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On Certain Generalized Bazilevic Type Functions Associated with Conic Regions

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Abstract. Let f and g be analytic in the open unit disc and, for α, $\beta \geq 0$, let

$$J(\alpha,\beta,f,g) = \frac{zf'(z)}{f^{1-\alpha}(z)g^{\alpha}(z)} + \beta\left(1+\frac{zf''(z)}{f'(z)}\right) - \beta(1-\alpha)\frac{zf'(z)}{f(z)} - \alpha\beta\frac{zg'(z)}{g(z)}.$$

The main aim of this paper is to study the class of analytic functions which map $J(\alpha,\beta,f,g)$ onto conic regions. Several interesting problems such as arc length, inclusion relationship, rate of growth of coefficient and Growth rate of Hankel determinant will be discussed.

1. Introduction

Let $\mathbf{A}$ denotes the class of functions f given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are anlytic in $E = \{z : |z| < 1\}$. Let

$$P = \{p : \mathrm{Re}(p(z)) > 0,\ z \in E\}$$

and

$$P(p_\kappa) = \{p \in \mathbf{A} : p(0) = 1\ \wedge p \prec p_\kappa\}, \tag{1.2}$$

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where $p_\kappa(z)$ are extremal functions for conic regions Ω_κ, where

$$\Omega_\kappa = \left\{ a + ib : a > \kappa\sqrt{(a-1)^2 + b^2} \right\}. \tag{1.3}$$

The regions Ω_κ $(\kappa = 0)$ represents right half plane, Ω_κ $(0 < \kappa < 1)$ represents hyperbola, Ω_κ $(\kappa = 1)$ represents a parabola and Ω_κ $(\kappa > 1)$ represents an ellipse. For $p_\kappa(z)$, $\kappa \in [0, \infty)$ we refer [6, 7]. Clearly, $P(p_\kappa) \subset P(\alpha)$, where $\alpha = \frac{\kappa}{\kappa+1}$,

$$P(\alpha) = \{p : \mathrm{Re}\,(p(z)) > \alpha,\ z \in E\}.$$

The class $P(p_\kappa)$ extended as follows [17];

Definition 1.1. Let $p \in \mathbf{A}$ in E with $p(0) = 1$. Then $p \in P_m(p_\kappa)$, $m \geq 2$, $\kappa \in [0, \infty)$ if and only if

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \tag{1.4}$$

where $p_1, p_2 \in P(p_\kappa)$. We note $P_m(p_\kappa) \subset P_m(\rho)$, $\rho = \frac{\kappa}{\kappa+1}$ and this class has been studied in [18]. When $\kappa = 0$, the class $P_m(p_0) = P_m$ which was introduced by Pinchuk in [19].

Related to the class $P_m(p_\kappa)$, we have:

$$\kappa - UV_m = \left\{ f \in A : \frac{(zf')'}{f'} \in P_m(p_\kappa);\ z \in E \right\}$$

$$\kappa - UR_m = \left\{ f \in A : \frac{zf'}{f} \in P_m(p_\kappa);\ z \in E \right\}.$$

Some special classes of these classes are as pointed out below.

(i) $0 - UV_m = V_m$ and $0 - UR_m = R_m$ which are respectively, the well-known classes [3] of functions with bounded boundary and bounded radius rotation. By choosing $m = 2$, we obtain $V_2 = C$, the class of convex functions and $R_2 = S^*$ contains starlike functions.

(ii) $\kappa - UV_2 = \kappa - UCV$ is the class of uniformly convex functions; see [7] and $\kappa - UR_2 = \kappa - ST$ contain uniformly starlike functions [6].

Now we define:

Definition 1.2. Let $f \in A$, $\alpha, \beta \geq 0$. Then $f \in M_g(\alpha, \beta, \kappa)$ if and only if

$$J(\alpha, \beta, f, g) = \frac{zf'(z)}{f^{1-\alpha}(z)g^{\alpha}(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta(1-\alpha)\frac{zf'(z)}{f(z)} - \alpha\beta\frac{zg'(z)}{g(z)}.$$

belongs to $P(p_\kappa)$ for some $g \in A$.

