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On Measure Chaotic Dynamical Systems

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Abstract. In this paper, we introduce chaotic measure for discrete and continuous dynamical systems and study some properties of measure chaotic systems. Also relationship between chaotic measure, ergodic and expansive measures is investigated. Finally, we prove a new version of variational principle for chaotic measure.

1. Introduction

Chaos is an interesting topic of dynamical systems. Unfortunately, there is not decisive mathematical definition of chaos. Devaney chaos is one of the most popular definitions of chaos in which such systems must exhibit sensitive dependence to initial conditions; topological transitivity, and dense periodic orbits. Later, in [2], it is proven that if a system is transitive with dense periodic orbits, then sensitivity dependence to initial condition is guaranteed. It is clear that the sensitive dependence indicates unpredictability of chaos phenomenon, so sensitivity is an essential condition of chaotic behavior. Accordingly, it is important to study what systems have sensitivity. Recently, stronger forms of sensitivity for dynamical systems is defined, such as, ergodically sensitive, syndetically sensitive, and cofinitely sensitive. Due to that ergodic-theoretic plays an important role in dynamical systems. In 2005, Cadre and Jacob introduced pairwise sensitivity and investigated the relationship between pairwise sensitivity and weakly mixing and positive entropy [4]. After that the notions of measurable sensitivity and weak measurable sensitivity have been introduced [7], which are ergodic-theoretic versions of strong sensitivity and sensitivity, respectively.

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After presenting definition of expansivity, Morales introduced general concept of it as measure expansivity as well [5, 9]. In this paper, we attempt to generalize a notion of sensitivity. We introduce chaotic measure for discrete and continuous dynamical systems. In Example 2.3, we show that sensitivity implies measure chaotic but the converse is not true, and we attempt to discover a relation between expansive measure and chaotic measure. In Remark 2.6, we give an example of a map which is ergodic and is not measure chaotic. In section 3, we will study some properties of measure chaotic systems. First, we show that the suspension flow of a μ-chaotic map is $\bar{\mu}$-chaotic flow and vice versa (Theorem 3.3). As well, we show that the chaotic measure is invariant under the conjugacy (Theorem 3.5). Investigation of μ-chaotic on product of flows is another result of this paper. Indeed we prove that product of flows is μ-chaotic if and only if one of them is μ-chaotic (Theorem 3.6). In the last part of this section, we prove the variational principle for chaotic measures (Theorem 3.11) and as a corolllary we show that an ergodic measure with the positive entropy in a probability space is a chaotic measure.

2. Preliminaries

Let (M, d) be a compact metric space and let $f : M \to M$ be a continuous map. We say that a point $x \in M$ is *sensitive* if, there is $\delta > 0$ such that for every $\epsilon > 0$ there exist $y \in B_\epsilon(x)$, and $n \geq 0$ with $d\left(f^n(x), f^n(y)\right) > \delta$. In this case δ is called sensitivity constant. We denote the set of all sensitive points by S. A map f is said to be *sensitive to initial condition* if there is $\delta > 0$ such that every point of M is sensitive with sensitive constant $\delta > 0$. Here, we show that the set of all sensetive points is measurable.

Proposition 2.1. *S, the set of all sensitive points of f, is measurable.*

Proof. We define

$$\begin{aligned} s(x) &= \inf_{\epsilon>0} \sup \left\{diam(f^n(B_\epsilon(x)) \ ; \ n \in \{0, 1, \ldots\}\right\} \\ &= \inf \left\{d_\infty(U) \ ; \ x \in U \text{open}\right\}, \end{aligned}$$

where $d_\infty(U) = \sup_{n\geq 0} \left\{d(f^n(y), f^n(z)); y, z \in U\right\}$. We can see that x is a sensitive point with sensitivity constant $\frac{s(x)}{4}$ if and only if $s(x) > 0$. By definition of $s(x)$, for every $\epsilon > 0$ there is open set U of x such that

$$d_\infty(U) < s(x) + \epsilon,$$

then for every $y \in U$, $s(y) \leq d_\infty(U) < s(x) + \epsilon$, hence $s : M \to \mathbb{R}$ is upper-semi continuous. This is well known that any upper-semi contonuous map is measurable, so s is measurable. Therefore $S = s^{-1}((0, +\infty))$ is a measurable set. $\square$

