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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 1
Pages: 15-34

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.123056.766

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 1 (2021), 15-34
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.123056.766

A Note on Some Results for C-controlled K-Fusion Frames in Hilbert Spaces

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Abstract. In this manuscript, we study the relation between K-fusion frame and its local components which leads to the definition of a C-controlled K-fusion frames, also we extend a theory based on K-fusion frames on Hilbert spaces, which prepares exactly the frameworks not only to model new frames on Hilbert spaces but also for deriving robust operators. In particular, we define the analysis, synthesis and frame operator for C-controlled K-fusion frames, which even yield a reconstruction formula. Also, we define dual of C-controlled K-fusion frames and study some basic properties and perturbation of them.

1. Introduction

Frames for Hilbert spaces were proposed by Duffin and Schaeffer in 1952 to study some difficulties in nonharmonic Fourier Series [8]. During the last 20 years frame theory has been growing quickly since several new applications have been developed, we refer to [4, 12, 14, 16, 19, 20] for an introduction to frame theory and its applications.

The notion of K-frames have been recently introduced by L. Gavruta to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of the K, where K is a bounded linear operator on a separable Hilbert space H. One of the newest generalization of frames is controlled frames, controlled frames for spherical wavelets have been introduced in [5] to

2010 *Mathematics Subject Classification.* 42C15; Secondary 42C40,41A58.
Key words and phrases. Frame, K-Fusion frame, Controlled fusion frame, Controlled K-fusion frame.
Received: 14 March 2020, Accepted: 24 October 2020.
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get a numerically more efficient approximation algorithm and the related theory for general frames were developed in [1, 3, 10, 11, 15, 17, 18].

In this paper, we will define the concept of the C-controlled K- fusion frames and will show some properties of C-controlled K- fusion frames. Also, we will study perturbation and Q-duality for Controlled K- fusion frames and characterize Q-duals of some Controlled K- fusion frame.

Throughout this paper, H, H_1 and H_2 are separable Hilbert spaces, $GL(H)$ is the set of all bounded linear operators which have bounded inverses, $\mathcal{B}(H)$ is the family of all bounded operators on H and $K \in \mathcal{B}(H)$ and $\mathcal{R}(T)$ denotes the range of the operator "T". Also, we denote the orthogonal projection of H onto a closed subspace $W \subset H$ by π_{W_i}. We consider the index set $\mathbb{I}$ to be countable.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Lemma 2.1 ([7]). *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$ *;*
(ii) $L_1 L_1^* \leq \lambda L_2 L_2^*$ *for some* $\lambda > 0$ *;*
(iii) *there exists* $u \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 u$.

Moreover, if those conditions are valid then there exists a unique operator u so that

(a) $\|u\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\ker L_1 = \ker u$;
(c) $\mathcal{R}(u) \subseteq \overline{\mathcal{R}(L_2^*)}$.

Definition 2.2. A sequence $\{f_i\}_{i\in\mathbb{I}}$ in H is a frame if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

The constants A, B are frame bounds; A is the lower bound and B is the upper bound. The frame is tight if $A = B$, it is called a Parseval frame if $A = B = 1$. If we only have the upper bound, we call $\{f_i\}_{i\in\mathbb{I}}$ a Bessel sequence with bound B.

Lemma 2.3 ([6]). *Let $\{f_i\}_{i\in\mathbb{I}}$ be a sequence in H and $B > 0$ be given. Then $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence with bound B if and only if*

$$T : \ell^2(I) \longmapsto H,$$
$$T(\{c_i\}) = \sum_{i\in\mathbb{I}} c_i f_i,$$

defines a bounded operator and $\|T\| \leq \sqrt{B}$.

Now, if the operator T is bounded, then the adjoin operator T^* is well-defined and bounded which

$$T^* : H \longmapsto \ell^2(I),$$
$$T^* f = \{\langle f, f_i\rangle\}_{i\in\mathbb{I}}\,.$$

So, we can define the frame operator as follow:

$$S : H \longmapsto H,$$
$$Sf := TT^* f = \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\, f_i.$$

These operators are called synthesis operator; analysis operator and frame operator, respectively. The representation space employed in this setting is

$$\ell^2(\mathbb{I}) = \left\{ \{c_i\}_{i\in\mathbb{I}} : c_i \in \mathbb{C}, \sum_{i\in\mathbb{I}} |c_i|^2 < \infty \right\}.$$

Definition 2.4 ([2])**.** Let $K \in \mathcal{B}(H)$. A sequence $\{f_i\}_{i\in\mathbb{I}} \subset H$ is called a K-frame for H, if there exist constants $0 < A \le B < \infty$ such that for each $f \in H$,

$$A\,\|K^* f\|^2 \le \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le B\|f\|^2,$$

Definition 2.5 ([9])**.** Let $\{W_i\}_{i\in\mathbb{I}}$ be a collection of closed subspaces in H and $\{v_i\}_{i\in\mathbb{I}}$ be a collection of weights, i.e. $v_i > 0$, $i \in \mathbb{I}$. The sequence $\{W_i\}_{i\in\mathbb{I}}$ is called a K-fusion frame for H if there exist constants $0 < A \le B < \infty$ such that for all $f \in H$,

$$A\,\|K^* f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2\, \|\pi_{W_i} f\|^2 \le B\|f\|^2,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

Definition 2.6 ([13])**.** Let $F := \{f_i\}_{i\in\mathbb{I}}$ be a family of vectors in H and $C, C' \in GL(H)$. Then F is called a frame controlled by C and C' or a (C, C')-Controlled frame if there exist constants $0 < A_{CC'} \le B_{CC'} < \infty$ such that for each $f \in H$,

$$A_{CC'}\|f\|^2 \le \sum_{i\in\mathbb{I}} \left|\langle f, Cf_i\rangle \left\langle C' f_i, f\right\rangle\right|^2 \le B_{CC'}\|f\|^2.$$

We call F a Parseval (C, C')-controlled frame if $A_{CC'} = B_{CC'} = 1$. If only the right hand inequality hold, then we call F a (C, C')-controlled Bessel sequence.