Special cases:

(i) For $\beta = 0$ and $g \in \kappa - UR_m$, we have the class $M_g(\alpha, 0, \kappa) = B_m(\alpha, \kappa)$ and when $m = 2$, $\kappa = 0$, $B_2(\alpha, 0) = B(\alpha)$ is the well-known class of Bazilevic functions of type α, see [21].

(ii) For $\beta = 0$, $\rho = \frac{\kappa}{\kappa+1}$ and $g \in R_m(\rho)$, we have $M_g(\alpha, 0, \kappa) = B_m(\alpha, \rho, \kappa)$ introduced by Noor et. al. [14].

(iii) With $g \in R_m$, $M_g(1, 0, 0) = T_m$, the class of generalized close-to-convex functions introduced and studied in [12]. For $m = 2$, we have $T_2 = K$, the well-known class of close-to-convex functions introduced in [8].

(iv) $M_g(0, \beta, 0) = M(\beta)$ is the class of $\beta-$starlike functions and in this case, $f \in M(\beta)$ implies

$$\left\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \in P, \quad z \in E.$$

(v) $M_g(1, 0, k) = \kappa - UCV$ is the class of k-uniformly convex functions, see [7].

2. Preliminary Results

Lemma 2.1 ([4])**.** *Let $h \in P$, $z \in E$ and $z = re^{i\phi}$. Then*

$$\int_0^{2\pi} \left|h\left(re^{i\phi}\right)\right|^\eta d\theta < c(\eta)\frac{1}{(1-r)^{\eta-1}},$$

where $\eta > 1$ and $c(\eta)$ is a constant depending only on λ.

Lemma 2.2 ([18])**.** *Let $g \in V_m(\rho)$. Then*

$$g'(z) = \left(g_1'(z)\right)^{1-\rho}, \quad g_1 \in V_m. \tag{2.1}$$

3. Main Results

Theorem 3.1. *Let $g \in \kappa - UR_m$. Then, for $m \geq 2$ and $\kappa \geq 0$*

$$M_g(\alpha, \beta, \kappa) \subset M_g(\alpha, 0, \kappa) = B_m(\alpha, \kappa).$$

Proof. Let

$$f \in M_g(\alpha, \beta, \kappa), \qquad g \in \kappa - UR_m. \tag{3.1}$$

and let

$$\frac{zf'(z)}{f^{(1-\alpha)}(z)g^\alpha(z)} = Q(z). \tag{3.2}$$

We note that $Q(z)$ is analytic in E and $Q(0) = 1$. By using (3.2), (3.1) and some simple calculations, we have

$$\left(Q(z) + \beta \frac{zQ'(z)}{Q(z)}\right) \prec p_\kappa(z). \tag{3.3}$$

Now, due to result of Miller Mocanu [9], it follows from (3.3) that

$$Q(z) \prec q_k(z) \prec p_\kappa(z),$$

where

$$q_k(z) = \left[\int_0^1 \left(\exp \int_0^{tz} \frac{p_\kappa(\zeta) - 1}{\zeta} d\zeta\right) dt\right]^{-1}$$

is best dominant. Therefore it follows that $f \in B_m(\alpha, \kappa)$, $z \in E$. □

Remark 3.2. As a partial converse case, with $\kappa = 0$,

$$B_m(\alpha, 0) \subset M_g(\alpha, \beta, 0) \text{ for } |z| < r_\beta,$$

where

$$r_\beta = \frac{1}{\left[2\beta + \sqrt{4\beta^2 - 2\beta + 1}\right]}. \tag{3.4}$$

As a proof, let

$$\frac{zf'(z)}{f^{(1-\alpha)}(z) g^\alpha(z)} = H(z).$$

Then $Q \in P$. Now using distortion results for the class P, see [3], we have

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zf'(z)}{f^{(1-\alpha)}(z)g^\alpha(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta(1-\alpha)\frac{zf'(z)}{f(z)} - \alpha\beta\frac{zg'(z)}{g(z)}\right\} \\ &\quad = \operatorname{Re}\left(H(z) + \beta\frac{zH'(z)}{H(z)}\right) > 0, \quad \text{for } |z| < r_\beta, \end{aligned}$$

where r_β is given by (3.4).

