Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 4 (2020), 71-84
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.124021.772

Some Properties of Certain Subclass of Meromorphic Functions Associated with (p,q)-derivative

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Abstract. In this paper, by making use of (p,q)-derivative operator we introduce a new subclass of meromorphically univalent functions. Precisely, we give a necessary and sufficient coefficient condition for functions in this class. Coefficient estimates, extreme points, convex linear combination, Radii of starlikeness and convexity and finally partial sum property are investigated.

1. Introduction

The q-theory has an important role in various branches of mathematics and physics as for example, in the areas of special functions, ordinary fractional calculus, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics (see for instance, [1, 2, 9, 14, 16, 17]).

The theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus as the q-integral and q-derivative were used to construct several subclasses of analytic functions (see, for example, [11–13]).

For convenience, we provide some basic definitions and concept details of fractionl q-calculus operators of complex-valued function $f(z)$ which are used in this paper.

2020 *Mathematics Subject Classification.* 34B24, 34B27.

Key words and phrases. Meromorphic function, (p,q)-derivative, Coefficient bound, Extreme point, Convex set, Partial sum, Hadamard product.

Received: 04 April 2020, Accepted: 24 October 2020.

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Let $\sum$ denotes the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1}, \tag{1.1}$$

which are analytic in the punctured unit disk

$$\triangle^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Jackson [6] defined the (p, q)-derivative of a function $f(z)$ in a given subset of $\mathbb{C}$ by

$$D_{p.q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0, 0 < q < p \leq 1, \tag{1.2}$$

and

$$D_{p,q} f(0) = f'(0).$$

From relationships (1.2) and (1.1), we get

$$D_{p,q} f(z) = \frac{-1}{pqz^2} + \sum_{k=1}^{\infty} [k-1]_{p,q} a_k z^{k-2}, \quad z \in \triangle^*, 0 < q < p \leq 1, \tag{1.3}$$

where

$$[k-1]_{p,q} := \frac{p^{k-1} - q^{k-1}}{p-q}. \tag{1.4}$$

Also

$$\lim_{p \to 1} [k-1]_{p,q} = \frac{1 - q^{k-1}}{1-q} = [k-1]_q.$$

Note also that for $p = 1$, the (p, q)-derivative of a function $f(z)$ of the form (1.1) reduces to the q-derivative as Gasper and Rahman [4] defined as follows

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \tag{1.5}$$

where $z \in \triangle^*$ and $0 < q < 1$. So we conclude

$$\lim_{q \to \bar{1}} D_q f(z) = f'(z), \quad z \in \triangle^*,$$

see, for details [3, 5, 7, 8, 10, 15] .

The object of this paper is to introduce a new subclass $\sum_{p,q}(\lambda, \alpha, \beta)$ of meromorphic analytic functions by (p, q)-derivative operator and we investigate coefficient estimates, extreme points, convex linear combination, Radii of starlikeness and convexity and partial sum property as defined above.

Now, we introduce new subclasess $\sum_{p,q}(\lambda, \alpha, \beta)$ of the class $\sum$ as follows.

Let $0 < q < p \leq 1, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1$ and $\beta > 0$. Then a function $f \in \sum$ given in (1.1) is said to be the subclass $\sum\limits_{p,q}(\lambda, \alpha, \beta)$ if and only if

$$\left| \frac{z^4 \left(D_{p,q} f(z)\right)'' + z^3 \left(D_{p,q} f(z)\right)' + \dfrac{4}{pq}}{\lambda z^2 \left(D_{p,q} f(z)\right) - \dfrac{1}{pq} + \dfrac{(1+\lambda)\alpha}{pq}} \right| < \beta. \tag{1.6}$$

Unless otherwise mentioned, we suppose throughout this paper that $0 < q < p \leq 1, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1$ and $\beta > 0$. First we state coefficient estimates on the class $\sum\limits_{p,q}(\lambda, \alpha, \beta)$.