Let β be the Borel σ-algebra on M. Denote by $\mathcal{M}(M)$ the set of all Borel's probability measures on M endowed with weak* topology. A Borel probability measure μ is called f-invariant if, for any Borel set B, it holds $\mu\left(f^{-1}(B)\right) = \mu(B)$. We set

$$\mathcal{M}_f(M) = \{\mu \in \mathcal{M}(M) \ : \ \mu \text{ is } f\text{-invariant}\}.$$

An f-invariant probability measure is called *ergodic* if any f-invariant Borel set has measure 0 or 1. Here we define the notion of the measure chaotic as a generalization of the notion of the sensitivity.

Definition 2.2. Let $\mu \in \mathcal{M}_f(M)$. We say that f is μ-chaotic if $\mu(S) = 1$.

It is easy to see that any sensitive map is μ-chaotic, but the converse is not true. In the following example, we show that f is not sensitive but it is μ-chaotic for some $\mu \in \mathcal{M}_f(M)$.

Example 2.3. Given $f : [0,1] \to [0,1]$ as follows: where f is identity

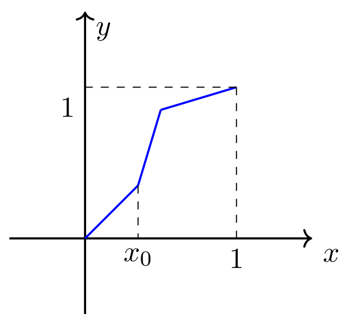

FIGURE 1. The graph of the map f

on $[0, x_0]$, clearly f is not sensitive and x_0 is a sensitive point. Consider

$$\mu = \delta_{x_0}(A) = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A. \end{cases}$$

since x_0 is a fixed point, so $\delta_{x_0} \in \mathcal{M}_f(M)$ and we have $\delta_{x_0}(S) = 1$; therefore f is μ-chaotic.

In the following, we give an example that shows f is μ-chaotic for every invariant measure μ, but it is not sensitive.

Example 2.4. Consider $f : [0,1] \cup \{2,3,\dots\} \to [0,1] \cup \{2,3,\dots\}$ as follows:

$$f(x) = \begin{cases} g(x), & x \in [0,1], \\ h(x), & x \in \{2,3,\ldots\}, \end{cases}$$

where

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ -2x+2, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

and $h(n) = n+1,\ n = 2,3,\ldots$.
The map $g(x)$ is sensitive to initial condition on $[0,1]$ and $h(x)$ is not sensitive, so f is not sensitive and the set of all sensitive points of f is equal to $[0,1]$. The space $[0,1] \cup \{2,3,\ldots\}$ is not compact, so we need to show $\mathbb{M}_f(M) \neq \emptyset$. In [8] example 6.4, the author proved that the Lebesgue measure is invariant under map $g(x)$, and since $Leb(\{n\}) = Leb(\{n+1\}) = 0$, so $Leb \in \mathbb{M}_f(M)$. Now we show that f is μ-chaotic for every $\mu \in \mathbb{M}_f(M)$. Let μ be an f-invariant probability measure. We claim that $\mu([0,1]) = 1$. If $0 \leq \mu([0,1]) < 1$, then there is $n \in \{2,3,\dots\}$ such that $\mu(n) > 0$. So $\mu(n+1) = \mu(f(n)) = \mu(n) > 0$. Therefore

$$\begin{aligned} \mu(\{2,3,\ldots\}) &= \mu\left(\bigcup_{n=2}^{\infty} \{n\}\right) \\ &= \sum_{n=2}^{\infty} \mu(n) \\ &= \infty, \end{aligned}$$

which is contradicts by probability of μ. Hence $\mu([0,1]) = 1$ and f is μ-chaotic.