Example 2.7 ([13])**.** Let $\{f_i\}_{i=1}^{35}$ be a frame for the Hilbert space $\mathbb{R}^2$ in which

$$f_i = e_1 = (1,0) : i : 1, 2, \ldots, 8,$$
$$f_i = e_2 = (0,1) : i : 9, 10, \ldots, 35.$$

It is easy to see that for all $f \in H$,

$$8\|f\|^2 \leq \sum_{i=1}^{m} |\langle f, f_i \rangle|^2 \leq 27\|f\|^2,$$

so the frame operator S is defined by the following diagonal matrix;

$$S = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}$$

Define $V(x,y) := (3x, 7y)$ and $W := V^{-1}$. Then it is easy to see that $\{f_i\}_{i=1}^{35}$ is a Parseval (C, C')-controlled frame for $\mathbb{R}^2$, in which,

$$C(x,y) = WS^{-\frac{1}{3}}(x,y) = \left(\frac{1}{6}x, \frac{1}{21}y\right),$$
$$C'(x,y) = VS^{-\frac{2}{3}}(x,y) = \left(\frac{3}{4}x, \frac{7}{9}y\right).$$

Definition 2.8 ([12])**.** Let $\{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of a Hilbert space H. Let $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, and let $C, C' \in GL(H)$. Then $W = \{(W_i, v_i)\}_{i\in\mathbb{I}}$ is called a fusion frame controlled by C and C' or (C, C')-controlled fusion frame if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i} C' f, \pi_{W_i} C f \right\rangle \leq B\|f\|^2. \tag{2.1}$$

The set W is called a tight controlled fusion frame, if the constants A, B can be chosen such that $A = B$, a Parseval fusion frame provided $A = B = 1$. We call W a C^2-controlled fusion frame if $C = C'$.

If only the right hand of (2.1) is required, we call C-controlled Bessel fusion sequence with the bound B. If W is a (C, C')-controlled fusion frame and $C^*\pi_{W_i}C'$ is a positive operator for each $i \in \mathbb{I}$, then $C^*\pi_{W_i}C' = C'^*\pi_{W_i}C$ and we have

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \left\|(C^*\pi_{W_i}C')^{\frac{1}{2}} f\right\|^2 \leq B\|f\|^2.$$

We note that, in inequality (2.1), the term

$$\sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i} C' f, \pi_{W_i} C f \right\rangle$$

is well defined. Because, $C, C' \in GL(H)$, therefore CC', C^* and C^{-1} are also in $GL(H)$. So, we have

$$C\pi_{W_i} = \pi_{W_i}C,$$

and

$$C^*\pi_{W_i} = \pi_{W_i}C^*.$$

Finally

$$\sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}C'f, \pi_{W_i}Cf \right\rangle,$$

is a series of real numbers.

Let $C^*\pi_{W_i}C'$ be a positive operator for each $i \in \mathbb{I}$. We define the controlled analysis operator by

$$\begin{aligned} &T_W^* : H \to \mathcal{K}_{2,W}, \\ &T_W^*(f) = \left\{ v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f \right\}_{i\in\mathbb{I}}, \end{aligned}$$

where

$$\mathcal{K}_{2,W} := \left\{ \left\{ v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f \right\}_{i\in\mathbb{I}} \ : \ f \in H \right\} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}.$$

It is easy to see that $\mathcal{K}_{2,W}$ is closed and T_W^* is well defined. Moreover T_W^* is a bounded linear operator with the adjoint (the controlled synthesis operator) operator T_W is defined by

$$\begin{aligned} &T_W : \mathcal{K}_{2,W} \to H, \\ &T_W \left\{ v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f \right\}_{i\in\mathbb{I}} = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}C'f. \end{aligned}$$

Therefore, we can define the controlled fusion frame operator S_W on H by

$$\begin{aligned} S_W f &= T_W T_W^*(f) \\ &= \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}C'f. \end{aligned}$$

Example 2.9 ([12])**.** Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$ and $W = \{(W_i, 1)\}_{i=1}^3$ be a 1-uniform fusion frame for it, in which $W_1 = \overline{span}\{e_1, e_2\}, W_2 = \overline{span}\{e_1, e_3\}, W_3 = \overline{span}\{e_3\}$. It is easy to see that for all $f \in \mathbb{R}^3$,

$$\|f\|^2 \le \sum_{i=1}^3 \|\pi_{W_i}f\|^2 \le 2\|f\|^2.$$

Let $C(x_1,x_2,x_3)=(5x_1,4x_2,5x_3)$ and $C'(x_1,x_2,x_3)=\left(\frac{1}{6}x_1,\frac{1}{3}x_2,\frac{1}{6}x_3\right)$ be two operators on $\mathbb{R}^3$. It is easy to see that $C,C'\in GL^+(\mathbb{R}^3)$, $CC'=C'C$, $CS_W=S_WC$ and $C'S_W=S_WC'$. Now an easy computation shows that for all $f\in\mathbb{R}^3$,

$$\frac{4}{3}\|f\|^2\leq\sum_{i=1}^{3}\left\langle\pi_{W_i}Cf,C'f\right\rangle\leq\frac{5}{3}\|f\|^2.$$

So $\{(W_i,1)\}_{i=1}^3$ is a (C,C')-controlled fusion frame for $\mathbb{R}^3$.

3. C-controlled K-fusion Frames

In this section, we introduce the notion of C-controlled K-fusion frames in Hilbert spaces and discuss on some of their properties. In particular, we present some approaches for identifying and constructing of C-controlled K-fusion frames. Let us start our consideration with formal definition of C-controlled K-fusion frames. Throughout the paper, $C\in GL^+(H)$ and $CK=KC$.

