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## On Certain Generalized Bazilevic Type Functions Associated with Conic Regions

Khalida Inayat Noor<sup>1</sup> and Shujaat Ali Shah<sup>2\*</sup>

ABSTRACT. Let  $f$  and  $g$  be analytic in the open unit disc and, for  $\alpha, \beta \geq 0$ , let

$$J(\alpha, \beta, f, g) = \frac{zf'(z)}{f^{1-\alpha}(z)g^\alpha(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \beta(1-\alpha) \frac{zf'(z)}{f(z)} - \alpha\beta \frac{zg'(z)}{g(z)}.$$

The main aim of this paper is to study the class of analytic functions which map  $J(\alpha, \beta, f, g)$  onto conic regions. Several interesting problems such as arc length, inclusion relationship, rate of growth of coefficient and Growth rate of Hankel determinant will be discussed.

### 1. INTRODUCTION

Let  $\mathbf{A}$  denotes the class of functions  $f$  given by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in  $E = \{z : |z| < 1\}$ . Let

$$P = \{p : \operatorname{Re}(p(z)) > 0, z \in E\}$$

and

$$(1.2) \quad P(p_\kappa) = \{p \in \mathbf{A} : p(0) = 1 \wedge p \prec p_\kappa\},$$

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where  $p_\kappa(z)$  are extremal functions for conic regions  $\Omega_\kappa$ , where

$$(1.3) \quad \Omega_\kappa = \left\{ a + ib : a > \kappa \sqrt{(a-1)^2 + b^2} \right\}.$$

The regions  $\Omega_\kappa$  ( $\kappa = 0$ ) represents right half plane,  $\Omega_\kappa$  ( $0 < \kappa < 1$ ) represents hyperbola,  $\Omega_\kappa$  ( $\kappa = 1$ ) represents a parabola and  $\Omega_\kappa$  ( $\kappa > 1$ ) represents an ellipse. For  $p_\kappa(z)$ ,  $\kappa \in [0, \infty)$  we refer [6, 7]. Clearly,  $P(p_\kappa) \subset P(\alpha)$ , where  $\alpha = \frac{\kappa}{\kappa+1}$ ,

$$P(\alpha) = \{p : \operatorname{Re}(p(z)) > \alpha, z \in E\}.$$

The class  $P(p_\kappa)$  extended as follows [17];

**Definition 1.1.** Let  $p \in \mathbf{A}$  in  $E$  with  $p(0) = 1$ . Then  $p \in P_m(p_\kappa)$ ,  $m \geq 2$ ,  $\kappa \in [0, \infty)$  if and only if

$$(1.4) \quad p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z),$$

where  $p_1, p_2 \in P(p_\kappa)$ . We note  $P_m(p_\kappa) \subset P_m(\rho)$ ,  $\rho = \frac{\kappa}{\kappa+1}$  and this class has been studied in [18]. When  $\kappa = 0$ , the class  $P_m(p_0) = P_m$  which was introduced by Pinchuk in [19].

Related to the class  $P_m(p_\kappa)$ , we have:

$$\begin{aligned} \kappa - UV_m &= \left\{ f \in A : \frac{(zf')'}{f'} \in P_m(p_\kappa); z \in E \right\} \\ \kappa - UR_m &= \left\{ f \in A : \frac{zf'}{f} \in P_m(p_\kappa); z \in E \right\}. \end{aligned}$$

Some special classes of these classes are as pointed out below.

- (i)  $0 - UV_m = V_m$  and  $0 - UR_m = R_m$  which are respectively, the well-known classes [3] of functions with bounded boundary and bounded radius rotation. By choosing  $m = 2$ , we obtain  $V_2 = C$ , the class of convex functions and  $R_2 = S^*$  contains starlike functions.
- (ii)  $\kappa - UV_2 = \kappa - UCV$  is the class of uniformly convex functions; see [7] and  $\kappa - UR_2 = \kappa - ST$  contain uniformly starlike functions [6].

Now we define:

**Definition 1.2.** Let  $f \in A$ ,  $\alpha, \beta \geq 0$ . Then  $f \in M_g(\alpha, \beta, \kappa)$  if and only if

$$\begin{aligned} J(\alpha, \beta, f, g) &= \frac{zf'(z)}{f^{1-\alpha}(z)g^\alpha(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta(1-\alpha) \frac{zf'(z)}{f(z)} \\ &\quad - \alpha\beta \frac{zg'(z)}{g(z)}. \end{aligned}$$

belongs to  $P(p_\kappa)$  for some  $g \in A$ .