The above counterexample motivates us to study the class of systems defined below.

Definition 2.5. The map f is called measure chaotic if f is μ-chaotic for every Borel invariant measure μ.

Remark 2.6. Ergodicity is not enough condition to imply measure chaotic. Indeed, consider irrational rotation $f(x) = x + \alpha$ on S^1. In [15], one can see that the irrational rotation on unit circle S^1 is ergodic with respect to the Lebesgue measure. Since f is isometry, so it does not have any sensitive point. Then, for Lebesgue invariant measure μ, $\mu(S) = 0$.

Weakly mixing is a stronger condition than ergodicity, and if f is a weakly mixing, then it is sensitive to initial condition (see [6]). Therefore we can conclude that f is measure chaotic.

In the next remark, we discuss the relationship between measure chaotic and measure expansive. We recall notion of measure expansive. An expansive measure of continuous map $f : M \to M$ of a metric space M is a Borel measure μ for which there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$, for all $x \in M$, where

$$\Gamma_\delta(x) = \left\{y \in M : d\left(f^i(x), f^i(y)\right) \leq \delta \text{ for all } i \in \mathbb{Z}\right\}.$$

Remark 2.7. In [1, Proposition 3.4], Arbieto and Morales proved that, for every expansive measures μ, the set of all Lyapunov stable points has measure zero, so μ is chaotic measure, but the converse does not hold. For this, we consider f as introduced in Example 2.3. By contradiction, let μ be expansive. Then there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$, for all $x \in M$. Consider $x = x_0 - \frac{\delta}{2}$, so $d\left(f^i(x), f^i(x_0)\right) = d(x, x_0) < \delta$ and hence $x_0 \in \Gamma_\delta(x)$. Therefore $\mu(\Gamma_\delta(x)) = 1$, that is a contradiction. Hence μ is not expansive, but in Example 2.3, we showed that μ is chaotic.

Here we extend notion of measure chaotic for continuous systems. A flow of M is a map $\varphi : M \times \mathbb{R} \to M$ satisfying $\varphi(x, 0) = x$ and $\varphi(\varphi(x, s), t) = \varphi(x, t + s)$ for all $t, s \in \mathbb{R}$ and $x \in M$. A flow is continuous if it is continuous with respect to the product metric of $M \times \mathbb{R}$. The *time t-map* $\varphi_t : M \to M$ defined by $\varphi_t(x) = \varphi(x, t)$ is a continuous map of M for all $t \in \mathbb{R}$. The set $O(x) = \{\varphi_t(x) \ : \ t \in \mathbb{R}\}$ is said to be the orbit of φ_t through the point x.

Definition 2.8. A point x is *sensitive*, if there exists $r > 0$ such that, for any neighborhood U of x, there are $y \in U$ and $t > 0$ such that $d(\varphi_t(x), \varphi_t(y)) \geq r$. We denote by S, the set of all sensitive points.

Definition 2.9. A point p is *Lyapunov stable*, if for given $\epsilon > 0$, there is $\delta > 0$ such that $d(p, q) < \delta$ implies that $d(\varphi_t(p), \varphi_t(q)) < \epsilon$ for all $t \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of all positive real numbers.

Let β be the Borel σ-algebra on M. A Borel probability measure μ is called φ-invariant if for any Borel set B, $\mu(\varphi_t(B)) = \mu(B)$ for all $t \in \mathbb{R}$.

$$\mathcal{M}_\varphi(M) = \{\mu \in \mathcal{M}(M) \ : \ \mu \text{ is } \varphi\text{-invariant}\}.$$

Definition 2.10. Let $\mu \in \mathcal{M}_\varphi(M)$. We say that φ is μ-chaotic if $\mu(S) = 1$.

We say that a flow φ is measure chaotic if, for every Borel invariant measure μ, flow φ is μ-chaotic.

3. Main Results

A useful property of chaotic measure is given by the following lemma.