Theorem 1.1. *Let $f(z) \in \sum$, then $f(z) \in \sum\limits_{p,q}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{+\infty} [k-1]_{p,q} \left((k-2)^2 + \lambda\beta\right) a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq}, \tag{1.7}$$

and the result is sharp for $G(z)$ given by

$$G(z) = \frac{1}{z} + \frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2 + \lambda\beta\right)} z^{k-1}. \tag{1.8}$$

Proof. Let $f(z) \in \sum\limits_{p,q}(\lambda, \alpha, \beta)$, then (1.6) holds true. So by replacing (1.3) in (1.6) we have

$$\left| \frac{\sum\limits_{k=1}^{+\infty} [k-1]_{p,q}(k-2)(k-3) a_k z^k + \sum\limits_{k=1}^{+\infty} [k-1]_{p,q}(k-2) a_k z^k}{-\dfrac{\lambda}{pq} + \sum\limits_{k=1}^{+\infty} \lambda [k-1]_{pq} a_k z^k - \dfrac{1}{pq} + \dfrac{(1+\lambda)\alpha}{pq}} \right| < \beta,$$

or

$$\left| \frac{\sum\limits_{k=1}^{+\infty} [k-1]_{p,q}(k-2)^2 a_k z^k}{\dfrac{(1+\lambda)}{pq}(1-\alpha) - \sum\limits_{k=1}^{+\infty} \lambda [k-1]_{p,q} a_z z^k} \right| < \beta.$$

Since $\mathrm{Re}(z) \leq |z|$ for all z, therefore

$$\mathrm{Re} \left\{ \frac{\sum\limits_{k=1}^{+\infty} [k-1]_{p,q}(k-2)^2 a_k z^k}{(\dfrac{1+\lambda}{pq})(1-\alpha) - \sum\limits_{k=1}^{+\infty} \lambda [k-1]_{pq} a_k z^k} \right\} < \beta.$$

By letting $z \to \overline{1}$ through real values, we have

$$\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq}.$$

Conversely, let (1.7) holds true, then by (1.6) it is enough to show that

$$X(f)=\left|\frac{z^4\left(D_{p,q}f(z)\right)''+z^3\left(D_{p,q}f(z)\right)'+\frac{4}{pq}}{\lambda z^2\left(D_{p,q}f(z)\right)-\frac{1}{pq}+\frac{(1+\lambda)\alpha}{pq}}\right| <\beta,$$

or

$$X(f)=\left|z^4\left(D_{p,q}f(z)\right)''+z^3\left(D_{p,q}f(z)\right)'+\frac{4}{pq}\right| -\beta\left|\lambda z^2\left(D_{p,q}f(z)\right)-\frac{1}{pq}+\frac{(1+\lambda)\alpha}{pq}\right| <0.$$

But for $0<|z|=r<1$ we have

$$\begin{aligned} X(f)=&\left|\sum_{k=1}^{+\infty}[k-1]_{p,q}(k-2)^2a_kz^k\right| \\ &-\beta\left|\frac{(1+\lambda)}{pq}(1-\alpha)-\lambda\sum_{k=1}^{+\infty}[k-1]_{p,q}a_kz^k\right| \\ \leq&\sum_{k=1}^{+\infty}[k-1]_{p,q}(k-2)^2|a_k|r^k \\ &-\frac{\beta(1+\lambda)(1-\alpha)}{pq}+\sum_{k=1}^{+\infty}\lambda\beta[k-1]_{p,q}|a_k|r^k \\ \leq&\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)|a_k|r^k-\frac{\beta(1+\lambda)(1-\alpha)}{pq}. \end{aligned}$$

Since the above inequality holds for all $r \quad (0<r<1)$, by letting $r \to \overline{1}$ and using (1.7) we obtain $X(f) \leq 0$, and this completes the proof. □

Next we obtain extreme points and convex linear combination property for functions $f(z) \in \sum_{p,q}(\lambda,\alpha,\beta)$.