Special cases:

- (i) For  $\beta = 0$  and  $g \in \kappa - UR_m$ , we have the class  $M_g(\alpha, 0, \kappa) = B_m(\alpha, \kappa)$  and when  $m = 2, \kappa = 0, B_2(\alpha, 0) = B(\alpha)$  is the well-known class of Bazilevic functions of type  $\alpha$ , see [21].
- (ii) For  $\beta = 0, \rho = \frac{\kappa}{\kappa+1}$  and  $g \in R_m(\rho)$ , we have  $M_g(\alpha, 0, \kappa) = B_m(\alpha, \rho, \kappa)$  introduced by Noor et. al. [14].
- (iii) With  $g \in R_m, M_g(1, 0, 0) = T_m$ , the class of generalized close-to-convex functions introduced and studied in [12]. For  $m = 2$ , we have  $T_2 = K$ , the well-known class of close-to-convex functions introduced in [8].
- (iv)  $M_g(0, \beta, 0) = M(\beta)$  is the class of  $\beta$ -starlike functions and in this case,  $f \in M(\beta)$  implies
 
$$\left\{ (1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \in P, \quad z \in E.$$
- (v)  $M_g(1, 0, k) = \kappa - UCV$  is the class of  $k$ -uniformly convex functions, see [7].

### 2. PRELIMINARY RESULTS

**Lemma 2.1** ([4]). *Let  $h \in P, z \in E$  and  $z = re^{i\phi}$ . Then*

$$\int_0^{2\pi} |h(re^{i\phi})|^\eta d\theta < c(\eta) \frac{1}{(1-r)^{\eta-1}},$$

where  $\eta > 1$  and  $c(\eta)$  is a constant depending only on  $\lambda$ .

**Lemma 2.2** ([18]). *Let  $g \in V_m(\rho)$ . Then*

$$(2.1) \quad g'(z) = (g_1'(z))^{1-\rho}, \quad g_1 \in V_m.$$

### 3. MAIN RESULTS

**Theorem 3.1.** *Let  $g \in \kappa - UR_m$ . Then, for  $m \geq 2$  and  $\kappa \geq 0$*

$$M_g(\alpha, \beta, \kappa) \subset M_g(\alpha, 0, \kappa) = B_m(\alpha, \kappa).$$

*Proof.* Let

$$(3.1) \quad f \in M_g(\alpha, \beta, \kappa), \quad g \in \kappa - UR_m.$$

and let

$$(3.2) \quad \frac{zf'(z)}{f^{(1-\alpha)}(z)g^\alpha(z)} = Q(z).$$

We note that  $Q(z)$  is analytic in  $E$  and  $Q(0) = 1$ . By using (3.2), (3.1) and some simple calculations, we have

$$(3.3) \quad \left( Q(z) + \beta \frac{zQ'(z)}{Q(z)} \right) \prec p_\kappa(z).$$

Now, due to result of Miller Mocanu [9], it follows from (3.3) that

$$Q(z) \prec q_k(z) \prec p_\kappa(z),$$

where

$$q_k(z) = \left[ \int_0^1 \left( \exp \int_0^{tz} \frac{p_\kappa(\zeta) - 1}{\zeta} d\zeta \right) dt \right]^{-1}$$

is best dominant. Therefore it follows that  $f \in B_m(\alpha, \kappa)$ ,  $z \in E$ .  $\square$

**Remark 3.2.** As a partial converse case, with  $\kappa = 0$ ,

$$B_m(\alpha, 0) \subset M_g(\alpha, \beta, 0) \text{ for } |z| < r_\beta,$$

where

$$(3.4) \quad r_\beta = \frac{1}{\left[ 2\beta + \sqrt{4\beta^2 - 2\beta + 1} \right]}.$$

As a proof, let

$$\frac{zf'(z)}{f^{(1-\alpha)}(z)g^\alpha(z)} = H(z).$$

Then  $Q \in P$ . Now using distortion results for the class  $P$ , see [3], we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zf'(z)}{f^{(1-\alpha)}(z)g^\alpha(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \beta(1-\alpha) \frac{zf'(z)}{f(z)} - \alpha\beta \frac{zg'(z)}{g(z)} \right\} \\ = \operatorname{Re} \left( H(z) + \beta \frac{zH'(z)}{H(z)} \right) > 0, \quad \text{for } |z| < r_\beta, \end{aligned}$$

where  $r_\beta$  is given by (3.4).