Definition 3.1. Let $K\in\mathcal{B}(H)$, $C\in GL^+(H)$ and $CK=KC$. Suppose $\{W_i\}_{i\in\mathbb{I}}$ is a collection of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights. The collection $W:=\{(W_i,v_i)\}_{i\in\mathbb{I}}$ is called a C-controlled K-fusion frame for H if there exist constants $0<A\leq B<\infty$ such that for all $f\in H$,

$$A\left\|C^{\frac{1}{2}}K^*f\right\|^2\leq\sum_{i\in\mathbb{I}}v_i^2\left\langle\pi_{W_i}f,\pi_{W_i}Cf\right\rangle\leq B\|f\|^2. \tag{3.1}$$

We call A and B lower and upper frame bounds for C-controlled K-fusion frame. If just the right hand side inequality in 3.1 is satisfied, then $\{W_i\}_{i\in\mathbb{I}}$ is called C-controlled fusion Bessel sequence for H.

We define

$$\left(\sum_{i\in\mathbb{I}}\bigoplus W_i\right)_{l^2}=\left\{\{f_i\}_{i\in\mathbb{I}}:f_i\in W_i,\sum_{i\in\mathbb{I}}\|f_i\|^2\leq\infty\right\},$$

where, with the inner product $\left\langle\{f_i\}_{i\in\mathbb{I}},\{g_i\}_{i\in\mathbb{I}}\right\rangle=\sum_{i\in\mathbb{I}}\left\langle\{f_i\},\{g_i\}\right\rangle$ is a Hilbert space.

Let $\{(W_i,v_i)\}_{i\in\mathbb{I}}$ be a C-controlled K-fusion frame for H and $C\pi_{W_i}$ is a positive operator. The synthesis operator and analysis operator are defined, respectively, by

$$T_W:\left(\sum_{i\in\mathbb{I}}\oplus W_i\right)_{\ell_2}\to H,\qquad\qquad T_W^*:H\to\left(\sum_{i\in\mathbb{I}}\oplus W_i\right)_{\ell_2},$$

$$T_W\left(\{g_i\}_{i\in\mathbb{I}}\right)=\sum_{i\in\mathbb{I}}v_i(C\pi_{W_i})^{\frac{1}{2}}g_i,\qquad T_W^*(f)=\left\{v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\}_{i\in\mathbb{I}}.$$

Also, the controlled fusion frame operator $S_C:H\to H$ is given by,

$$\begin{aligned}S_Cf&=T_WT_W^*f\\&=\sum_{i\in\mathbb{I}}v_i^2C\pi_{W_i}f.\end{aligned}$$

Hence for each $f_1,f_2\in H$,

$$\langle S_Cf_1,f_2\rangle=\left\langle\sum_{i\in\mathbb{I}}v_i^2C\pi_{W_i}f_1,f_2\right\rangle=\sum_{i\in\mathbb{I}}v_i^2\left\langle C\pi_{W_i}f_1,f_2\right\rangle.$$

Notice that for each $f\in H$,

$$\begin{aligned}\langle S_Cf,f\rangle=\|T_W^*f\|^2&\geq A\left\|C^{\frac{1}{2}}K^*f\right\|^2\\&=A\left\langle CKK^*f,f\right\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}ACKK^*&\leq S_C\\&\leq BI_H,\end{aligned}$$

further the operator CKK^* is self-adjoint, since

$$\begin{aligned}(CKK^*)^*&=KK^*C^*\\&=KK^*C\\&=KCK^*\\&=CKK^*.\end{aligned}$$

Theorem 3.2. *Let $W:=\{(W_i,v_i)\}_{i\in I}$ be a C-controlled K- fusion frame with frame operator S_C and $K\in\mathcal{B}(H)$ with closed range, then S_C is invertible on the subspace $\mathcal{R}(K)\subset H$.*

Proof. Since $\mathcal{R}(K)$ is closed, there exists the pseudo-inverse $K^\dagger$ of K, such that for all $f\in\mathcal{R}(K)$,

$$KK^\dagger f=f.$$

Namely

$$KK^\dagger|_{\mathcal{R}(K)}=I_{\mathcal{R}(K)},$$

so we have

$$I^*_{\mathcal{R}(K)}=(K^\dagger|_{\mathcal{R}(K)})^*K^*.$$

Hence for any $f\in\mathcal{R}(K)$, we obtain

$$\|f\|=\left\|(K^\dagger|_{\mathcal{R}(K)})^*C^{-\frac{1}{2}}C^{\frac{1}{2}}K^*f\right\|$$

$$\leq \left\|K^{\dagger}\right\| \left\|C^{-\frac{1}{2}}\right\| \left\|C^{\frac{1}{2}}K^{*}f\right\|,$$

that is

$$\left\|C^{\frac{1}{2}}K^{*}f\right\|^{2} \geq \left\|K^{\dagger}\right\|^{-2} \left\|C^{-\frac{1}{2}}\right\|^{-2} \|f\|^{2}.$$

Combined above inequality with C-controlled K-fusion frame definition, for all $f \in \mathcal{R}(K)$ we have

$$\begin{aligned}\langle S_{C}f, f\rangle &\geq A\left\|C^{\frac{1}{2}}K^{*}f\right\|^{2}\\ &\geq A\left\|K^{\dagger}\right\|^{-2} \left\|C^{-\frac{1}{2}}\right\|^{-2} \|f\|^{2}.\end{aligned}$$

So

$$\begin{aligned}A\left\|K^{\dagger}\right\|^{-2} \left\|C^{-\frac{1}{2}}\right\|^{-2} \|f\| &\leq \|S_{C}f\|\\ &\leq B\|f\|,\end{aligned}$$

which implies that $S_C : \mathcal{R}(K) \longrightarrow S_C(\mathcal{R}(K))$ is a homeomorphism. Furthermore, we have for all $f \in S_C(\mathcal{R}(K))$

$$\begin{aligned}B^{-1}\|f\| &\leq \left\|S_{C}^{-1}f\right\|\\ &\leq A^{-1}\left\|K^{\dagger}\right\|^{2} \left\|C^{-\frac{1}{2}}\right\|^{2} \|f\|.\end{aligned}$$