Theorem 1.2. *The function $f(z)$ of the form* (1.1) *belongs to* $\sum_{p,q}(\lambda,\alpha,\beta)$ *if and only if it can be expressed as* $f(z)=\sum_{k=0}^{\infty}\sigma_k f_k(z)$, $\sum_{k=0}^{\infty}\sigma_k=1$, $\sigma_k\geq 0$, *where*

$$f_0(z)=\frac{1}{z},$$

and

$$f_k(z)=\frac{1}{z}+\frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)}z^{k-1},\quad (k=1,2,\ldots).$$

Proof. Let

$$\begin{aligned}f(z)=&\sum_{k=0}^{\infty}\sigma_k f_k(z)\\ =&\sigma_0 f_0(z)+\sum_{k=1}^{\infty}\sigma_k\left[\frac{1}{z}+\frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)}z^{k-1}\right],\end{aligned}$$

or

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)}\sigma_k z^{k-1}.$$

Now by using Theorem 1.1 we conclude that $f(z)\in\sum_{p,q}(\lambda,\alpha,\beta)$.

Conversely, if $f(z)$ given by (1.1) belongs to $\sum_{p,q}(\lambda,\alpha,\beta)$, by letting

$$\sigma_0=1-\sum_{k=1}^{+\infty}\sigma_k,$$

where

$$\sigma_k=\frac{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)}{\beta(1+\lambda)(1-\alpha)}a_k,\quad (k=1,2,\ldots),$$

we conclude the required result. □

Theorem 1.3. *Let for $n=1,2,\ldots,m$,* $f_n(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_{k,n}z^{k-1}$ *belongs to* $\sum_{p,q}(\lambda,\alpha,\beta)$, *then* $F(z)=\sum_{n=1}^{m}\sigma_n f_n(z)$ *also belongs in the same class, where* $\sum_{n=1}^{m}\sigma_n=1$. *(Hence* $\sum_{p,q}(\lambda,\alpha,\beta)$ *is a convex set.)*

Proof. According to Theorem 1.1 for every $n = 1, 2, \dots, m$, we have

$$\sum_{n=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)a_{k,n} \le \frac{\beta(1+\lambda)(1-\alpha)}{pq}.$$

But

$$\begin{aligned} F(z) &= \sum_{n=1}^{m}\sigma_n f_n(z) \\ &= \sum_{n=1}^{m}\sigma_n\left(\frac{1}{z}+\sum_{k=1}^{\infty}a_{k,n}z^{k-1}\right) \\ &= \frac{1}{z}\sum_{n=1}^{m}\sigma_n+\sum_{k=1}^{\infty}\left(\sum_{n=1}^{m}\sigma_n a_{k,n}\right)z^{k-1} \\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\left(\sum_{n=1}^{m}\sigma_n a_{k,n}\right)z^{k-1}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\left(\sum_{n=1}^{m}\sigma_n a_{k,n}\right) \\ &\quad= \sum_{n=1}^{m}\sigma_n\left(\sum_{k=1}^{\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\right)a_{k,n} \\ &\quad\le \sum_{n=1}^{m}\sigma_n\left(\frac{\beta(1+\lambda)(1-\alpha)}{pq}\right) \\ &\quad= \frac{\beta(1+\lambda)(1-\alpha)}{pq}\sum_{n=1}^{m}\sigma_n \\ &\quad= \frac{\beta(1+\lambda)(1-\alpha)}{pq}, \end{aligned}$$

then by Theorem 1.1 the proof is complete. □

2. Radii Condition and partial sum property

In this section, we obtain Radii of starlikeness and convexity and investigate the partial sum property.

