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Fixed Point Results for Extensions of Orthogonal Contraction on Orthogonal Cone Metric Space

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Abstract. In this paper, some fixed point results of self mapping which is defined on orthogonal cone metric spaces are given by using extensions of orthogonal contractions. And by taking advantage of these results, the necessary conditions for self mappings on orthogonal cone metric space to have P property are investigated. Also an example is given to illustrate the main results.

1. Introduction and Preliminaries

The cone metric is obtained by selecting an ordered Banach space instead of real numbers for the range of the metric mapping. The primary studies on this subject are given by Huang and Zhang. (see [8]) The authors introduced cone metric spaces and proved some fixed point theorems of contractive mappings on cone metric spaces in their research article. Then, different fixed point theorems are obtained on cone metric spaces by many researchers. (see [1, 2, 9, 10, 13–16]).

On the other hand, the notion of orthogonal set and orthogonal metric spaces are introduced by Gordji.(see [7]) Then, some researchers present the generalizations of the theorems on this type sets. (see [3, 6, 11, 12])

Very recently, a new concept of orthogonal cone metric spaces, orthogonal completeness and orthogonal continuity are defined in [4] and illustrative examples are given for this new definitions. Also, an example is given for orthogonal complete cone metric space which is not complete cone metric space. Furthermore, fixed point theorems and their corollaries are proved on orthogonal cone metric spaces and also

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some fixed point theorems for contractive mappings are presented on ordered orthogonal cone metric spaces in [5].

In this paper, some fixed point results of self mapping which is defined on orthogonal cone metric spaces are given by using extensions of orthogonal contractions. And by taking advantage of these results, the necessary conditions for self mappings on orthogonal cone metric space to have P property are investigated. Also an example is given to illustrate the main results.

In the sequel, respectively, $\mathbb{Q}, \mathbb{Q}^c, \mathbb{Z}, \mathbb{R}$ denote rational numbers, irrational numbers, integers and real numbers.

Definition 1.1 ([7])**.** Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp y), \tag{1.1}$$

it is called an orthogonal set (shortly O-set), $(X, \perp)$ is called an O-set and the element x_0 is called an orthogonal element.

Example 1.2 ([6])**.** Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.

By the following example, we can see that x_0 is not necessarily unique.

Example 1.3 ([6])**.** Let $X = [0, \infty)$, we define $x \perp y$ if $xy \in \{x, y\}$, then by setting $x_0 = 0$ or $x_0 = 1$, $(X, \perp)$ is an O-set.

Definition 1.4 ([7])**.** Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.

Definition 1.5 ([7])**.** A sequence $\{x_n\}$ is called an orthogonal sequence (shortly O-sequence) if

$$(\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \vee (\forall n \in \mathbb{N}; x_{n+1} \perp x_n). \tag{1.2}$$

Similarly, a Cauchy sequence $\{x_n\}$ is said to be an orthogonally Cauchy sequence (shortly O-Cauchy sequence) if

$$(\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \vee (\forall n \in \mathbb{N}; x_{n+1} \perp x_n). \tag{1.3}$$

Definition 1.6 ([7])**.** Let $(X, \perp)$ be an orthogonal set and d be an usual metric on X. Then $(X, \perp, d)$ is called an orthogonal metric space (shortly O-metric space).

Definition 1.7 ([8])**.** Let E be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, nonempty, $P \neq \{\theta_E\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta_E$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \prec\prec y$ will stand for $y - x \in \text{int}P$, where int$P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \preceq x \preceq y$ implies $\|x\|_E \leq K \|y\|_E$.