Lemma 3.1. *If a continuous flow φ is μ-chaotic, then, for all $t \neq 0$, the flow φ_t is μ-chaotic.*

Proof. Let $t_0 \in \mathbb{R} \setminus \{0\}$ be arbitrary and fixed. Take $\varphi_{t_0} = f$ and denote the set of all sensitive points of φ and the set of all sensitive points of f by S and S', respectively. The flow φ is μ-chaotic so $\mu(S^c) = 0$, therefore it is enough to show that $(S')^c \subseteq S^c$. Given $\epsilon > 0$, by continuity of φ there is $\eta > 0$ such that if $d(a,b) < \eta$ then, $d(\varphi_s(a), \varphi_s(b)) < \epsilon$ for all $s \in [0, t_0]$. Let $x \in (S')^c$, then there exists $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f^n(x), f^n(y)) < \eta$ for every $n \in \mathbb{N}$. For all $t \in \mathbb{R}^+$, we have $nt_0 \leq t \leq (n+1)t_0$ for some $n \in \mathbb{Z}^+$, and

$$\begin{aligned} d(\varphi_t(x), \varphi_t(y)) &= d\left(\varphi_{t-nt_0}\left(f^n(x)\right), \varphi_{t-nt_0}\left(f^n(y)\right)\right) \\ &< \epsilon. \end{aligned}$$

Hence $x \in S^c$. □

Let (M, d) be a metric space and let $f : M \to M$ be a homeomorphism. Consider

$$\Omega = M \times [0,1]/ \sim$$

where $\sim$ is the identification of $(x,1)$ with $(f(y),0)$ and give it the usual quotient topology. The standard *suspension flow of f* is the flow ψ_t on Ω defined by

$$\psi_t(y,s) = (y, t+s), \quad \text{for } 0 \leq \mathrm{t+s} < 1.$$

We consider metric D on Ω, known as the *Bowen–Walters metric*[3], as follows: We shall consider horizontal and vertical segments. Given $x, y \in M$ and $t \in [0,1]$, we define the length of the *horizontal segment* $[(x,t),(y,t)]$ by

$$d_1((x,t),(y,t)) = (1-t)\, d(x,y) + t\,(f(x), f(y)).$$

Given $(x,t),(y,s) \in \Omega$ on the same orbit. We define the length of *vertical segment* $[(x,t),(y,s)]$ by

$$d_2((x,t),(y,s)) = \inf\{|r| \ : \ \psi_r(x,t) = (y,s) \text{ and } \mathrm{r} \in \mathbb{R}\}.$$

Finally, given two point $(x,t),(y,s) \in \Omega$, consider all finite chains $\{(z_i,t_i)\}_{i=1}^n$ of elements of Ω such that $(z_1,t_1) = (x,t)$, $(z_n,t_n) = (y,s)$, and for each $1 \leq i \leq n$, either $[(z_i,t_i),(z_{i+1},t_{i+1})]$ is a horizontal segment or $[(z_i,t_i),(z_{i+1},t_{i+1})]$ is a vertical segment.

$$D((x,t),(y,s)) = \inf\{\text{length of all chains between } (x,t) \text{ and } (y,s)\}.$$

For a subset $A \subset \Omega$, we denote

$$\pi(A) = \{x \in M \ : \ (x,t) \in A, \text{ for some } 0 \leq t < 1\}.$$

Lemma 3.2. *If S_M and S_Ω are the set of all sensitive pints of f and ψ_t (where ψ_t is the suspension flow of f), respectively. Then $\pi(S_\Omega^c) \subseteq S_M^c$.*