As special case, if $f \in M(\beta)$ implies $f \in S^*$ for $|z| = r_1 < \frac{1}{2+\sqrt{3}}$.

Theorem 3.3. *Let $f \in B_m(\alpha, \kappa)$. Then, for $\alpha \in (0, 1]$ and $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$, we have*

$$\underset{|z|=r}{L_f} = O(1) M_r^{(1-\alpha)}(f) \left(\frac{1}{1-r}\right)^\gamma, \quad \gamma = \frac{\alpha}{\kappa+1}\left(\frac{m}{2}+1\right) + \sigma - 1, \tag{3.5}$$

where $M_r(f) = \max_{|z|=r} |f(z)|$, $\underset{|z|=r}{L_f}$ the length of the image of the circle $|z| = r$ under f and $O(1)$ denotes a constant depending on κ, m and α.

Proof. As we know that, for $z = re^{i\theta}$, $0 < r < 1$

$$\underset{|z|=r}{L_f} = \int_0^{2\pi} \left| zf'(z) \right| d\theta. \tag{3.6}$$

Since $f \in B_m(\alpha, \kappa)$, we have

$$zf'(z) = f^{(1-\alpha)}(z) g^{\alpha}(z) h(z), \tag{3.7}$$

where $g \in k - UR_m \subset R_m\left(\frac{\kappa}{\kappa+1}\right)$, $h \in P(p_\kappa)$. Using Lemma 2.2 and a result of Brannan [1] for the generalized case, we can write

$$\frac{g(z)}{z} = \frac{\left(\frac{g_1(z)}{z}\right)^{\left(\frac{1}{\kappa+1}\right)\left(\frac{m}{4}+\frac{1}{2}\right)}}{\left(\frac{g_2(z)}{z}\right)^{\left(\frac{1}{\kappa+1}\right)\left(\frac{m}{4}-\frac{1}{2}\right)}}, \quad g_1, g_2 \in S^*. \tag{3.8}$$

Also $h \in P(p_\kappa)$ can be written as

$$h(z) = p^{\sigma}(z), \quad p \in P, \qquad \sigma = \frac{2}{\pi} \tan^{-1} \frac{1}{\kappa}. \tag{3.9}$$

From (3.6)-(3.9), we obtain

$$\underset{|z|=r}{L_f} \le \frac{M_r^{(1-\alpha)}(f)}{r^{\frac{\alpha}{\kappa+1}}} \int_0^{2\pi} \left| \frac{(g_1(z))^{\left(\frac{m}{4}+\frac{1}{2}\right)}}{(g_2(z))^{\left(\frac{m}{4}-\frac{1}{2}\right)}} \right|^{\frac{\alpha}{\kappa+1}} . \left|p(z)\right|^{\sigma} d\theta$$

Using distortion result for starlike function $g_2(z)$, to get

$$\underset{|z|=r}{L_f} \le \frac{2^{\frac{\alpha}{\kappa+1}\left(\frac{m}{2}+1\right)} M_r^{(1-\alpha)}(f)}{r^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)}} \int_0^{2\pi} \left|g_1(z)\right|^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)} . \left|p(z)\right|^{\sigma} d\theta. \tag{3.10}$$

Using Holder's inequality, we note that

$$\begin{aligned} \int_0^{2\pi} \left|g_1(z)\right|^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)} . \left|p(z)\right|^{\sigma} d\theta &\le \left(\int_0^{2\pi} \left|g_1(z)\right|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} \right)^{\frac{2-\sigma}{2}} \\ &\quad \times \left(\int_0^{2\pi} \left|p(z)\right|^2 \right)^{\frac{\sigma}{2}} d\theta. \end{aligned} \tag{3.11}$$