The least positive number satisfying above inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq \cdots \preceq y, \tag{1.4}$$

for some $y \in E$, then there exists $x \in E$ such that $\|x_n - x\|_E \to 0 (n \to \infty)$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with int$P \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

Definition 1.8 ([8])**.** Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

- (d_1) $\theta_E \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta_E$ if and only if $x = y$.
- (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Lemma 1.9 ([15])**.** *Let (X, d) be a cone metric space. Then for each $\theta \prec\prec c$, $c \in E$, there exists $\delta > 0$ such that $c - x \in \text{int}P$ whenever $\| x \| < \delta$, $x \in E$.*

Definition 1.10 ([4])**.** Let $(X, \perp)$ be an orthogonal set and d be a cone metric on X. Then $(X, \perp, d)$ is called an orthogonal cone metric space (briefly O-cone metric space).

Example 1.11 ([4])**.** Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$ and $X = \mathbb{Z}$ and $d : X \times X \to E$, $d(x, y) = (| x - y |, \alpha | x - y |)$ where $\alpha \geq 0, \alpha \in \mathbb{R}$. Assume that binary relation $\perp$ on $X = \mathbb{Z}$ as in the Example 1.2, then $(X, d, \perp)$ is an orthogonal cone metric space.

Example 1.12 ([4])**.** Let $q, b \in \mathbb{R}$ where $q \geq 1, b > 1$, $E = \{\{x_n\} \mid x_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} (| x_n |)^q < \infty\}$ and $P = \{\{x_n\} \in E \mid x_n \geq 0, \forall n \in \mathbb{N}\}$.

Assume that $(X, \perp, \rho)$ is an orthogonal metric space, then the mapping

$$d : X \times X \to E, d(x, y) = \left(\frac{\rho}{b^n}\right)^{\frac{1}{q}}, \tag{1.5}$$

can be defined on X and this mapping is an orthogonal cone metric. So $(X, \perp, d)$ is an orthogonal cone metric space.

Example 1.13 ([4])**.** Let $E = (C_{\mathbb{R}}[0, \infty), ||\ .\ ||_\infty)$ and $P = \{f \in \mathbb{E} \mid f(t) \geq 0\}$. Assume that $(X, \perp, \rho)$ is an orthogonal metric space, then the mapping

$$d : X \times X \to E, d(x, y) = f_{x,y} \text{ where } f_{x,y}(t) = \rho(x, y)t, \tag{1.6}$$

can be defined on X and this mapping is an orthogonal cone metric. So $(X, \perp, d)$ is an orthogonal cone metric space.

Definition 1.14 ([4])**.** Let $(X, \perp, d)$ be an O-cone metric space. Let $\{x_n\}$ be an O-sequence in X and $x \in X$. If for any $c \in E$ with $\theta \prec\prec c$ there is $N \in \mathbb{N}$ such that for all $n \geq N(n \in \mathbb{N})$, $d(x_n, x) \prec\prec c$, then x_n is said to be convergent and the sequence $\{x_n\}$ converges to x(or x is the limit of $\{x_n\}$). We denote this by

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x(n \to \infty). \tag{1.7}$$

Definition 1.15 ([4])**.** Let $(X, \perp, d)$ be an O-cone metric space. Let $\{x_n\}$ be an O-sequence in X. If for any $c \in E$ with $\theta \prec\prec c$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N(n, m \in \mathbb{N})$, $d(x_n, x_m) \prec\prec c$, then O-sequence x_n is called an O-Cauchy sequence in X.

Definition 1.16 ([4])**.** Let $(X, \perp, d)$ be an O-cone metric space. If every O-Cauchy sequence in X is convergent, then $(X, \perp, d)$ is called an O-complete cone metric space.

Lemma 1.17 ([4])**.** *Let $(X, \perp, d)$ be an O-cone metric space, $\{x_n\}$ be an O-sequence in X. If the sequence $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is O-Cauchy sequence.*

Definition 1.18 ([4])**.** Let $(X, \perp, d)$ be an O-cone metric space. If for any O-sequence $\{x_n\}$ in X, there is an O-subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X, then $(X, \perp, d)$ is called a sequently compact O-cone metric space.

