|_\varphi \longrightarrow 0$. It follows that,

$$\frac{1}{\|e\|^2} f_n\left(\frac{e}{\|e\|}\right) g_n\left(\frac{e}{\|e\|}\right) = (f_n\varphi g_n\varphi)\left(\frac{e}{\|e\|}\right) \longrightarrow 0. \tag{3.1}$$

Hence by (3.1) we can conclude that,

$$\begin{aligned}
\|T(f_n)\cdot T(g_n)\|_\varphi &= \left\| f_n\left(\frac{e}{\|e\|}\right)\varphi g_n\left(\frac{e}{\|e\|}\right)\right\|_\varphi \\
&= \left\| f_n\left(\frac{e}{\|e\|}\right)\varphi g_n\left(\frac{e}{\|e\|}\right)\varphi\right\|_\infty \\
&= \left| f_n\left(\frac{e}{\|e\|}\right) g_n\left(\frac{e}{\|e\|}\right)\right| \|\varphi\|_\infty^2 \\
&\longrightarrow 0.
\end{aligned}$$

This shows that $T : \big(C^{b\varphi}(K), \|\cdot\|_\varphi\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\varphi\big)$ is strongly zero-product preserving. A similar argument can be applied to show that $T : \big(C^{b\varphi}(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ is also strongly zero-product preserving.

Proposition 3.4. *Let $T : \big(C^{b\varphi}(K), \|\cdot\|_\infty\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\infty\big)$ be a strongly zero-product preserving map such that $T(f\varphi) = T(f)\varphi$ for all $f \in C^{b\varphi}(K)$. Then $T : \big(C^{b\varphi}(K), \|\cdot\|_\varphi\big) \longrightarrow \big(C^{b\varphi}(K), \|\cdot\|_\varphi\big)$ is strongly zero-product preserving.*

Proof. Let $f_n \cdot g_n \xrightarrow{\|\cdot\|_\varphi} 0$. So $f_n \cdot (g_n\varphi) \xrightarrow{\|\cdot\|_\infty} 0$. It follows that $T(f_n) \cdot T(g_n\varphi) \xrightarrow{\|\cdot\|_\infty} 0$. Hence $T(f_n)\varphi T(g_n)\varphi \xrightarrow{\|\cdot\|_\infty} 0$. Thus $T(f_n) \cdot T(g_n) \xrightarrow{\|\cdot\|_\varphi} 0$. □

The following proposition is a result concerning algebraic homomorphisms on $C^b(K)$ and $C^{b\varphi}(K)$.

Proposition 3.5. *Let $T : C^b(K) \longrightarrow C^b(K)$ be an algebraic homomorphism such that $T(\varphi) = \varphi$. Then $T : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ is so.*

Also if $T : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ is an algebraic homomorphism such that $T(1_K) = 1_K$ then $T : C^b(K) \longrightarrow C^b(K)$ is so.

Proof. Let $T : C^b(K) \longrightarrow C^b(K)$ be an algebraic homomorphism and $T(\varphi) = \varphi$. So,

$$\begin{aligned}
T(f\cdot g) &= T(f\varphi g) \\
&= T(f)T(\varphi)T(g) \\
&= T(f)\varphi T(g) \\
&= T(f)\cdot T(g),
\end{aligned}$$

for all $f, g \in C^{b\varphi}(K)$. Thus T is an algebraic homomorphism on $C^{b\varphi}(K)$. Also let $T : C^{b\varphi}(K) \longrightarrow C^{b\varphi}(K)$ be an algebraic homomorphism such that $T(1_K) = 1_K$. So,

$$\begin{aligned} T(f)\varphi T(g) &= T(f \cdot g) \\ &= T((fg) \cdot 1_K) \\ &= T(fg) \cdot T(1_K) \\ &= T(fg) \cdot 1_K \\ &= T(fg)\varphi, \end{aligned}$$

for all $f, g \in C^b(K)$. It follows that $\left(T(f)T(g) - T(fg)\right)\varphi = 0$. Hence, by Proposition 2.1 we can conclude that $T(fg) = T(f)T(g)$ for all $f, g \in C^b(K)$. Therefore, T is an algebraic homomorphism on $C^b(K)$. $\square$

Question 3.6. Let $T : \left(C^{b\varphi}(K), \|\cdot\|_\infty\right) \longrightarrow \left(C^{b\varphi}(K), \|\cdot\|_\infty\right)$ be a strongly zero-product preserving map.

Is necessarily $T : \left(C^b(K), \|\cdot\|_\infty\right) \longrightarrow \left(C^b(K), \|\cdot\|_\infty\right)$ a strongly zero-product preserving map?

4. Conclusions

If $\dim A > 1$ then there are non-equivalent norms on $C^b(K)$. The norm $\|\cdot\|_\varphi$ is not an algebraic norm on $C^b(K)$, whereas it is an algebraic norm on $C^{b\varphi}(K)$. The pair $\left(C^{b\varphi}(K), \|\cdot\|_\infty\right)$ is a Banach algebra, whereas $\left(C^{b\varphi}(K), \|\cdot\|_\varphi\right)$ is a non-complete normed algebra. So $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are non-equivalent norms on $C^{b\varphi}(K)$. The zero-product preserving maps on $\left(C^b(K), \|\cdot\|_\infty\right)$ and $\left(C^{b\varphi}(K), \|\cdot\|_\infty\right)$ are the same, but it is not the case for strongly zero-product preserving maps.

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