Theorem 2.1. *If $f(z) \in \sum_{p,q}(\lambda, \alpha, \beta)$, then f is a meromorphically univalent starlike of order γ in disk $|z| < R_1$, and it is a meromerphically* *univalent convex of order γ in disk $|z| < R_2$ where*

$$R_1 = \inf_k \left\{ \frac{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)(1-\gamma)}{\beta(1+\lambda)(1-\alpha)(k+2+\gamma)} \right\}^{\frac{1}{k}}, \tag{2.1}$$

and

$$R_2 = \inf_k \left\{ \frac{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)(1-\gamma)}{\beta(k-1)(1+\lambda)(1-\alpha)(k+2+\gamma)} \right\}^{\frac{1}{k}}. \tag{2.2}$$

Proof. For starlikeness it is enough to show that

$$\left| \frac{zf'}{f} + 1 \right| < 1-\gamma,$$

but

$$\begin{aligned}\left| \frac{zf'}{f} + 1 \right| =& \left| \frac{\sum_{k=1}^{+\infty} k a_k z^k}{1+\sum_{k=1}^{+\infty} a_k z^k} \right| \\ \leq& \frac{\sum_{k=1}^{+\infty} k a_k |z|^k}{1-\sum_{k=1}^{+\infty} a_k |z|^k} \\ \leq& 1-\gamma,\end{aligned}$$

or

$$\sum_{k=1}^{+\infty} k a_k |z|^k \leq 1-\gamma-(1-\gamma)\sum_{k=1}^{+\infty} a_k |z|^k,$$

or

$$\sum_{k=1}^{+\infty} \frac{k+2+\gamma}{1-\gamma} a_k |z|^k \leq 1.$$

By using (1.7) we obtain

$$\begin{aligned}\sum_{k=1}^{+\infty} \frac{k+2+\gamma}{1-\gamma} a_k |z|^k \leq& \sum_{k=1}^{+\infty} \frac{\beta(1+\lambda)(1-\alpha)(k+2+\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)(1-\alpha)} |z|^k \\ \leq& 1.\end{aligned}$$

So, it is enough to suppose

$$|z|^k \leq \frac{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)(1-\alpha)}{\beta(1+\lambda)(1-\alpha)(k+2+\alpha)}.$$

Hence we get the required result (2.1). For convexity, by using the Alexander's Theorem (If f be an analytic function in the unit disk and normalized by $f(0) = f^{'}(0) - 1 = 0$, then $f(z)$ is convex if and only if $zf^{'}(z)$ is starlike.) and applying an easy calculation we conclude the required result (2.2). So the proof is complete. □

Theorem 2.2. *Let* $f(z) \in \sum$ *and define*

$$S_1(z) = \frac{1}{z}, \qquad S_m(z) = \frac{1}{z} + \sum_{k=1}^{m-1} a_k z^{k-1}, \quad (m = 2, 3, \ldots). \tag{2.3}$$

Also suppose $\sum\limits_{k=1}^{+\infty} e_k a_k \leq 1$, *where*

$$e_k = \frac{pq[k-1]_{p,q}\left((k-2)^2 + \lambda\beta\right)}{\beta(1+\lambda)(1-\alpha)}, \tag{2.4}$$

then

$$\mathrm{Re}\left(\frac{f(z)}{S_m(z)}\right) > 1 - \frac{1}{e_m}, \qquad \mathrm{Re}\left(\frac{S_m(z)}{f(z)}\right) > \frac{e_m}{1+e_m}. \tag{2.5}$$

Proof. Since $\sum\limits_{k=1}^{+\infty} e_k a_k \leq 1$, then by Theorem 1.1, $f(z) \in \sum\limits_{p,q}(\lambda, \alpha, \beta)$.

Also by (1.4) we have $\frac{[k-1]_{p,q}}{1-\alpha} \geq 1$, so

$$e_k > \frac{pq\left((k-2)^2 + \lambda\beta\right)}{\beta(1+\lambda)}, \tag{2.6}$$

and $\{e_k\}$ is an increasing sequence, therefore we obtain

$$\begin{aligned} \sum_{k=1}^{m-1} a_k + e_m \sum_{k=m}^{+\infty} a_k &\leq \sum_{k=1}^{+\infty} e_k a_k \\ &\leq 1. \end{aligned} \tag{2.7}$$