As special case, if  $f \in M(\beta)$  implies  $f \in S^*$  for  $|z| = r_1 < \frac{1}{2+\sqrt{3}}$ .

**Theorem 3.3.** Let  $f \in B_m(\alpha, \kappa)$ . Then, for  $\alpha \in (0, 1]$  and

$\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$ , we have

$$(3.5) \quad L_f = O(1)M_r^{(1-\alpha)}(f) \left( \frac{1}{1-r} \right)^\gamma, \quad \gamma = \frac{\alpha}{\kappa+1} \left( \frac{m}{2} + 1 \right) + \sigma - 1,$$

where  $M_r(f) = \max_{|z|=r} |f(z)|$ ,  $L_f$  the length of the image of the circle  $|z| = r$  under  $f$  and  $O(1)$  denotes a constant depending on  $\kappa$ ,  $m$  and  $\alpha$ .

*Proof.* As we know that, for  $z = re^{i\theta}$ ,  $0 < r < 1$

$$(3.6) \quad L_f = \int_{|z|=r}^{2\pi} |zf'(z)| d\theta.$$

Since  $f \in B_m(\alpha, \kappa)$ , we have

$$(3.7) \quad zf'(z) = f^{(1-\alpha)}(z)g^\alpha(z)h(z),$$

where  $g \in k - UR_m \subset R_m\left(\frac{\kappa}{\kappa+1}\right)$ ,  $h \in P(p_\kappa)$ . Using Lemma 2.2 and a result of Brannan [1] for the generalized case, we can write

$$(3.8) \quad \frac{g(z)}{z} = \frac{\left(\frac{g_1(z)}{z}\right)^{\left(\frac{1}{\kappa+1}\right)\left(\frac{m}{4}+\frac{1}{2}\right)}}{\left(\frac{g_2(z)}{z}\right)^{\left(\frac{1}{\kappa+1}\right)\left(\frac{m}{4}-\frac{1}{2}\right)}}, \quad g_1, g_2 \in S^*.$$

Also  $h \in P(p_\kappa)$  can be written as

$$(3.9) \quad h(z) = p^\sigma(z), \quad p \in P, \quad \sigma = \frac{2}{\pi} \tan^{-1} \frac{1}{\kappa}.$$

From (3.6)-(3.9), we obtain

$$L_f \leq \frac{M_r^{(1-\alpha)}(f)}{r^{\frac{\alpha}{\kappa+1}}} \int_0^{2\pi} \left| \frac{(g_1(z))^{\left(\frac{m}{4}+\frac{1}{2}\right)}}{(g_2(z))^{\left(\frac{m}{4}-\frac{1}{2}\right)}} \right|^{\frac{\alpha}{\kappa+1}} \cdot |p(z)|^\sigma d\theta$$

Using distortion result for starlike function  $g_2(z)$ , to get

$$(3.10) \quad L_f \leq \frac{2^{\frac{\alpha}{\kappa+1}\left(\frac{m}{2}+1\right)} M_r^{(1-\alpha)}(f)}{r^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)}} \int_0^{2\pi} |g_1(z)|^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)} \cdot |p(z)|^\sigma d\theta.$$

Using Holder's inequality, we note that

$$(3.11) \quad \int_0^{2\pi} |g_1(z)|^{\frac{\alpha}{\kappa+1}\left(\frac{m}{4}+\frac{1}{2}\right)} \cdot |p(z)|^\sigma d\theta \leq \left( \int_0^{2\pi} |g_1(z)|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} d\theta \right)^{\frac{2-\sigma}{2}} \times \left( \int_0^{2\pi} |p(z)|^2 d\theta \right)^{\frac{\sigma}{2}}.$$