Definition 1.19 ([4])**.** Let $(X, \perp, d)$ be an O-cone metric space and $\lambda \in \mathbb{R}, 0 < \lambda < 1$. A mapping $f : X \to X$ is said to be an orthogonal contraction (shortly $\perp$-contraction) with Lipschitz constant λ when

$$d(fx, fy) \preceq \lambda d(x, y) \text{ if } x \perp y. \tag{1.8}$$

Definition 1.20 ([4]). Let $(X, \perp, d)$ be an O-cone metric space. A mapping $f : X \to X$ is called orthogonal preserving (shortly $\perp$-preserving) when

$$fx \perp fy \text{ if } x \perp y. \tag{1.9}$$

Definition 1.21 ([4]). Let $(X, \perp, d)$ be an O-cone metric space. A mapping $f : X \to X$ is called orthogonal continuous (shortly $\perp$-continuous) at $x \in X$ if for each O-sequence $\{x_n\}$ in X such that $x_n \to x$ then $f(x_n) \to f(x)$. Also f is $\perp$-continuous on X if f is $\perp$-continuous in each $x \in X$.

The following remarks note which are given in [4].

Remark 1.22 ([4]).
- (i) It is easy to see that every Lipschitz contraction is O-Lipschitz contraction. The converse of the statement is not true in general.
- (ii) There are $\perp$-preserving or not $\perp$-preserving mappings on O-cone metric space.
- (iii) It is easy to see that every continuous mapping is $\perp$-continuous. The converse of the statement is not true in general.
- (iv) Every complete cone metric space is an O-complete cone metric space. The converse of the statement is not true in general.

Also the following fixed point results are given in [4].

Theorem 1.23 ([4]). *Let $(X, \perp, d)$ be an O-complete cone metric space (not necessarily complete cone metric space) and $\lambda \in \mathbb{R}, 0 < \lambda < 1$. Let $f : (X, \perp, d) \to (X, \perp, d)$ be an $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n(x_0)\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.*

Corollary 1.24 ([4]). *Let $(X, \perp, d)$ be an O-complete cone metric space, P be a normal cone with normal constant K and $\lambda \in \mathbb{R}, 0 < \lambda < 1$. For $c \in E$ with $0 \prec\prec c$ and any $x_0 \in X$, define $B(x_0, c) = \{x \in X : d(x_0, x) \leq c\}$. Let $f : (X, \perp, d) \to (X, \perp, d)$ be an $\perp$-contraction with Lipschitz constant λ for all $x, y \in B(x_0, c)$, $\perp$-preserving on $B(x_0, c)$ and $d(fx_0, x_0) \preceq (1 - k)c$. In this case, there exists a point $x^* \in B(x_0, c)$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n(x_0)\}$ converges to this point. Also, if f is $\perp$-continuous on $B(x_0, c)$, then $x^* \in B(x_0, c)$ is a unique fixed point of f.*

2. Main Results

It is useful to examine the following note and definition before starting:

Remark 2.1 ([2])**.** It is an obvious fact that, if f is a map which has a fixed point p, then p is also a fixed point of f^n for every natural number n. However, the converse is false. For example, consider $X = [0, 1]$ and f defined by $fx = 1 - x$. Then, f has a unique fixed point at $\frac{1}{2}$, but every even iterate of f is the identity map, which has every point of $[0, 1]$ as a fixed point. On the other hand, if $X = [0, \pi], fx = \cos x$, then every iterate of f has the same fixed point as f.

Definition 2.2 ([2])**.** If a map f satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ denotes the set of all fixed point of f, then it is said to have property P. We shall say that f and g have property Q if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$.

Now, we are ready to give and prove our main results.