Proof. Let $x \in \pi\left(S^c_\Omega\right)$ be arbitrary. So $(x,t) \in S^c_\Omega$ for some $0 \leq t < 1$. Given $\epsilon > 0$, since $(x,t) \in S^c_\Omega$, there exists $\delta > 0$ such that for all $(y,t') \in B_\delta(x,t)$,

$$D\left(\psi_s(x,t), \psi_s\left(y,t'\right)\right) < \epsilon. \tag{3.1}$$

By continuity of f, there exists $0 < \delta' < \delta$ such that if $d(x,y) < \delta'$, then $d(f(x), f(y)) < \frac{\delta}{2}$. Since

$$\begin{aligned} D((x,t),(y,t)) &= (1-t)\, d(x,y) + t d(f(x), f(y)) \\ &< (1-t)\,\delta + t\delta \\ &= \delta, \end{aligned}$$

so $(y,t) \in B_\delta(x,t)$. In particular the inequality (3.1) holds for $s = n-t$ and for all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} d\left(f^n(x), f^n(y)\right) &= (1-0)d\left(f^n(x), f^n(y)\right) + 0d\left(f^{n+1}(x), f^{n+1}(y)\right) \\ &= D\left(f^n(x),0),(f^n(y),0)\right) \\ &= D\left((x, n-t+t), (y, n-t+t)\right) \\ &= D\left(\psi_{n-t}(x,t), \psi_{n-t}(y,t)\right) \\ &< \epsilon. \end{aligned}$$

Therefore $x \in S^c_M$. □

Let f be μ-chaotic and ψ_t be the suspension flow of f. We define $\bar{\mu}$ as follows:

$$\int_\Omega \xi d\bar{\mu} := \int_M \int_0^1 \xi(x,t)\, dt d\mu, \qquad \text{for all } \xi \in C^0(\Omega).$$

If μ is an invariant measure of f, then $\bar{\mu}$ is an invariant measure of ψ_t; see [11].

Theorem 3.3. *The map f is μ-chaotic if and only if ψ is $\bar{\mu}$-chaotic.*

Proof. We show that if f is $\mu-$chaotic, then the suspension flow ψ_t is $\bar{\mu}$-chaotic.

Let $\bar{\mu}\left(S^c_\Omega\right) > 0$, where S^c_Ω is the set of all stable points of ψ_t in Ω. Then

$$\begin{aligned} 0 &< \bar{\mu}\left(S^c_\Omega\right) \\ &= \int_\Omega \chi_{S^c_\Omega} d\bar{\mu} \\ &= \int_M \int_0^1 \chi_{S^c_\Omega}(x,t)\, dt d\mu \\ &\leq \int_M \chi_{\pi\left(S^c_\Omega\right)}(t)\, d\mu \end{aligned}$$

$$= \mu\left(\pi\left(S^c_\Omega\right)\right).$$

By Lemma 3.2, $\mu\left(\pi\left(S^c_\Omega\right)\right) \leq \mu\left(S^c_M\right)$, so $\mu\left(S^c_M\right) > 0$, which is a contradiction, since f is μ-chaotic.

Conversely, assume that the suspension flow ψ of f is $\bar{\mu}$-chaotic. Fix $0 < T < 1$. For every $x \in M$ and $0 \leq s \leq T$

$$\begin{aligned}\psi_1(x,s) &= (x, s+1)\\ &= (f(x), s)\\ &= (f \times Id)(x,s),\end{aligned}$$

where Id is the identity map on $[0,T]$. Lemma 3.1 implies that $\bar{\mu}$ is chaotic for ψ_1, so $f \times Id : M \times [0,T] \to M \times [0,T]$ is $\bar{\mu}$-chaotic. So $\bar{\mu}\left(S^c_{\psi_1}\right) = \mu\left(S^c_f\right) \times Leb\left([0,T]\right) = 0$. Since $Leb\left([0,T]\right) \neq 0$, hence μ is chaotic for f. $\square$

Recall that if M and N are metric spaces, the flows $\varphi_t : M \to M$ and $\psi_t : N \to N$ are *conjugate* if there is a homeomorphism $h : M \to N$ mapping orbits of φ_t onto orbits of ψ_t.