□

Example 3.3. Take $H = \mathbb{R}^3$ and define $K \in \mathcal{B}(H)$ as

$$Ke_1 = e_1, \qquad ke_2 = e_3, \qquad Ke_3 = e_2,$$

where $\{e_i\}_{i=1}^{3}$ is the standard orthonormal basis of $\mathbb{R}^3$. Obviously $K = K^*$. Suppose $C(x_1, x_2, x_3) = (2x_1, x_2, 2x_3)$ is an operator on $\mathbb{R}^3$ and $C \in GL^{+}(H)$. It can be shown that $C^{\frac{1}{2}}$ exists. Also, let

$$W_1 = span\{e_1, e_2\}, \qquad W_2 = span\{e_1, e_3\}, \qquad W_3 = span\{e_2, e_3\}$$

and $v_i = 1$, for all $1 \leq i \leq 3$. Then $\{(W_i, 1)\}_{i=1}^{3}$ is a C-controlled K-fusion frame for H with bounds 1 and 4.

Theorem 3.4. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a C-controlled fusion Bessel sequence for H with bound B if and only if the operator*

$$T_W : \left(\sum_{i\in\mathbb{I}} \oplus W_i\right)_{l_2} \to H,$$

$$T_W(\{g_i\}) = \sum_{i\in\mathbb{I}} v_i (C\pi_{W_i})^{\frac{1}{2}} g_i,$$

is well-defined and bounded operator with $\|T_W\| \leq \sqrt{B}$.

Proof. The necessary condition follows from the definition of C-controlled fusion Bessel sequence. We only need to prove that the sufficient condition hold. Let T_W be a well-defined and bounded operator with $\|T_W\| \leq \sqrt{B}$. For any $f \in H$, we have,

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle (C\pi_{W_i}) f, f\rangle \\ &= \left\langle T_W(v_i (C\pi_{W_i})^{\frac{1}{2}} f), f\right\rangle \\ &\leq \|T_W\| \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\| \|f\|.\end{aligned}$$

But

$$\begin{aligned}\left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\|^2 &= \sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle \\ &= \|T^* f\|^2 .\end{aligned}$$

It follows that

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle \leq B\|f\|^2,$$

and this means that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled fusion Bessel sequence for H. □

Theorem 3.5. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled K-fusion frame for H with bounds A,B. Let $M, K \in \mathcal{B}(H)$ with $\mathcal{R}(M) \subset \mathcal{R}(K)$ and C commutes with M and K both. Then $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a C-controlled M-fusion frame for H.*

Proof. Suppose $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled K-fusion frame for H with bounds A and B. Then for $f \in H$,

$$\begin{aligned}A\left\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f\right\rangle &\leq \sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle \\ &\leq B\langle f, f\rangle .\end{aligned} \tag{3.2}$$

Since $\mathcal{R}(M) \subset \mathcal{R}(K)$, from Lemma 2.1, there exists some $\lambda > 0$ such that $MM^* \leq \lambda KK^*$. So we have ,

$$\left\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\right\rangle \leq \lambda \left\langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\right\rangle,$$

multiplying the above inequality by A, we get,

$$\frac{A}{\lambda}\left\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\right\rangle \leq A\left\langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\right\rangle.$$

From (3.2), we have,

$$\begin{aligned}\frac{A}{\lambda}\left\langle MM^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\right\rangle &\leq \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle \\ &\leq B\left\langle f, f\right\rangle.\end{aligned}$$

Therefore, $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled M-fusion frame for H with bounds $\dfrac{A}{\lambda}$ and B, respectively. □

Theorem 3.6. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a C-controlled K-fusion frame and $C \in GL^+(H)$. Then $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a K-fusion frame for H.*

Proof. Suppose that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled K-fusion frame with bounds A and B. For $f \in H$,

$$A\left\|C^{\frac{1}{2}}K^*f\right\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle \leq B\|f\|^2,$$

thus for the lower bound we have,

$$\begin{aligned}A\left\|K^*f\right\|^2 &= A\left\|C^{-\frac{1}{2}}C^{\frac{1}{2}}K^*f\right\|^2 \\ &\leq A\left\|C^{\frac{1}{2}}\right\|^2\left\|C^{-\frac{1}{2}}K^*f\right\|^2 \\ &\leq \left\|C^{\frac{1}{2}}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}C^0f\right\rangle \\ &= \left\|C^{\frac{1}{2}}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}f\right\|^2.\end{aligned}$$

Hence,

$$A\left\|C^{\frac{1}{2}}\right\|^{-2}\left\|K^*f\right\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}f\right\|^2.$$

On the other hand for every $f \in H$,

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}C^{-\frac{1}{2}}C^{\frac{1}{2}}f\right\|^2 &\leq \left\|C^{-\frac{1}{2}}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}C^{\frac{1}{2}}f\right\|^2 \\ &= \left\|C^{-\frac{1}{2}}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}C^{\frac{1}{2}}f, \pi_{W_i}C^{\frac{1}{2}}f\right\rangle \\ &= \left\|C^{-\frac{1}{2}}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle \\ &\leq B\left\|C^{-\frac{1}{2}}\right\|^2 \|f\|^2.\end{aligned}$$

These inequalities yields that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a K-fusion frame with bounds $A\left\|C^{\frac{1}{2}}\right\|^{-2}$ and $B\left\|C^{-\frac{1}{2}}\right\|^{2}$. □

Theorem 3.7. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a K-fusion frame for H, then $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled K-fusion frame for H.*

Proof. Suppose that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a K-fusion frame for H with bounds A', B'. Then for all $f \in H$,

$$A'\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\pi_{W_i} f\|^2 \leq B'\|f\|^2.$$