Now, it is known [3] for  $p \in P$  that

$$(3.12) \quad \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2},$$

and subordination principle together with Lemma 2.1, gives us

$$(3.13) \quad \int_0^{2\pi} |g_1(z)|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} d\theta \leq \int_0^{2\pi} \left| \frac{r}{1-re^{i\theta}} \right|^{\frac{\alpha\left(\frac{m}{2}+1\right)}{(\kappa+1)(2-\sigma)}} d\theta$$

$$\leq c(\alpha, m, \kappa) \left[ \frac{1}{1-r} \right]^{\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} - 1},$$

where  $c(\alpha, m, \kappa)$  is a constant and  $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$ . Thus, using (3.12)-(3.13), we obtain from (3.10) that

$$L_f = O(1) M_r^{(1-\alpha)}(f) \left( \frac{1}{1-r} \right)^\gamma, \quad \gamma = \frac{\alpha \left( \frac{m}{2} + 1 \right)}{(\kappa+1)} + \sigma - 1.$$

□

**Corollary 3.4.** *Let  $\kappa = 0 \Rightarrow \sigma = 1$  and  $\alpha = \frac{1}{2}$ . Then, for  $m > \left( \frac{1}{\alpha} - 2 \right)$  and  $r_0 = \frac{1}{1-r}$*

$$L_f = O(1) M_r^{(1-\alpha)}(f) r_0^{\alpha \left( \frac{m}{2} + 1 \right)}.$$

**Corollary 3.5.** *Let  $\kappa = 1 \Rightarrow \sigma = \frac{1}{2}$ . Then, for  $m > \left( \frac{3}{\alpha} - 2 \right)$  and  $r_0 = \frac{1}{1-r}$*

$$L_f = O(1) M_r^{(1-\alpha)}(f) r_0^{\left[ \alpha \left( \frac{m}{4} + \frac{1}{2} \right) - \frac{1}{2} \right]}.$$

For  $\alpha = 1$ , we have

$$L_f = O(1) M_r^{(1-\alpha)}(f) r_0^{\frac{m}{4}}.$$

**Corollary 3.6.** *Let  $\kappa = 1$ ,  $m = 4$ ,  $\sigma = 1$  and  $r_0 = \frac{1}{1-r}$ . Then  $\alpha \in \left( \frac{1}{2}, 1 \right]$  and we have*

$$L_f = O(1) M_r^{(1-\alpha)}(f) r_0^{\frac{3\alpha-1}{2}}.$$

The case when  $\alpha > 1$  is similar and is stated as following;

**Theorem 3.7.** *Let  $f \in B_m(\alpha, \kappa)$ . Then, for  $\alpha > 1$  and  $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$ , we have*

$$(3.14) \quad L_f = O(1) m_r^{(1-\alpha)}(f) \left( \frac{1}{1-r} \right)^\gamma, \quad \gamma = \frac{\alpha}{\kappa+1} \left( \frac{m}{2} + 1 \right) + \sigma - 1,$$

where  $m_r(f) = \min_{|z|=r} |f(z)|$  and  $O(1)$  denotes a constant depending on  $\kappa$ ,  $m$  and  $\alpha$ .

**Corollary 3.8.** *For  $\kappa = 0 \Rightarrow \sigma = 1$ ,  $\alpha = 2$  and  $r_0 = \frac{1}{1-r}$ , we have*

$$L_f = O(1) m_r^{(1-\alpha)}(f) r_0^{(m+2)}.$$

**Corollary 3.9.** For  $\kappa = 1 \Rightarrow \sigma = \frac{1}{2}$ ,  $\alpha = 2$  and  $r_0 = \frac{1}{1-r}$ , we have

$$L_f = O(1)m_r^{(1-\alpha)}(f)r_0^{\left(\frac{m}{2}+1\right)} \Big|_{|z|=r}.$$

**Theorem 3.10.** Let  $f \in B_m(\alpha, \kappa)$ . Then, for  $\frac{\alpha(m+2)}{(\kappa+1)(2-\sigma)} > 1$ , we have

$$a_n = \begin{cases} O(1)M^{1-\alpha}(f)n^{\gamma-1}; & 0 < \alpha \leq 1 \\ O(1)m^{1-\alpha}(f)n^{\gamma-1}; & \alpha > 1, \end{cases} \quad (n \rightarrow \infty),$$

where  $M(f)$ ,  $m(f)$ ,  $\gamma$  and  $O(1)$  are same as defined before.