Definition 3.4. Given measurable spaces (M, Σ_1) and (N, Σ_2), a measurable mapping $f : M \to N$ and a measure $\mu : \Sigma_1 \to [0, +\infty)$, the push-forward of μ is defined to be the measure $f_*\mu : \Sigma_2 \to [0, \infty)$ given by

$$f_*\mu(B) = \mu(f^{-1}(B)), \quad \text{for B} \in \Sigma_2.$$

The next theorem implies that the property of having chaotic measures is conjugacy invariant.

Theorem 3.5. *If f is an equivalence between φ and ψ, then μ is chaotic for φ if and only if $f_*\mu$ is chaotic for ψ.*

Proof. Denote by S_φ and S_ψ the set of sensitive points of φ and ψ, respectively. Let μ be chaotic for φ, so $\mu\left(S^c_\varphi\right) = 0$. We show that $f_*\mu\left(S^c_\psi\right) = 0$. Since $f_*\mu\left(S^c_\psi\right) = \mu\left(f^{-1}\left(S^c_\psi\right)\right)$, it is enough to prove $f^{-1}\left(S^c_\psi\right) \subseteq S^c_\varphi$. Given $\epsilon > 0$, by continuity of f^{-1} there exists $\alpha > 0$ such that if $d(a,b) < \alpha$, then $d\left(f^{-1}(a), f^{-1}(b)\right) < \epsilon$. Let $x \in f^{-1}\left(S^c_\psi\right)$, then there is $z \in S^c_\psi$ such that $f(x) = z$. Since $z \in S^c_\psi$, there exists $\delta > 0$ such that $d(z,w) < \delta$ implies $d\left(\psi_t(z), \psi_t(w)\right) < \alpha$. Since f is cotinuous, there is $\delta' > 0$ such that $d(x,y) < \delta'$ implies $d(f(x), f(y)) < \delta$. By putting $f(y) = w$ for some y, we have

$$\begin{aligned}d\left(\varphi_t(x), \varphi_t(y)\right) &= d\left(\varphi_t\left(f^{-1}(z)\right), \varphi_t\left(f^{-1}(w)\right)\right)\\ &= d\left(f^{-1}\left(\psi_t(z)\right), f^{-1}\left(\psi_t(w)\right)\right)\end{aligned}$$

$$< \epsilon.$$

Therefor $x \in S_{\varphi}^{c}$. For the converse, replace f by f^{-1}. □

Takure and Das in [12] discused sensitivity problems on product of semi flows. They proved that φ_{∞} is sensitive (multi sensitive or ergodically sensitive) if and only if there exists a positive integer k such that φ_k is sensitive (multi sensitive or ergodically sensitive), where $\varphi_{\infty} = \prod_{i=1}^{\infty} \varphi_i : \mathbb{R} \times \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{\infty} X_i$. This result motivate us to investigate the same property for measure chaotic flows.

Theorem 3.6. *$\varphi \times \psi$ is $\mu \otimes \nu$-chaotic if and only if either φ is μ-chaotic or ψ is ν-chaotic.*

Proof. Consider $S = \{(x, y) \mid (x, y) \text{ is sensitive point for } \varphi \times \psi\}$, $S_{\varphi} = \{x \mid x \text{ is sensitive point for } \varphi\}$ and $S_{\psi} = \{y \mid y \text{ is sensitive point for } \psi\}$.

Let $\varphi \times \psi$ be $\mu \otimes \nu$-chaotic, by contradiction let φ be not μ-chaotic and ψ is not ν-chaotic, i.e. $\mu\left(S_{\varphi}^{c}\right) > 0$ and $\nu\left(S_{\psi}^{c}\right) > 0$. Take $S_{x}^{c} = \{y \mid (x, y) \in S^{c}\}$, let $y \in S_{\psi}^{c}$ be arbitrary. Since $\mu\left(S_{\varphi}^{c}\right) > 0$ so S_{φ}^{c} is nonempty and we can take a point $x \in S_{\varphi}^{c}$. Given $\epsilon > 0$, there are δ_1 and δ_2 as the definition of stable point for x and y. Take $\delta = \min\{\delta_1, \delta_2\}$. For all $(x', y') \in B_{\delta}(x, y)$ we have $x' \in B_{\delta_1}(x)$ and $y' \in B_{\delta_2}(y)$ therefore,

$$\begin{aligned} d\left(\varphi \times \psi\left(t, (x, y)\right), \varphi \times \psi\left(t, (x', y')\right)\right) &= \max\left\{d\left(\varphi_t(x), \varphi_t(x')\right), d\left(\psi_t(y), \psi_t(y')\right)\right\} \\ &< \epsilon. \end{aligned}$$