Now, it is known [3] for $p \in P$ that

$$\int_0^{2\pi} \left|p(z)\right|^2 d\theta \le \frac{1+3r^2}{1-r^2}, \tag{3.12}$$

and subordination principle together with Lemma 2.1, gives us

$$\int_0^{2\pi} \left|g_1(z)\right|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} d\theta \le \int_0^{2\pi} \left| \frac{r}{1-re^{i\theta}} \right|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} d\theta \tag{3.13}$$

$$\leq c(\alpha, m, \kappa)\left[\frac{1}{1-r}\right]^{\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)}-1},$$

where $c(\alpha, m, \kappa)$ is a constant and $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$. Thus, using (3.12)-(3.13), we obtain from (3.10) that

$$\underset{|z|=r}{L_f} = O(1)M_r^{(1-\alpha)}(f)\left(\frac{1}{1-r}\right)^{\gamma}, \qquad \gamma = \frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)} + \sigma - 1.$$

□

Corollary 3.4. *Let $\kappa = 0 \Rightarrow \sigma = 1$ and $\alpha = \frac{1}{2}$. Then, for $m > \left(\frac{1}{\alpha} - 2\right)$ and $r_0 = \frac{1}{1-r}$*

$$\underset{|z|=r}{L_f} = O(1)M_r^{(1-\alpha)}(f)\, r_0^{\alpha\left(\frac{m}{2}+1\right)}.$$

Corollary 3.5. *Let $\kappa = 1 \Rightarrow \sigma = \frac{1}{2}$. Then, for $m > \left(\frac{3}{\alpha} - 2\right)$ and $r_0 = \frac{1}{1-r}$*

$$\underset{|z|=r}{L_f} = O(1)M_r^{(1-\alpha)}(f)\, r_0^{\left[\alpha\left(\frac{m}{4}+\frac{1}{2}\right)-\frac{1}{2}\right]}.$$

For $\alpha = 1$, we have

$$\underset{|z|=r}{L_f} = O(1)M_r^{(1-\alpha)}(f)\, r_0^{\frac{m}{4}}.$$

Corollary 3.6. *Let $\kappa = 1$, $m = 4$, $\sigma = 1$ and $r_0 = \frac{1}{1-r}$. Then $\alpha \in \left(\frac{1}{2}, 1\right]$ and we have*

$$\underset{|z|=r}{L_f} = O(1)M_r^{(1-\alpha)}(f)\, r_0^{\frac{3\alpha-1}{2}}.$$

The case when $\alpha > 1$ is similar and is stated as following;

Theorem 3.7. *Let $f \in B_m(\alpha, \kappa)$. Then, for $\alpha > 1$ and $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$, we have*

$$\underset{|z|=r}{L_f} = O(1)m_r^{(1-\alpha)}(f)\left(\frac{1}{1-r}\right)^{\gamma}, \quad \gamma = \frac{\alpha}{\kappa+1}\left(\frac{m}{2}+1\right) + \sigma - 1, \tag{3.14}$$

where $m_r(f) = \min_{|z|=r}|f(z)|$ and $O(1)$ denotes a constant depending on κ, m and α.

Corollary 3.8. *For $\kappa = 0 \Rightarrow \sigma = 1$, $\alpha = 2$ and $r_0 = \frac{1}{1-r}$, we have*

$$\underset{|z|=r}{L_f} = O(1)m_r^{(1-\alpha)}(f)\, r_0^{(m+2)}.$$

Corollary 3.9. *For $\kappa = 1 \Rightarrow \sigma = \frac{1}{2}$, $\alpha = 2$ and $r_0 = \frac{1}{1-r}$, we have*

$$\underset{|z|=r}{L_f} = O(1) m_r^{(1-\alpha)}(f) r_0^{\left(\frac{m}{2}+1\right)}.$$

Theorem 3.10. *Let $f \in B_m(\alpha, \kappa)$. Then, for $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$, we have*

$$a_n = \begin{cases} O(1)M^{1-\alpha}(f)n^{\gamma-1}; & 0 < \alpha \leq 1 \\ O(1)m^{1-\alpha}(f)n^{\gamma-1}; & \alpha > 1, \end{cases} \qquad (n \to \infty),$$

where $M(f)$, $m(f)$, γ and $O(1)$ are same as defined before.