For the lower bound we have,

$$\begin{aligned} A'\left\|C^{\frac{1}{2}}K^*f\right\|^2 &= A'\left\|K^*C^{\frac{1}{2}}f\right\|^2 \\ &\leq \sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}C^{\frac{1}{2}}f\right\|^2 \\ &= \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}C^{\frac{1}{2}}f, \pi_{W_i}C^{\frac{1}{2}}f\right\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle. \end{aligned}$$

On the other hand for every $f \in H$,

$$\begin{aligned} \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \left\langle \pi_{W_i}C^{\frac{1}{2}}f, \pi_{W_i}C^{\frac{1}{2}}f\right\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \left\|\pi_{W_i}C^{\frac{1}{2}}f\right\|^2 \\ &\leq B'\left\|C^{\frac{1}{2}}f\right\|^2 \\ &\leq B'\left\|C^{\frac{1}{2}}\right\|^2 \|f\|^2. \end{aligned}$$

Therefore $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a C-controlled K-fusion frame for H with bounds $A', B'\left\|C^{\frac{1}{2}}\right\|^2$. □

Theorem 3.8. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a C-controlled fusion Bessel sequence with bound $B > 0$ and $J \subsetneq \mathbb{I}$ with T_W is the associated synthesis operator of $\{(W_i, v_i)\}_{i\in\mathbb{I}\diagup J}$. Let $a, b \geq 0$ and $K \in B(H)$ satisfying*

$$\left\|(C^{\frac{1}{2}}K^* - T_W T_W^*)f\right\| \leq a\left\|C^{\frac{1}{2}}K^*f\right\| + b\|T_W^*\|.$$

Then $\{(W_i, v_i)\}_{i\in I\diagup J}$ forms a C-controlled K-fusion frame with bounds $\left(\dfrac{1-a}{b+\|T_W\|}\right)^2$ and B if $a < 1$.

Proof. For all $f \in H$, we have,

$$\begin{aligned}\left\|C^{\frac{1}{2}}K^*f\right\| &\leq \left\|(C^{\frac{1}{2}}K^* - T_W T_W^*)f\right\| + \|T_W T_W^* f\| \\ &\leq a\left\|C^{\frac{1}{2}}K^*f\right\| + (b + \|T_W\|)\,\|T_W^* f\|.\end{aligned}$$

Therefore for all $f \in H$,

$$\begin{aligned}\left(\frac{1-a}{b+\|T_W\|}\right)^2 \left\|C^{\frac{1}{2}}K^*f\right\|^2 &\leq \sum_{i\in I \diagup J} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle \\ &\leq \sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle \\ &\leq B\|f\|^2.\end{aligned}$$

The proof is completed. □

Theorem 3.9. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a C-controlled fusion Bessel sequence with bound $B > 0$ and let $J \subsetneq \mathbb{I}$ so that the associated synthesis operator of $\{(W_i, v_i)\}_{i\in\mathbb{I}\diagup J}$ is T_W. Let $a, b, d \geq 0$ and $K \in \mathcal{B}(H)$ be a closed range operator such that*

$$\left\|(C^{\frac{1}{2}}K^* - T_W T_W^*)f\right\| \leq a\left\|C^{\frac{1}{2}}K^*f\right\| + b\,\|T_W^*\| + d\|f\|.$$

If

$$a + d\left\|K^\dagger\right\| < 1,$$

then $\{(W_i, v_i)\}_{i\in\mathbb{I}\diagup J}$ is a C-controlled K-fusion frame for $\mathcal{R}(K)$ with bounds $\left(\frac{1-a-d\left\|K^\dagger\right\|}{b+\|T_W\|}\right)^2$ and B.

Proof. For all $f \in \mathcal{R}(K)$ we have,

$$\begin{aligned}\left\|C^{\frac{1}{2}}K^*f\right\| &. \left\|(C^{\frac{1}{2}}K^* - T_W T_W^*)f\right\| + \|T_W T_W^* f\| \\ &\leq a\left\|C^{\frac{1}{2}}K^*f\right\| + (b + \|T_W\|)\,\|T_W^*\| + d\|f\|.\end{aligned}$$

So,

$$\left(1 - a - d\left\|K^\dagger\right\|\right) \leq \left\|C^{\frac{1}{2}}K^*f\right\| \leq (b + \|T_W\|)\,\|T_W^*\|.$$

Therefore for all $f \in \mathcal{R}(K)$, we have the following :

$$\begin{aligned}\left(\frac{1-a-d\left\|K^\dagger\right\|}{b+\|T_W\|}\right)^2 \left\|C^{\frac{1}{2}}K^*f\right\| &\leq \|T_W^*\|^2 \\ &= \sum_{i\in I \diagup J} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle\end{aligned}$$

$$\leq \sum_{i\in\mathbb{I}} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} Cf\rangle$$
$$\leq B\|f\|^2.$$

Consequently , our declaration is sustainable. □

Theorem 3.10. *Let $W := \{(W_i, v_i)\}_{i\in I}$ and $Z := \{(Z_i, v_i)\}_{i\in I}$ be two C-controlled fusion Bessel sequences for H with bounds B_1 and B_2, respectively and $KC = CK$. Suppose that T_W^* and T_Z^* be their controlled analysis operators such that $T_Z T_W^* = CK^*$. Then, both W and Z are C-controlled K and K^*-fusion frames, respectively.*