Theorem 2.3. *Let $(X, \perp, d)$ be an O-cone metric space, $f : (X, \perp, d) \to (X, \perp, d)$ be an $\perp$-preserving mapping. Assume that, one of the following conditions holds:*
(i) for all $x \in X$ which satisfying $x \perp fx$, there exists $\lambda \in \mathbb{R}, 0 < \lambda < 1$,

$$d(fx, f^2x) \preceq \lambda d(x, fx), \tag{2.1}$$

(ii) for all $x \in X$ which satisfying $x \perp fx$ and $x \neq fx$,

$$d(fx, f^2x) \prec d(x, fx). \tag{2.2}$$

If $F(f) \neq \emptyset$ and $u \perp fu$ for all $u \in F(f^n)$, then f has property P.

Proof. Firstly from the assumption $F(f) \neq \emptyset$ and using Remark 2.1, we know $F(f) \subseteq F(f^n), \forall n \in \mathbb{N}$. For this reason $F(f^n) \neq \emptyset$. Thus if we prove $F(f^n) \subseteq F(f), \forall n \in \mathbb{N}$, then the proof is completed. Now, we can choose $u \in F(f^n)$. In this case $f^n(u) = u$ and from the assumption for this element $u \in F(f^n)$, $u \perp fu$.
Suppose that (i) is satisfied. Then, from the inequality (2.1)

$$\begin{aligned} d(u, fu) &= d(f(f^{n-1}u), f^2(f^{n-1}u)) \\ &\preceq \lambda d(f^{n-1}u, f^n u) \\ &\;\;\vdots \\ &\preceq \lambda^n d(u, fu). \end{aligned} \tag{2.3}$$

From the definition of $\preceq$, $\lambda^n d(u, fu) - d(u, fu) = (\lambda^n - 1)d(u, fu) \in P$ and so $d(u, fu) = \theta$ that is $u \in F(u)$.

Suppose that (ii) is satisfied. Assume that $u \neq fu$ and then by using inequality (2.2),

$$d(u, fu) = d(f(f^{n-1}u), f^2(f^{n-1}u)) \prec d(f^{n-1}u, f^n u) \vdots \prec d(u, fu). \tag{2.4}$$

This is a contradiction. So, $u = fu$ that is $u \in F(u)$.

Thus, in both cases f has property P. □

Theorem 2.4. *Let $(X, \perp, d)$ be an O-complete cone metric space (it not necessarily complete cone metric space) and $\alpha, \beta, \gamma \in \mathbb{R}, 0 < \alpha + 2\beta + 2\gamma < 1$. Let $f : (X, \perp, d) \to (X, \perp, d)$ be an $\perp$-preserving which satisfies the following inequality*

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \text{ if } x \perp y. \tag{2.5}$$

In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n(x_0)\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.

Proof. Because $(X, \perp)$ is an O-set,

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp y). \tag{2.6}$$

And since, f is a self mapping on X, for any orthogonal element $x_0 \in X$, $x_1 \in X$ can be chosen as $x_1 = f(x_0)$. Thus,

$$x_0 \perp f(x_0) \vee f(x_0) \perp x_0 \quad \Rightarrow \quad x_0 \perp x_1 \vee x_1 \perp x_0. \tag{2.7}$$

Then, if we continue in the same way

$$x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \dots, x_n = f(x_{n-1}) = f^n(x_0), \tag{2.8}$$

so $\{f^n(x_0)\}$ is an iteration sequence. Since f is $\perp$-preserving, $\{f^n(x_0)\}$ is an O-sequence and by using (2.5),

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\preceq \alpha d(x_n, x_{n-1}) + \beta[d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})] \\ &\quad + \gamma[d(x_n, fx_{n-1}) + d(x_{n-1}, fx_n)] \\ &= \alpha d(x_n, x_{n-1}) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + \gamma[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ &\preceq \alpha d(x_n, x_{n-1}) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned} \tag{2.9}$$

And so, we get

$$d(x_{n+1}, x_n) \preceq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} d(x_n, x_{n-1}). \tag{2.10}$$