*Proof.* With  $z = re^{i\theta}$ , we use Cauchy Theorem to have

$$\begin{aligned} (3.15) \quad n|a_n| &= \frac{1}{2\pi r^n} \left| \int_0^{2\pi} z f'(z) e^{-in\theta} d\theta \right| \\ &\leq \frac{1}{2\pi r^n} \int_0^{2\pi} |z f'(z)| d\theta \\ &= \frac{1}{2\pi r^n} L_f \Big|_{|z|=r}. \end{aligned}$$

We can easily obtain our required result from (3.5), (3.14) and (3.15). □

#### 4. HANKEL DETERMINANT PROBLEM

Let  $f \in \mathbf{A}$  and given by (1.1). Then for  $q \geq 1$ ,  $n \geq 1$ , the  $q$ th hankel determinant  $H_q(n)$  is defined as;

$$(4.1) \quad H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}$$

Several authors have discussed rate of growth of  $H_q(n)$  as  $n \rightarrow \infty$  for well-known classes, see [10, 11, 13, 15, 16]. In [20] Pommerenke, studied it for starlike functions. Hayman [5] proved that  $H_2(n) = O(1).n^{\frac{1}{2}}$  as  $n \rightarrow \infty$  and  $f$  is univalent. The exponent  $\frac{1}{2}$  is best possible and  $O(1)$  is constant. Here we discuss this problem for  $f \in B_m(\alpha, \kappa)$ ,  $m \geq 2$ ,  $\kappa \geq 0$  as  $n \rightarrow \infty$ . To prove our main result of this section, we shall need the following two lemmas.



**Lemma 4.1** ([10]). *Let  $f \in A$  and let the Hankel determinant of  $f(z)$  be defined by (4.1). Then, writing  $\Delta_j(n) = \Delta_j(n, z_1, f)$ , we have*

$$(4.2) \quad H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \dots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \dots & \Delta_{q-2}(n+q) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & \dots & \Delta_0(n+2q-2) \end{vmatrix},$$

where, with  $\Delta_0(n, z_1, f) = a_n$ , we define for  $j \geq 1$ ,

$$\Delta_j(n, z_1, f) = \Delta_{j-1}(n, z_1, f) - z_1 \Delta_{j-1}(n+1, z_1, f)$$

**Lemma 4.2** ([10]). *With  $z_1 = \left(\frac{n}{n+1}y\right)$  and  $v \geq 0$  be any integer*

$$\Delta_j(n+v, x, z f'(z)) = \sum_{l=0}^j \binom{j}{l} \frac{y^l (v - (l-1)n)}{(n+1)^l} \Delta_{j-l}(n+v+l, y, f(z)).$$

**Theorem 4.3.** *Let  $f \in B_m(\alpha, \kappa)$ . Then, for  $M(r, f) = \max_{|z|=r} |f(z)|$ ,*

*$q \geq 1$ ,  $n \geq 1$  and  $m > \left(\frac{4q-2}{\alpha_1} - 2\right)$*

$$H_q(n) = O(1) M_r^{(1-\alpha)}(f) \begin{cases} n^{\beta_1}; & q = 1, & m > \left[\frac{2}{\alpha_1} - 2\right] \\ n^{\beta_2}; & q > 1, & m > \left[\frac{4q-2}{\alpha_1} - 2\right] \end{cases}$$

where

$$\beta_1 = \alpha_1 \left(\frac{m}{2} - 1\right) + \sigma - 2, \quad \alpha_1 = \alpha(1 - \rho).$$

and

$$\beta_2 = \left(\alpha_1 \left(\frac{m}{2} + 1\right) + \sigma - 1\right) q - q^2.$$

*Proof.* Let  $zG'(z) = g(z)$ . Then  $\frac{(zG'(z))'}{G'(z)} \in P_m(p_\kappa) \subset P_m(\rho)$ ,  $\rho = \frac{\kappa}{\kappa+1}$ . This implies  $G \in V_m(\rho)$ , so from (2.1) we have

$$(4.3) \quad G'(z) = (G_1'(z))^{(1-\rho)}, \quad G_1 \in V_m \quad (z \in E).$$

Since  $f \in B_m(\alpha, \kappa)$ , so we can write

$$(4.4) \quad z f'(z) = f^{(1-\alpha)}(z) g^\alpha(z) h^\sigma(z), \quad \text{for } h \in P.$$