Proof. For each $f \in H$, we have

$$\begin{aligned}
\left\|C^{\frac{1}{2}}K^*f\right\|^4 &= \left|\left\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\right\rangle\right|^2 \\
&= |\langle CK^*f, K^*f\rangle|^2 \\
&= \left|\left\langle CK^*f, C^{-\frac{1}{2}}C^{\frac{1}{2}}K^*f\right\rangle\right|^2 \\
&= \left|\left\langle T_Z T_W^* f, C^{-\frac{1}{2}}C^{\frac{1}{2}}K^*f\right\rangle\right|^2 \\
&\leq \|T_Z\|^2 \|T_W^* f\|^2 \left\|C^{-\frac{1}{2}}\right\|^2 \left\|C^{\frac{1}{2}}K^*f\right\|^2 \\
&= B_2. \|T_W^* f\|^2 \left\|C^{-\frac{1}{2}}\right\|^2 \left\|C^{\frac{1}{2}}K^*f\right\|^2 \\
&= B_2. \left(\sum_{i\in I} v_i^2 \langle \pi_W f, \pi_W Cf\rangle\right). \left\|C^{-\frac{1}{2}}\right\|^2 \left\|C^{\frac{1}{2}}K^*f\right\|^2.
\end{aligned}$$

Thus,

$$B_2^{-1}. \left\|C^{-\frac{1}{2}}\right\|^{-2} \left\|C^{\frac{1}{2}}K^*f\right\|^2 \leq \sum_{i\in I} v_i^2 \langle \pi_W f, \pi_W Cf\rangle.$$

Therefore W is a C-controlled K-fusion frame for H. Similarly, via $T_W T_Z^* = KC$, then Z is a C-controlled K^*-fusion frame with the lower bound $B_1^{-1}. \left\|C^{-\frac{1}{2}}\right\|^{-2}$. □

4. Perturbation and Q-Duality on Controlled K-fusion Frame

Perturbation of frames is an important and useful objects to construct new frames from a given one or to compute the tolerance of a frame against unwanted mutations. At the first, the problem of perturbation studied by Paley and Wiener for bases and then extended to frames. The most general result obtained by Casazza and Christensen [6].

In this section, we study stability conditions of C-controlled K-fusion frames under perturbations and we introduce the notion of duality for C-controlled K-fusion frame and characterize dual of some C-controlled K-fusion frames.

Theorem 4.1. *Let $W=\{(W_i,v_i)\}_{i\in I}$ be a C-controlled fusion Bessel sequences for H with bound B. Suppose that there exists $0<\lambda<1$ such that*

$$\left\|C^{\frac{1}{2}}K^*f-T_W^*f\right\|^2\leq\lambda\left\|C^{\frac{1}{2}}K^*f\right\|^2. \tag{4.1}$$

Then W is a C-controlled K-fusion frame for H.

Proof. For each $f\in H$, we have

$$\begin{aligned}\|T_W^*f\|^2&\geq\left\|C^{\frac{1}{2}}K^*f\right\|^2-\left\|C^{\frac{1}{2}}K^*f-T_W^*f\right\|^2\\&\geq(1-\lambda)\left\|C^{\frac{1}{2}}K^*f\right\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}(1-\lambda)\left\|C^{\frac{1}{2}}K^*f\right\|^2&\leq\|T_W^*f\|^2\\&=\langle T_W^*f,T_W^*f\rangle\\&=\left(\sum_{i\in I}v_i^2\langle\pi_Wf,\pi_WCf\rangle\right),\end{aligned}$$

hence,

$$(1-\lambda).\left\|C^{\frac{1}{2}}K^*f\right\|^2\leq\left(\sum_{i\in I}v_i^2\langle\pi_Wf,\pi_WCf\rangle\right).$$

Therefore, W is a C-controlled K-fusion frame for H. □

Theorem 4.2. *Let $K_1,K_2\in\mathcal{B}(H)$ and $\{(W_i,v_i)\}_{i\in I}$ be a C-controlled K_1-fusion frame for H. Suppose $\lambda\geq0$ and $0\leq\mu<1$ such that for all $f\in H$,*

$$\left\|C^{\frac{1}{2}}(K_1^*-K_2^*)f\right\|\leq\lambda\left\|C^{\frac{1}{2}}K_1^*f\right\|+\mu\left\|C^{\frac{1}{2}}K_2^*f\right\|.$$

Then $\{(W_i,v_i)\}_{i\in I}$ is a C-controlled K_2-fusion frame for H.

Proof. Since $\{(W_i,v_i)\}_{i\in I}$ is a C-controlled K_1-fusion frame for H, there exist $A,B>0$ such that for all $f\in H$ we have

$$A\left\|C^{\frac{1}{2}}K_1^*f\right\|^2\leq\sum_{i\in I}v_i^2\langle\pi_{W_i}f,\pi_{W_i}Cf\rangle\leq B\|f\|^2.$$

So, we obtain

$$\begin{aligned}\left\|C^{\frac{1}{2}}K_2^*f\right\| &\le \left\|C^{\frac{1}{2}}(K_1^*-K_2^*)f\right\| + \left\|C^{\frac{1}{2}}K_1^*f\right\| \\ &\le (1+\lambda)\left\|C^{\frac{1}{2}}K_1^*f\right\| + \mu\left\|C^{\frac{1}{2}}K_2^*f\right\|.\end{aligned}$$

Therefore for all $f \in H$, we have

$$\begin{aligned}A\left(\frac{1-\mu}{1+\lambda}\right)^2\left\|C^{\frac{1}{2}}K_2^*f\right\| &\le A\left\|C^{\frac{1}{2}}K_1^*f\right\| \\ &\le \sum_{i\in I} v_i^2\left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle \\ &\le B\|f\|^2,\end{aligned}$$

and this completes the proof. □

Theorem 4.3. *Let $W := \{(W_i, v_i)\}_{i\in I}$ be a C-controlled K-fusion frame for H, also $0 \le \lambda_1, \lambda_2 < 1$ and $\epsilon > 0$. If*

$$\left\|v_i(C\pi_{W_i} - C\pi_{Z_i})^{\frac{1}{2}}f\right\| \le \lambda_1\left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| + \lambda_2\left\|v_i(C\pi_{Z_i})^{\frac{1}{2}}f\right\| + \epsilon v_i\left\|C^{\frac{1}{2}}K^*f\right\|.$$

Then $\{(Z_i, v_i)\}_{i\in I}$ is a C-controlled K-fusion frame for H.