Now by putting

$$E(z) = e_m \left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{e_m})\right], \tag{2.8}$$

and making use of (2.7) we obtain

$$\begin{aligned} \mathrm{Re}\left(\frac{E(z)-1}{E(z)+1}\right) &\leq \left|\frac{E(z)-1}{E(z)+1}\right| \\ &= \left|\frac{e_m f(z) - e_m S_m(z)}{e_m f(z) - e_m S_m(z) + 2S_m(z)}\right|, \end{aligned}$$

or

$$\begin{aligned}\operatorname{Re}\left(\frac{E(z)-1}{E(z)+1}\right) \leq& \left|\frac{x_m \sum\limits_{k=m}^{+\infty} a_k z^k}{x_m \sum\limits_{k=m}^{+\infty} a_k z^k + 2\left(\frac{1}{z} + \sum\limits_{k=1}^{m-1} a_k z^k\right)}\right| \\ \leq& \frac{e_m \sum\limits_{k=m}^{+\infty} |a_k|}{2 - 2\sum\limits_{k=1}^{m-1} |a_k| - e_m \sum\limits_{k=m}^{+\infty} |a_k|} \\ \leq& 1.\end{aligned}$$

By a simple calculation we get $\operatorname{Re}(E(z)) > 0$, therefore

$$\operatorname{Re}\left(\frac{E(z)}{e_m}\right) > 0,$$

or equivalently $\operatorname{Re}\left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{e_m})\right] > 0$, and this gives the first inequality in (2.5).

For the second inequality we consider

$$G(z) = (1 + e_m)\left[\frac{S_m(z)}{f(z)} - \frac{e_m}{1 + e_m}\right],$$

and by using (2.7) we have $\left|\frac{G(z)-1}{G(z)+1}\right| \leq 1$, and hence $\operatorname{Re}(G(z)) > 0$, therefore $\operatorname{Re}\left(\frac{G(z)}{1+e_m}\right) > 0$, or equivalently $\operatorname{Re}\left[\frac{S_m(z)}{f(z)} - \frac{e_m}{1+e_m}\right] > 0$, and this shows the second inequality in (2.5). So the proof is complete. □

3. Some Properties of $\sum_{p,q}(\lambda, \alpha, \beta)$

Theorem 3.1. *Let $f(z), g(z) \in \sum_{p,q}(\lambda, \alpha, \beta)$ and are given by*

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1}, \qquad g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} b_k z^{k-1}.$$

Then the function $h(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2) z^{k-1}$ is also in $\sum_{p,q}(\gamma, \alpha, \beta)$ where $\gamma \leq \frac{\lambda}{2} - \frac{(k-2)^2}{2\beta}$.

Proof. Since $f(z), g(z) \in \sum_{p,q}(\lambda, \alpha, \beta)$ therefore we have

(3.1)
$$\sum_{k=1}^{+\infty} \left[[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\right]^2 a_k^2 \leq \left[\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right) a_k\right]^2 \leq \left[\frac{\beta(1+\lambda)(1-\alpha)}{pq}\right]^2,$$

and

(3.2)
$$\sum_{k=1}^{+\infty} \left[[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\right]^2 b_k^2 \leq \left[\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right) b_k\right]^2 \leq \left[\frac{\beta(1+\lambda)(1-\alpha)}{pq}\right]^2.$$

The above inequalities yield

$$\sum_{k=1}^{+\infty} \frac{1}{2}\left[[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\right]^2 (a_k^2+b_k^2) \leq \left[\frac{\beta(1+\lambda)(1-\alpha)}{pq}\right]^2.$$

Now we must show

$$\sum_{k=1}^{+\infty} \left[[k-1]_{p,q}\left((k-2)^2+\gamma\beta\right)\right]^2 (a_k^2+b_k^2) \leq \left[\frac{\beta(1+\lambda)(1-\alpha)}{pq}\right]^2.$$