Since $\alpha, \beta, \gamma \in \mathbb{R}, 0 \leq \alpha, \beta, \gamma$ and $\alpha + 2\beta + 2\gamma < 1$, if $t = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ is chosen then $t \in (0,1)$ and

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq t d(x_n, x_{n-1}) \\ &\preceq t^2 d(x_{n-1}, x_{n-2}) \\ &\ \vdots \\ &\preceq t^n d(x_1, x_0). \end{aligned} \tag{2.11}$$

If for all $n \in \mathbb{N}, x_n = x_{n+1}$ then we get $x_n = f(x_n)$ and so f has a fixed point. Assume that $\forall n, n+1 \in \mathbb{N}, x_n \neq x_{n+1}$. In this case, $\forall n, m \in \mathbb{N}, n > m$,

(2.12)

$$\begin{aligned} \theta \preceq d(x_n, x_m) &\preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\preceq t^{n-1} d(x_1, x_0) + t^{n-2} d(x_1, x_0) + \cdots + t^m d(x_1, x_0) \\ &\preceq \frac{t^m}{1-t} d(x_1, x_0). \end{aligned}$$

In the sequel there are two cases:
Case I: If P is a normal cone with normal constant K, from the inequality 2.12,

$$\begin{aligned} \|d(x_n, x_m)\| &\leq K \left\| \frac{t^m}{1-t} d(x_1, x_0) \right\| \\ &\leq \frac{t^m}{1-t} K \|d(x_1, x_0)\|. \end{aligned} \tag{2.13}$$

Using the above equation, since $0 < t < 1$, $d(x_n, x_m) \to \theta (n, m \to \infty)$ and so $\{x_n\} = \{f^n(x_0)\}$ is an O-Cauchy sequence.

Case II: If P is not a normal cone, let $c \in E$ such that $\theta \prec\prec c$. Then $c \in \mathrm{int}P$. Also $\delta > 0$ can be chosen such that $c + N_\delta(\theta) \subset P$ where $N_\delta(\theta) = \{y \in E : \| y - \theta \| < \delta\}$. Since $0 < t < 1$,

$$\left\| \frac{t^m}{1-t} d(x_1, x_0) \right\| = \frac{t^m}{1-t} \|d(x_1, x_0)\| \to \theta (m \to \infty). \tag{2.14}$$

Since the choosing of δ, $\left\| \frac{t^m}{1-t} d(x_1, x_0) \right\| < \delta$ and using the Lemma 1.9 we get

$$c - \frac{t^m}{1-t} d(x_1, x_0) \in \mathrm{int}P \text{ that is } \frac{t^m}{1-t} d(x_1, x_0) \prec\prec c (m \to \infty). \tag{2.15}$$

Thus, for all $n, m \in \mathbb{N}$ such that $n \geq m$, we obtain that $d(x_n, x_m) \leq \frac{t^m}{1-t} d(x_1, x_0) \prec\prec c$, so $\{x_n\} = \{f^n(x_0)\}$ is an O-Cauchy sequence.

In both cases, since $(X, \perp, d)$ is an O-complete cone metric space, there exists $x^* \in X$ such that $\{x_n\} = \{f^n(x_0)\}$ converges to this point. Now, assume that f is $\perp$-continuous at $x^* \in X$ and let $c \in E$ such that $\theta \prec\prec c$. Because of $\{x_n\} = \{f^n(x_0)\}$ converges to $x^* \in X$ and f is $\perp$-continuous at $x^* \in X$, there exists $n_0 \in \mathbb{N}$ and for all $n \geq n_0$,

$$d(fx_n, x^*) \prec\prec \frac{c}{2} \quad \text{and} \quad d(fx_n, fx^*) \prec\prec \frac{c}{2}. \tag{2.16}$$