Proof. Let $f \in H$. We have

$$\begin{aligned}\left\|v_i(C\pi_{Z_i})^{\frac{1}{2}}f\right\| &= \left\|v_i(C\pi_{Z_i} - C\pi_{W_i})^{\frac{1}{2}}f + v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| \\ &\le \left\|v_i(C\pi_{Z_i} - C\pi_{W_i})^{\frac{1}{2}}f\right\| + \left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| \\ &\le \lambda_1\left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| + \lambda_2\left\|v_i(C\pi_{Z_i})^{\frac{1}{2}}f\right\| \\ &\quad + \epsilon v_i\left\|C^{\frac{1}{2}}K^*f\right\| + \left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\|.\end{aligned}$$

Hence,

$$(1-\lambda_2)\left\|(v_i(C\pi_{Z_i})^{\frac{1}{2}}f\right\| \le (1+\lambda_1)\left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| + \epsilon v_i\left\|C^{\frac{1}{2}}K^*f\right\|.$$

Since W is a C-controlled K-fusion frame with bounds A and B, then

$$\left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\|^2 = \sum_{i\in I} v_i^2\left\langle \pi_{W_i}f, \pi_{W_i}Cf\right\rangle \le B\|f\|^2.$$

So,

$$\begin{aligned}\left\|v_i(C\pi_{Z_i})^{\frac{1}{2}}f\right\| &\le \frac{(1+\lambda_1)\left\|v_i(C\pi_{W_i})^{\frac{1}{2}}f\right\| + \epsilon v_i\left\|C^{\frac{1}{2}}K^*f\right\|}{1-\lambda_2} \\ &\le \left(\frac{(1+\lambda_1)\sqrt{B} + \epsilon v_i\left\|C^{\frac{1}{2}}K^*\right\|}{1-\lambda_2}\|f\|\right).\end{aligned}$$

Thus

$$\sum_{i\in I} v_i^2 \langle \pi_{Z_i} f, \pi_{Z_i} C f\rangle = \left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\|^2$$
$$\leq \left(\frac{(1+\lambda_1)\sqrt{B} + \epsilon v_i \left\| C^{\frac{1}{2}} K^*\right\|}{1-\lambda_2} \|f\| \right)^2 .$$

Now, for the lower bound, we have

$$\left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\| = \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f - v_i (C\pi_{W_i} - C\pi_{Z_i})^{\frac{1}{2}} f\right\|$$
$$\geq \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\| - \left\| v_i (C\pi_{Z_i} - C\pi_{Z_i})^{\frac{1}{2}} f\right\|$$
$$\geq \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\| - \lambda_1 \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\|$$
$$- \lambda_2 \left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\| - \epsilon v_i \left\| C^{\frac{1}{2}} K^* f\right\| .$$

Therefore,

$$(1+\lambda_2) \left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\| \geq (1-\lambda_1) \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\| - \epsilon v_i \left\| C^{\frac{1}{2}} K^* f\right\| ,$$

or

$$\left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\| \geq \frac{(1-\lambda_1) \left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\| - \epsilon v_i \left\| C^{\frac{1}{2}} K^* f\right\|}{1+\lambda_2}.$$

Hence, since W is a C-controlled K-fusion frame with bounds A and B, we get,

$$\left\| v_i (C\pi_{W_i})^{\frac{1}{2}} f\right\|^2 = \sum_{i\in I} v_i^2 \langle \pi_{W_i} f, \pi_{W_i} C f\rangle$$
$$\geq A \left\| C^{\frac{1}{2}} K^* f\right\|^2 .$$

So,

$$\left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\| \geq \left(\frac{(1-\lambda_1)\sqrt{A} - \epsilon v_i}{1+\lambda_2} \left\| C^{\frac{1}{2}} K^* f\right\| \right) .$$

Thus,

$$\sum_{i\in I} v_i^2 \langle \pi_{Z_i} f, \pi_{Z_i} C f\rangle = \left\| v_i (C\pi_{Z_i})^{\frac{1}{2}} f\right\|^2$$
$$\geq \left(\frac{(1-\lambda_1)\sqrt{A} - \epsilon v_i}{1+\lambda_2} \left\| C^{\frac{1}{2}} K^* f\right\| \right)^2 ,$$

and the proof is completed. □

Definition 4.4. Suppose $\{(W_i, v_i)\}_{i\in I}$ is a C-controlled K-fusion frame for H and $CK = KC$. A C-controlled fusion Bessel sequence $\{(Z_i, v_i)\}_{i\in I}$ is called Q-dual C-controlled K-fusion frame of $\{(W_i, v_i)\}_{i\in I}$ (or C-QK-dual for $\{(W_i, v_i)\}_{i\in I}$) if there exsits a bounded linear operator $Q : \left(\sum_{i\in I} \oplus W_i\right)_{l_2} \to \left(\sum_{i\in I} \oplus Z_i\right)_{l_2}$ such that

$$T_W Q^* T_Z^* = KC. \tag{4.2}$$

The following theorem presents equivalent conditions of the definition:

Theorem 4.5. *Let $\{(Z_i, v_i)\}_{i\in I}$ be a C-QK-dual for $\{(W_i, v_i)\}_{i\in I}$. The following conditions are equvalent:*

(I) $T_W Q^* T_Z^* = KC$;
(II) $T_Z Q T_W^* = C^* K^*$;
(III) *for each $h, h' \in H$, we have*

$$\begin{aligned}\langle KCh, h'\rangle &= \langle T_Z^*(h), QT_W^*(h')\rangle\\ &= \langle Q^* T_Z^*(h), T_W^*(h')\rangle.\end{aligned}$$