Proof. With $z = re^{i\theta}$, we use Cauchy Theorem to have

$$\begin{aligned} n|a_n| &= \frac{1}{2\pi r^n}\left|\int_0^{2\pi} zf'(z)e^{-\iota n\theta}d\theta\right| \\ &\leq \frac{1}{2\pi r^n}\int_0^{2\pi} \left|zf'(z)\right| d\theta \\ &= \frac{1}{2\pi r^n} \underset{|z|=r}{L_f}. \end{aligned} \tag{3.15}$$

We can easily obtain our required result from (3.5), (3.14) and (3.15). □

4. Hankel Determinant Problem

Let $f \in \mathbf{A}$ and given by (1.1). Then for $q \geq 1$, $n \geq 1$, the qth hankel determinant $H_q(n)$ is defined as;

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & .. & a_{n+2q-2} \end{vmatrix} \tag{4.1}$$

Several authors have discussed rate of growth of $H_q(n)$ as $n \to \infty$ for well-known classes, see [10, 11, 13, 15, 16]. In [20] Pommerenke, studied it for starlike functions. Hayman [5] proved that $H_2(n) = O(1).n^{\frac{1}{2}}$ as $n \to \infty$ and f is univalent. The exponent $\frac{1}{2}$ is best possible and $O(1)$ is constant. Here we discuss this problem for $f \in B_m(\alpha, \kappa)$, $m \geq 2$, $\kappa \geq 0$ as $n \to \infty$. To prove our main result of this section, we shall need the following two lemmas.

Lemma 4.1 ([10])**.** *Let $f \in A$ and let the Hankel determinant of $f(z)$ be defined by (4.1). Then, writing $\Delta_j(n) = \Delta_j(n, z_1, f)$, we have*

(4.2)
$$H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & ... & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & ... & \Delta_{q-2}(n+q) \\ . & . & . & . \\ . & . & . & . \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & .. & \Delta_0(n+2q-2) \end{vmatrix},$$

where, with $\Delta_0(n, z_1, f) = a_n$, we define for $j \geq 1$,

$$\Delta_j(n, z_1, f) = \Delta_{j-1}(n, z_1, f) - z_1\Delta_{j-1}(n+1, z_1, f)$$

Lemma 4.2 ([10])**.** *With $z_1 = \left(\frac{n}{n+1}y\right)$ and $v \geq 0$ be any integer*

$$\Delta_j(n+v, x, zf'(z)) = \sum_{l=0}^{j} \binom{j}{i} \frac{y^l(v-(l-1)n)}{(n+1)^l} \Delta_{j-l}(n+v+l, y, f(z)).$$

Theorem 4.3. *Let $f \in B_m(\alpha, \kappa)$. Then, for $M(r, f) = \max_{|z|=r} |f(z)|$, $q \geq 1$, $n \geq 1$ and $m > \left(\frac{4q-2}{\alpha_1} - 2\right)$*

$$H_q(n) = O(1)M_r^{(1-\alpha)}(f) \begin{cases} n^{\beta_1}; & q = 1, \quad m > \left[\frac{2}{\alpha_1} - 2\right] \\ n^{\beta_2}; & q > 1, \quad m > \left[\frac{4q-2}{\alpha_1} - 2\right] \end{cases}$$

where

$$\beta_1 = \alpha_1\left(\frac{m}{2} - 1\right) + \sigma - 2, \qquad \alpha_1 = \alpha(1-\rho).$$

and

$$\beta_2 = \left(\alpha_1\left(\frac{m}{2} + 1\right) + \sigma - 1\right)q - q^2.$$