So, for all $n \geq n_0$, $d(fx^*, x^*) \preceq d(fx^*, fx_n) + d(fx_n, x^*) \prec\prec c$. On the other hand, for $m \in \mathbb{N}, m \geq 1$ we obtain $0 < \frac{1}{m} \leq 1$. Using $c \in \text{int}P$ and $\lambda \text{int}P \subseteq \text{int}P (\lambda \in \mathbb{R}, \lambda > 0)$ we get $\frac{c}{m} \in \text{int}P$. Thus, for all $n \geq n_0$ and for $m \in \mathbb{N}, m \geq 1$ we have $d(fx^*, x^*) \prec\prec \frac{c}{m}$, then $\frac{c}{m} - d(fx^*, x^*) \in P$. Using the cone P is a closed set, where taking limit $m \to \infty$ we get

$$\lim_{m \to \infty} \left(\frac{c}{m} - d(fx^*, x^*) \right) = -d\left(fx^*, x^*\right) \in P. \tag{2.17}$$

Besides $\theta \preceq d(fx^*, x^*)$ that is $d(fx^*, x^*) \in P$. So, because P is cone $d(fx^*, x^*) = \theta$ that is $fx^* = x^*$, so $x^* \in X$ is a fixed point of f.
Now, we show the uniqueness of the fixed point. Suppose that there exist two distinct fixed points x^* and y^*.Then,
(i) If $x^* \perp y^* \vee y^* \perp x^*$,

$$d(x^*, y^*) = d(fx^*, fy^*) \preceq \lambda d(x^*, y^*) \prec d(x^*, y^*). \tag{2.18}$$

This is a contradiction and thus $x^* \in X$ is an unique fixed point of f.
(ii) If not $x^* \perp y^* \vee y^* \perp x^*$, for the chosen orthogonal element $x_0 \in X$,

$$[(x_0 \perp x^*) \wedge (x_0 \perp y^*)] \vee [(x^* \perp x_0) \wedge (y^* \perp x_0)], \tag{2.19}$$

and since f is $\perp$- preserving,

$$[(f(x_n) \perp x^*) \wedge (f(x_n) \perp y^*)] \vee [(x^* \perp f(x_n)) \wedge (y^* \perp f(x_n))], \tag{2.20}$$

is obtained. So,

$$\begin{aligned} d(x^*, y^*) &\preceq d(x^*, fx_{n+1}) + d(fx_{n+1}, y^*) \\ &= d(fx^*, f(fx_n)) + d(f(fx_n), fy^*) \\ &\preceq \lambda[d(x^*, fx_n) + d(fx_n, y^*)] \\ &= \lambda[d(x^*, x_{n+1}) + d(x_{n+1}, y^*)], \end{aligned} \tag{2.21}$$

and taking limit $n \to \infty$, we get that $-d(x^*, y^*) \in P$. Besides $\theta \preceq d(x^*, y^*)$ that is $d(x^*, y^*) \in P$. So, because of P is a cone, $d(x^*, y^*) = \theta$ that is $x^* = y^*$. Thus, $x^* \in X$ is a unique fixed point of f. □

Theorem 2.5. *Let* $(X, \perp, d)$ *bes an* O*-complete cone metric space and* $\alpha, \beta, \gamma \in \mathbb{R}, 0 < \alpha + 2\beta + 2\gamma < 1$. *Let* $f : (X, \perp, d) \to (X, \perp, d)$ *be* $\perp$*-preserving which satisfies the following inequality*

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \text{ if } x \perp y. \tag{2.22}$$

If $u \perp fu$ *for all* $u \in F(f^n)$*, then* f *has property* P.