Proof. Straightforward. □

Theorem 4.6. *If $\{(Z_i, v_i)\}_{i\in I}$ is a C-QK-dual for $\{(W_i, v_i)\}_{i\in I}$, then $\{(Z_i, v_i)\}_{i\in I}$ is a C-controlled K^*-fusion frame for H.*

Proof. Let $h \in H$ and B be an upper bound of $\{(W_i, v_i)\}_{i\in I}$. Therefore

$$\begin{aligned}\left\|C^{\frac{1}{2}}Kh\right\|^4 &= \left|\left\langle C^{\frac{1}{2}}Kh, C^{\frac{1}{2}}Kh\right\rangle\right|^2\\ &= |\langle CKh, Kh\rangle|^2\\ &= |\langle T_W Q^* T_Z^* h, Kh\rangle|^2\\ &= \left|\left\langle T_Z^* h, QT_W^* C^{-\frac{1}{2}} C^{\frac{1}{2}} Kh\right\rangle\right|^2\\ &\le \|T_Z^* h\|^2 \|Q\|^2 B \left\|C^{-\frac{1}{2}}\right\|^2 \left\|C^{\frac{1}{2}}Kh\right\|^2\\ &= \|Q\|^2 B \left\|C^{-\frac{1}{2}}\right\|^2 \left\|C^{\frac{1}{2}}Kh\right\|^2 \sum_{i\in I} v_i^2 \langle \pi_{Z_i} h, \pi_{Z_i} Ch\rangle.\end{aligned}$$

Hence

$$\|Q\|^{-2} B^{-1} \left\|C^{-\frac{1}{2}}\right\|^{-2} \left\|C^{\frac{1}{2}}Kh\right\|^2 \le \sum_{i\in I} v_i^2 \langle \pi_{Z_i} h, \pi_{Z_i} Ch\rangle.$$

Now, by appling the Definition 4.4, the proof is completed. □

Corollary 4.7. *If E_{op} and F_{op} are the optimal bounds of $\{(Z_i, v_i)\}_{i\in I}$, then*

$$E_{op} \ge B_{op}^{-1} \|Q\|^{-2} \left\|C^{-\frac{1}{2}}\right\|^{-2}, \qquad F_{op} \ge A_{op}^{-1} \|Q\|^{-2} \left\|C^{-\frac{1}{2}}\right\|^{-2},$$

which A_{op} and B_{op} are the optimal bounds of $\{(W_i, v_i)\}_{i\in I}$, respectively.

Suppose that $W := \{(W_i, v_i)\}_{i\in I}$ is a C-controlled K-fusion frame for H. Since $S_C \geq ACKK^*$, by Lemma 2.1 there exists an operator $X \in B(H, (\sum_{i\in I} \bigoplus W_i)_{l^2})$ such that

$$T_{W_i} X = K. \tag{4.3}$$

Now, we denote the i-th component of Xf by $X_i f = (Xf)_i$ for each $f \in H$. It is clear that $X_i \in B(H, C^* W_i)$. In the next theorem , we show that by these operators one may construct some QK-duals for W.

Theorem 4.8. *Let $W := \{(W_i, v_i)\}_{i\in I}$ be a C-controlled K-fusion frame for H. Also, let $K \in GL(H)$ and K commute with C and $C^{\frac{1}{2}}$. Furthermore, assume that X is an operator as in* (4.3)*, and $\widetilde{W} := \left\{(\widetilde{W}_i, v_i)\right\}_{i\in I}$ is a C-controlled K-fusion frame for H, where $\widetilde{W}_i = \{C^* X_i^* W_i\}_{i\in I}$. Then $\widetilde{W}$ is a C-QK-dual for $\{(W_i, v_i)\}_{i\in I}$.*

Proof. Define the mapping

$$U_0 : R(T^*_{\widetilde{W}}) \to \left(\sum_{i\in I} \bigoplus W_i\right)_{l^2},$$
$$U_0\left(T^*_{\widetilde{W}} f\right) = XCf.$$

Then U_0 is well-defined, since $T^*_{\widetilde{W}} f$ is injective because $K \in GL(H)$. Moreover,

$$\begin{aligned}
\|U_0\| &= \sup_{f\neq 0} \frac{\left\|U_0(T^*_{\widetilde{W}} f)\right\|}{\left\|(T^*_{\widetilde{W}} f)\right\|} \\
&\leq \sup_{f\neq 0} \frac{\|XCf\|}{\sqrt{A_{\widetilde{W}}} \left\|C^{\frac{1}{2}} K^* f\right\|} \\
&\leq \sup_{f\neq 0} \frac{\|X\|\|C\|\|f\|}{\sqrt{A_{\widetilde{W}}} \left\|C^{-\frac{1}{2}}\right\|^{-1} \|K^{-1}\|^{-1} \|f\|} \\
&\leq \frac{\|X\|\|C\|}{\sqrt{A_{\widetilde{W}}} \left\|C^{-\frac{1}{2}}\right\|^{-1} \|K^{-1}\|^{-1}} \\
&\leq \infty,
\end{aligned}$$

where $A_{\widetilde{W}}$ is a lower frame bound of $\left\{(\widetilde{W}_i, v_i)\right\}_{i\in I}$.
Hence , U_0 can be uniquely extended to $\overline{R(T^*_{\widetilde{W}})}$. Assume that,

$$U = \begin{cases} U_0, & \text{on } \overline{R(T^*_{\widetilde{W}})}, \\ 0, & \text{on } \overline{R(T^*_{\widetilde{W}})}^{\perp}. \end{cases}$$

So, U is well-defined and bounded. If we let $U = Q^*$, then we have

$$Q^* \in \mathcal{B}\left(\sum_{i\in I} \bigoplus C^* X_i^* W_i, \sum_{i\in I} \bigoplus W_i\right)$$

and

$$\begin{aligned} T_W Q^* T^*_{\widetilde{W}} &= T_W X C \\ &= KC. \end{aligned}$$

□

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