Hence $(x, y) \in S^{c}$ so $S_{\psi}^{c} \subseteq S_{x}^{c}$ and we have $0 < \nu\left(S_{\psi}^{c}\right) \leq \nu\left(S_{x}^{c}\right)$. Since for every $x \in S_{\varphi}^{c}$ $\nu\left(S_{x}^{c}\right) > 0$, so $\mu \otimes \nu\left(S^{c}\right) = \int_M \nu\left(S_{x}^{c}\right) d\mu(x) > 0$ which is a contradiction.

Conversely, let φ be μ-chaotic $\mu\left(S_{\varphi}^{c}\right) = 0$. We show that $S^{c^y} \subseteq S_{\varphi}^{c}$ where $S^{c^y} = \{x \mid (x, y) \in S^{c}\}$. Fix $\epsilon > 0$ and let $x \in S^{c^y}$. There is $y \in X$ such that $(x, y) \in S^{c}$, so there is $\delta > 0$ such that for all $(x', y) \in B_{\delta}(x, y)$ and $t \in \mathbb{R}^{+}$ we have

$$\begin{aligned} d\left(\varphi \times \psi\left(t, (x, y)\right), \varphi \times \psi\left(t, (x', y)\right)\right) &= \max\left\{d\left(\varphi_t(x), \varphi_t(x')\right), d\left(\psi_t(y), \psi_t(y)\right)\right\} \\ &< \epsilon, \end{aligned}$$

hence $x \in S_{\varphi}^{c}$.

$$\begin{aligned} \mu \times \nu\left(S^{c}\right) &= \int \mu\left(S^{c^y}\right) d\nu(y) \\ &\leq \int \mu\left(S_{\varphi}^{c}\right) d\nu(y) \\ &= 0, \end{aligned}$$

therefor $\varphi \times \psi$ is $\mu \times \nu$-chaotic. □

3.1. **Variational Principle for Chaotic Measure.** Let $f : M \to M$ be a continuous map on a compact metric space M. For every $n \in \mathbb{N}$ we define a map $d_n : M \times M \to \mathbb{R}$ by setting

$$d_n\left(x, y\right) = \max_{0 \leq i < n} \left\{ d\left(f^i\left(x\right), f^i\left(y\right)\right) \qquad \text{for every } x, y \in M \right\}.$$

Let $n \in \mathbb{N}$ and $\epsilon > 0$. We say that a set $E \subseteq M$ is $(n, \epsilon)-$separated if for every $x, y \in E$ with $x \neq y$, we have $d_n\left(x, y\right) \geq \epsilon$. Let $sep\left(n; \epsilon; f\right) = max\left\{|E| \ : E \subset M;\ E \text{ is } (n, \epsilon)\text{-separated}\right\}$ where $|E|$ is the cardinality of E. Now we can define topological entropy.

Definition 3.7. [14] Put $h\left(f, \epsilon\right) = \limsup_{n\to\infty} \frac{1}{n} \log sep\left(n, \epsilon, f\right)$. Then the topological entropy of the map f is defined as

$$h\left(f\right) = \lim_{\epsilon \to 0} h\left(f, \epsilon\right).$$

Let $(M, \mathcal{B}, \mu)$ be a probability space and $f : M \to M$ be a measure preserving map. Let $\alpha = \{A_1, \cdots, A_k\}$ be a finite partitions of M. For two partition α , β, set $\alpha \vee \beta := \{A \cap B \quad A \in \alpha,\ B \in \beta\}$. So, elements of $\bigvee_{i=0}^{n-1} f^{-i}\alpha$ are sets of the form

$$\left\{x \ : \ x \in A_{i_0},\ fx \in A_{i_1}, \cdots, f^{n-1}x \in A_{i_{n-1}}\right\},$$

for some $(i_0, i_1, \cdots, i_{n-1})$.