But the above inequalities hold if

$$[k-1]_{p,q}\left((k-2)^2+\gamma\beta\right) \leq \frac{1}{2}\left[[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\right],$$

or equivalently

$$2(k-2)^2+2\gamma\beta \leq (k-2)^2+\lambda\beta,$$

or

$$\gamma \leq \frac{\lambda}{2} - \frac{(k-2)^2}{2\beta}.$$

□

Theorem 3.2. *The class* $\sum_{p,q}(\lambda, \alpha, \beta)$ *is a convex set.*

Proof. Let

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1},$$

and

$$g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} b_k z^{k-1},$$

be in the class $\sum_{p,q}(\lambda, \alpha, \beta)$. For $t \in (0,1)$, it is enough to show that the function $h(z) = (1-t)f(z) + tg(z)$ is in the class $\sum_{p,q}(\lambda, \alpha, \beta)$. Since

$$h(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} \left((1-t)a_k + tb_k\right) z^{k-1}, \tag{3.3}$$

then

$$\sum_{k=1}^{\infty} \left[[k-1]_{p,q}\left((k-2)^2 + \lambda\beta\right)\right]\left((1-t)a_k + tb_k\right) \le \frac{\beta(1+\lambda)(1-\alpha)}{pq},$$

so $h(z) \in \sum_{p,q}(\lambda, \alpha, \beta)$. □

Corollary 3.3. *Let*

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_{k,j} z^{k-1}, \quad (j = 1, 2, \ldots, n),$$

be in the class $\sum_{p,q}(\lambda, \alpha, \beta)$*, then the function* $F(z) = \sum_{j=1}^{n} c_j f_j(z)$ *is also in* $\sum_{p,q}(\lambda, \alpha, \beta)$ *where* $\sum_{j=1}^{n} c_j = 1$.

4. Hadamard Product

For functions $f(z)$, $g(z)$ belonging to $\sum$, is given by (1.1), we denote by $(f * g)(z)$ the Hadamard product (or convolution) of the functions $f(z), g(z)$, that is

$$\begin{aligned}(f * g)(z) &= \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1} \\ &= (g * f)(z).\end{aligned}$$

Theorem 4.1. *If* $f(z), g(z)$ *defined by* (1.1) *is in the class* $\sum_{p,q}(\lambda, \alpha, \beta)$*, then* $(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1}$ *is in the class* $\sum_{p,q}(\gamma, \alpha, \beta)$ *where*

$$\gamma \le \frac{[k-1]_{p,q} pq \left((k-2)^2 + \lambda\beta\right)^2}{\beta^2(1+\lambda)(1-\alpha)} - \frac{(k-2)^2}{\beta}.$$

Proof. Since $f(z), g(z) \in \sum_{p,q}(\lambda, \alpha, \beta)$, so by (1.7)

$$\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq}, \tag{4.1}$$

and

$$\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)b_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq}. \tag{4.2}$$

We must find the smallest γ such that

$$\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\gamma\beta\right)a_k b_k \leq \frac{\beta(1+\gamma)(1-\alpha)}{pq}.$$

By using (4.1), (4.2) and the Cauchy-Schwarts inequality we have

$$\sum_{k=1}^{+\infty}[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)\sqrt{a_k b_k} \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq}. \tag{4.3}$$

Now it is enough to show that

$$[k-1]_{p,q}\left((k-2)^2+\gamma\beta\right)a_k b_k \leq [k-1]_{p,q}\left((k-2)^2+\gamma\beta\right)\sqrt{a_k b_k}, \tag{4.4}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{(k-2)^2+\lambda\beta}{(k-2)^2+\gamma\beta}.$$

But from (4.3), we have

$$\sqrt{a_k b_k} \leq \frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)},$$

so it is enough to have

$$\frac{\beta(1+\lambda)(1-\alpha)}{pq[k-1]_{p,q}\left((k-2)^2+\lambda\beta\right)} \leq \frac{(k-2)^2+\lambda\beta}{(k-2)^2+\gamma\beta},$$

or

$$\gamma \leq \frac{[k-1]_{p,q}pq\left((k-2)^2+\lambda\beta\right)^2}{\beta^2(1+\lambda)(1-\alpha)} - \frac{(k-2)^2}{\beta}.$$

□

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