Proof. From Theorem 2.4, f has a unique fixed point. By Remark 2.1, we know $F(f) \subseteq F(f^n), \forall n \in \mathbb{N}$. For this reason $F(f^n) \neq \emptyset$. Thus if we prove $F(f^n) \subseteq F(f), \forall n \in \mathbb{N}$, then the proof is completed. Now, we can choose $u \in F(f^n)$ that is $f^n(u) = u$. By using (2.22),

$$\begin{aligned} d(u, fu) &= d(f(f^{n-1}u), f(f^n u)) \\ &\preceq \alpha d(f^{n-1}u, f^n u) + \beta[d(f^{n-1}u, f^n u) + d(f^n u, f^{n+1}u)] \\ &\quad + \gamma[d(f^{n-1}u, f^{n+1}u) + d(f^n u, f^n u)] \\ &\preceq \alpha d(f^{n-1}u, f^n u) + \beta[d(f^{n-1}u, f^n u) + d(f^n u, f^{n+1}u)] \\ &\quad + \gamma[d(f^{n-1}u, f^n u) + d(f^n u, f^{n+1}u)] \\ &= \alpha d(f^{n-1}u, u) + \beta[d(f^{n-1}u, u) + d(u, fu)] \\ &\quad + \gamma[d(f^{n-1}u, u) + d(u, fu)]. \end{aligned} \tag{2.23}$$

And so, we get

$$d(u, fu) \preceq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(f^{n-1}u, u). \tag{2.24}$$

Since $\alpha, \beta, \gamma \in \mathbb{R}, 0 \leq \alpha, \beta, \gamma$ and $\alpha + 2\beta + 2\gamma < 1$, if $t = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ is chosen then $t \in (0, 1)$ and

$$\begin{aligned} d(u, fu) &= d(f^n u, f^{n+1}u) \\ &\preceq t d(f^{n-1}u, f^n u) \\ &\;\;\vdots \\ &\preceq t^n d(u, fu). \end{aligned} \tag{2.25}$$

From the definition of $\preceq$, $t^n d(u, fu) - d(u, fu) = (t^n - 1)d(u, fu) \in P$ and so $d(u, fu) = \theta$ that is $u \in F(u)$. Thus, we say f has property P. $\square$

Example 2.6. Let $E = \mathbb{R}^2$ be the Euclidean plane, $P = \{(x, y) \in E : x, y \geq 0\}$ be a cone in E and $X = \{(x, 0) \in E : 0 \leq x < 1\}$. Define the binary relation $\perp$ on E such that

$$(x_1, y_1) \perp (x_2, y_2) \iff \langle (x_1, y_1), (x_2, y_2) \rangle_e \in \{|| (x_1, y_1) ||_e, || (x_2, y_2) ||_e\}. \tag{2.26}$$

(Here $\langle ., . \rangle_e$ denotes the Euclidean inner product and $|| \ . \ ||_e$ denotes Euclide norm.) In this case, $(X, \perp)$ is an O-set. The mapping $d : X \times X \to E$ is defined by

$$d((x,0),(y,0)) = \left(\frac{3}{2} |x-y|, |x-y| \right). \tag{2.27}$$

Then,$(X, \perp, d)$ is an O-complete cone metric space. Consider $f : (X, \perp, d) \to (X, \perp, d)$ with

$$f(x,0) = \left(\frac{x}{3}, 0 \right). \tag{2.28}$$

It can be seen easily f is an $\perp$-preserving and $\perp$-continuous mapping on X which satisfies inequality 2.22 with $\alpha = \frac{1}{5}, \beta = \frac{1}{5}$ and $\gamma = \frac{1}{7}$. Thus, all hypothesis of Theorem 2.4 are satisfied and so, f has an unique fixed point in X. On the other hand, $u \perp fu$ for all $u \in F(f^n)$ and from Theorem 2.5 f has property P.

3. Conclusion

It is an obvious fact that, if f is a map which has a fixed point p, then p is also a fixed point of f^n for every natural number n. However, the converse is false. If a map f satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ denotes the set of all fixed point of f, then it is said to have property P. In this paper, some fixed point results of self mapping which is defined on orthogonal cone metric spaces are given by using extensions of orthogonal contractions. And by taking advantage of these results, the necessary conditions for self mappings on orthogonal cone metric space to have P property are investigated. Also an example is given to illustrate the main results.

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