Definition 3.8. [16] The metric entropy of f, is defined as follows:

$$H\left(\alpha\right) := H\left(\mu(A_1), \cdots, \mu(A_k)\right), \quad \text{where} \quad H\left(p_, \cdots, p_k\right) = -\sum p_i \log p_i,$$

$$h_\mu\left(f, \alpha\right) := \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right),$$

$$h_\mu(f) := \sup_\alpha h_\mu\left(f, \alpha\right).$$

The basic relationship between topological entropy and measure-theoretic entropy is proved in [14] as follows.

Theorem 3.9 (variational principle)**.** *Let $f : M \to M$ be a continuous map of a compact metric space M; then $h(f) = \sup\{h_\mu \ : \ \mu \in \mathcal{M}_f(M)\}$.*

Let us mention one of the interesting consequences, which guarantees that the the variational principle holds for ergodic measures.

Corollary 3.10. *[14] Let $f : M \to M$ be a continuous map on a compact metric space M. Then $h\left(f\right) = \sup\left\{h_\mu \ : \ \mu \in \mathcal{E}_f\left(M\right)\right\}$, where $\mathcal{E}_f\left(M\right)$ denotes the collection of all ergodic members of $\mathcal{M}_f\left(M\right)$.*

In the following, we attempt to conclude the variational principle for chaotic measure. For this, we use the definition of entropy given in [13].

Theorem 3.11. *Let φ be a flow on a compact metric space M which has no fixed point. Then*

$$h(\varphi) = \sup\{h_\mu(\varphi) \ : \ \mu \text{ is a chaotic measure}\}. \tag{3.2}$$

Proof. We consider $\varphi_1 = f$. In [10, Corollary 1], Sun and Vargas proved that $h_\mu(\varphi) = h_\mu(\varphi_1)$ for any ergodic φ invariant probability measure μ. Now, if $h(f) = 0$, then (3.2) holds. Otherwise, if $h(f) > 0$, then, by Corollary 3.10, there exists an ergodic measure μ such that

$$0 < h(f) - \epsilon < h_\mu(f).$$

So it is enough to show $\mu(S) = 1$. On the contrary, let $\mu(S^c) > 0$. Since the set of recurrent points is full measure, we can consider $x \in S^c \cap R$, where R is the set of recurrent points. Hence, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$; implies $d\left(f^i(x), f^i(y)\right) < \epsilon$ for $i = 1, 2, \ldots$.
Define the *dynamical ball* $B(x, n, \epsilon)$ as the set

$$B(x, n, \epsilon) = \left\{y \in M \ : \ d\left(f^i(x), f^i(y)\right) < \epsilon \text{ for every } 0 \leq i \leq n-1\right\}.$$

So $N_\delta(x) \subseteq B(x, n, \epsilon)$ for all $n \in \mathbb{N}$. Define

$$R_n(x, \epsilon) = \inf\left\{k \geq 1 \ : \ f^k(x) \in B(x, n, \epsilon)\right\};$$

since x is a recurrent point, so there exists k_0 such that $f^{k_0}(x) \in N_\delta(x)$, and since x is Lyapunov stable, then $R_n(x, \epsilon) \leq k_0$. Now by [13, Theorem A], we have

$$\begin{aligned} h_\mu(f) &= \bar{h}(f, x) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R_n(x, \epsilon) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log k_0 \\ &= 0, \end{aligned}$$

that is a contradiction. □

Using the proof of the previous theorem, one can obtain the following corollary.

Corollary 3.12. *An ergodic measure with the positive entropy in a probability space is a chaotic measure.*

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