Proof. Let $zG'(z) = g(z)$. Then $\frac{(zG'(z))'}{G'(z)} \in P_m(p_\kappa) \subset P_m(\rho)$, $\rho = \frac{\kappa}{\kappa+1}$. This implies $G \in V_m(\rho)$, so from (2.1) we have

$$G'(z) = \left(G_1'(z)\right)^{(1-\rho)}, \qquad G_1 \in V_m \quad (z \in E). \tag{4.3}$$

Since $f \in B_m(\alpha, \kappa)$, so we can write

$$zf'(z) = f^{(1-\alpha)}(z)g^{\alpha}(z)h^{\sigma}(z), \quad \text{for } h \in P. \tag{4.4}$$

From (4.4) and result due to Brannan [1], the above equation implies

$$zf'(z) = z^{\alpha}f^{(1-\alpha)}(z)\left[\frac{(g_1'(z))^{\left(\frac{m}{4}+\frac{1}{2}\right)}}{(g_2'(z))^{\left(\frac{m}{4}-\frac{1}{2}\right)}}\right]^{\alpha(1-\rho)} .h^{\sigma}(z), \quad g_1, g_2 \in C. \tag{4.5}$$

For any univalent function s, we can choose $z_1 = z_1(r)$ with $|z_1| = r$ such that

$$\max_{|z|=r} |(z - z_1)\, s(z)| \leq \frac{2r^2}{1 - r^2}, \quad \text{(see [2])}. \tag{4.6}$$

Thus, from (4.5) with $zg_i' = s_i \in S$ and $m \geq \left[\frac{2+4j}{\alpha(1-\rho)} - 2\right]$, we have

$$\left|\Delta_j(n, z_1, zf')\right| \leq \frac{M^{1-\alpha}(r, f)}{2\pi r^{n+j-\alpha}} \left(\frac{2r^2}{1 - r^2}\right)^j (2)^{\alpha_1\left(\frac{m}{2}-1\right)}$$
$$\times \int_0^{2\pi} |s_1(z)|^{\alpha_1\left(\frac{m}{4}+\frac{1}{2}\right)-j} |h^\sigma(z)|\, d\theta.$$

where we have used distortion result for starlike function s_2, we can rewrite above inequality as;

$$\left|\Delta_j(n, z_1, zf')\right| = O(1).M^{1-\alpha}(r, f) \left(\frac{1}{1 - r}\right)^j \times \left[\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\alpha_1\left(\frac{m}{4}+\frac{1}{2}\right)-j} |h^\sigma(z)|\, d\theta\right], \tag{4.7}$$

where $O(1)$ denotes a constant.
By making use of Holder's inequality, we have

(4.8)

$$\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\alpha_1\left(\frac{m}{4}+\frac{1}{2}\right)-j} |h^\sigma(z)|\, d\theta$$
$$\leq \left[\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\left\{\alpha_1\left(\frac{m}{4}+\frac{1}{2}\right)-j\right\}\frac{2}{2-\sigma}} d\theta\right]^{\frac{2-\sigma}{2}} \times \left[\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2\, d\theta\right]^{\frac{\sigma}{2}}.$$

Now, from (4.7), (4.8), Lemma 2.1 and subordination for starlike functions, we obtain for $m > \left[\frac{2(1+2j)}{\alpha_1} - 2\right]$

$$\Delta_j(n, z_1, zf') = O(1).M_r^{(1-\alpha)}(f) \left(\frac{1}{1 - r}\right)^{\alpha_1\left(\frac{m}{2}-1\right)-j+\sigma-1}, \quad (r \to 1),$$

and using Lemma 4.2, $r = 1 - \frac{1}{n}$, we get

$$\Delta_j(n, z_1, f) = O(1).M_r^{(1-\alpha)}(f)\, n^{\alpha_1\left(\frac{m}{2}-1\right)-j+\sigma-2}, \quad (n \to \infty).$$