From (4.4) and result due to Brannan [1], the above equation implies

$$(4.5) \quad z f'(z) = z^\alpha f^{(1-\alpha)}(z) \left[ \frac{(g_1'(z))^{\left(\frac{m}{4} + \frac{1}{2}\right)}}{(g_2'(z))^{\left(\frac{m}{4} - \frac{1}{2}\right)}} \right]^{\alpha(1-\rho)} \cdot h^\sigma(z), \quad g_1, g_2 \in C.$$

For any univalent function  $s$ , we can choose  $z_1 = z_1(r)$  with  $|z_1| = r$  such that

$$(4.6) \quad \max_{|z|=r} |(z - z_1) s(z)| \leq \frac{2r^2}{1 - r^2}, \quad (\text{see [2]}).$$

Thus, from (4.5) with  $z g'_i = s_i \in S$  and  $m \geq \left[ \frac{2+4j}{\alpha(1-\rho)} - 2 \right]$ , we have

$$\begin{aligned} |\Delta_j(n, z_1, z f')| &\leq \frac{M^{1-\alpha}(r, f)}{2\pi r^{n+j-\alpha}} \left( \frac{2r^2}{1 - r^2} \right)^j (2)^{\alpha_1(\frac{m}{2}-1)} \\ &\quad \times \int_0^{2\pi} |s_1(z)|^{\alpha_1(\frac{m}{4}+\frac{1}{2})-j} |h^\sigma(z)| d\theta. \end{aligned}$$

where we have used distortion result for starlike function  $s_2$ , we can rewrite above inequality as;

$$(4.7) \quad |\Delta_j(n, z_1, z f')| = O(1) \cdot M^{1-\alpha}(r, f) \left( \frac{1}{1 - r} \right)^j \times \left[ \frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\alpha_1(\frac{m}{4}+\frac{1}{2})-j} |h^\sigma(z)| d\theta \right],$$

where  $O(1)$  denotes a constant.

By making use of Holder's inequality, we have

$$(4.8) \quad \begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\alpha_1(\frac{m}{4}+\frac{1}{2})-j} |h^\sigma(z)| d\theta \\ &\leq \left[ \frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\{\alpha_1(\frac{m}{4}+\frac{1}{2})-j\} \frac{2}{2-\sigma}} d\theta \right]^{\frac{2-\sigma}{2}} \times \left[ \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \right]^{\frac{\sigma}{2}}. \end{aligned}$$

Now, from (4.7), (4.8), Lemma 2.1 and subordination for starlike functions, we obtain for  $m > \left[ \frac{2(1+2j)}{\alpha_1} - 2 \right]$

$$\Delta_j(n, z_1, z f') = O(1) \cdot M_r^{(1-\alpha)}(f) \left( \frac{1}{1 - r} \right)^{\alpha_1(\frac{m}{2}-1)-j+\sigma-1}, \quad (r \rightarrow 1),$$

and using Lemma 4.2,  $r = 1 - \frac{1}{n}$ , we get

$$\Delta_j(n, z_1, f) = O(1) \cdot M_r^{(1-\alpha)}(f) n^{\alpha_1(\frac{m}{2}-1)-j+\sigma-2}, \quad (n \rightarrow \infty).$$

For  $j > 0$ , we have  $q > 1$ , follow the similar technique used in [10] along with Lemma 4.1, we get for  $m > \left( \frac{4q-2}{\alpha_1} - 2 \right)$

$$H_q(n) = O(1) M_r^{(1-\alpha)}(f) \cdot n^{\beta_2},$$

where  $\beta_2 = (\alpha_1 (\frac{m}{2} + 1) + \sigma - 1) q - q^2$ . If  $j = 0$ , we have  $q = 1$  and  $\Delta_0(n, z_1, f) = a_n$ . This gives us, for  $m > \left[ \frac{2}{\alpha_1} - 2 \right]$

$$H_1(n) = O(1)M_r^{(1-\alpha)}(f) .n^{\beta_1},$$

where  $\beta_1 = \alpha_1 (\frac{m}{2} - 1) + \sigma - 2$  and  $\alpha_1 = \alpha(1 - \rho)$ . □

With suitable choices of parameters, we obtain some known results; see [12, 13, 17].

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