For $j > 0$, we have $q > 1$, follow the similar technique used in [10] along with Lemma 4.1, we get for $m > \left(\frac{4q-2}{\alpha_1} - 2\right)$

$$H_q(n) = O(1)M_r^{(1-\alpha)}(f)\, .n^{\beta_2},$$

where $\beta_2 = \left(\alpha_1\left(\frac{m}{2}+1\right)+\sigma-1\right)q - q^2$. If $j=0$, we have $q=1$ and $\Delta_0(n, z_1, f) = a_n$. This gives us, for $m > \left[\frac{2}{\alpha_1}-2\right]$

$$H_1(n) = O(1)M_r^{(1-\alpha)}(f).n^{\beta_1},$$

where $\beta_1 = \alpha_1\left(\frac{m}{2}-1\right)+\sigma-2$ and $\alpha_1 = \alpha(1-\rho)$. □

With suitable choices of parameters, we obtain some known results; see [12, 13, 17].

References

1. D.A. Brannan, *On functions of bounded boundary rotation*, Proc. Edinburg Math. Soc., 16 (1969), pp. 339-347.
2. G.M. Golusin, *On distortion theorem and coefficients of univalent functions*. Math. Sb., 19 (1946), pp. 183-203.
3. A.W. Goodman, *Univalent Functions*, Vols. I & II, Polygonal Publishing House, Washington, New Jersey, (1983).
4. W. K. Hayman, *On functions with positive real part*, J. London Math. Soc., 36 (1961), pp. 34-48.
5. W.K. Hayman, *On the second Hankel determinant of mean univalent functions*. Proc. London Math. Soc., 18 (1968), pp. 77-84.
6. S. Kanas and A. Wisniowska, *Conic domain and starlike functions*, Rev. Roumaine Math. Pures Appl., 45 (2000), pp. 647-657.
7. S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Math., 105 (1999), pp. 327-336.
8. W. Kaplan, *Close-to-convex Schlicht functions*, Mich. Math. J., 1 (1952), pp. 169-185.
9. S.S. Miller and P.T. Mocanu, *Differential subordinations theory and applications*, Marcel Dekker, Inc., New York, Basel, (2000).
10. J.W. Noonan and D.K. Thomas, *On the Hankel determinant of areally mean p-valent functions*, Proc. London Math. Soc., 25 (1972), pp. 503-524.
11. K.I. Noor, *Hankel determinant problem for functions of bounded boundary rotations*, Rev. Roum. Math. Pures Appl., 28 (1983), pp. 731-739.
12. K.I. Noor, *On a generalization of close-to-convexity*, Int. J. Math. Math. Sci., 6 (1983), pp. 327-334.
13. K.I. Noor, *On the Hankel determinant of close-to-convex univalent functions*, Inter. J. Math. Sci., 3 (1980), pp. 447-481.
14. K.I. Noor, K. Ahmad, *On higher order Bazilevic functions*, Int. J. Mod. Phys. B, 27(2013), 14 pages.
15. K.I. Noor, *On the Hankel determinant problem for strongly close-to-convex functions*, J. Natu. Geom., 11 (1997), pp. 29-34.

16. K.I. Noor and Al-Naggar, *Hankel determinant problem*, J. Natu. Geom., 14 (1998), pp. 133-140.
17. K.I. Noor and M.A. Noor, *Higher order close-to-convex functions related with conic domains*. Appl. Math. Inf. Sci., 8 (2014), pp. 2455-2463.
18. K.S. Padmanabhan and R. Parvatham, *Properties of a class of functions with bounded boundary rotation*, Ann. Polon. Math., 31 (1975), pp. 311-323.
19. B. Pinchuk, *Functions with bounded boundary rotation*, Israel J. Math., 10 (1971), pp. 7-16.
20. Ch. Pommerenke, *On starlike and close-to-convex functions*, Proc. London Math. Soc., 13 (1963), pp. 290-304.
21. D.K. Thomas, *On Bazilevič functions*, Trans. Amer. Math. Soc., 132 (1968), pp. 353-361.

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