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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 18
Number: 1
Pages: 35-46

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2020.127585.801

Sahand Communications in Mathematical Analysis (SCMA) Vol. 18 No. 1 (2021), 35-46
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2020.127585.801

On Approximation of Some Mixed Functional Equations

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Abstract. In this paper, we have improved some of the results in [C. Choi and B. Lee, *Stability of Mixed Additive–Quadratic and Additive–Drygas Functional Equations.* Results Math. **75** no. 1 (2020), Paper No. 38]. Indeed, we investigate the Hyers–Ulam stability problem of the following functional equations

$$2\varphi(x+y)+\varphi(x-y)=3\varphi(x)+3\varphi(y)$$
$$2\psi(x+y)+\psi(x-y)=3\psi(x)+2\psi(y)+\psi(-y).$$

We also consider the Pexider type functional equation

$$2\psi(x+y)+\psi(x-y)=f(x)+g(y),$$

and the additive functional equation

$$2\psi(x+y)+\psi(x-y)=3\psi(x)+\psi(y).$$

1. Introduction

In [2] the authors investigated the Hyers–Ulam stability problem of the following functional equations

$$2f(x+y)+f(x-y)-3f(x)-3f(y)=0, \tag{1.1}$$
$$2f(x+y)+f(x-y)-3f(x)-2f(y)-f(-y)=0, \tag{1.2}$$

on a set $\Omega \subseteq \mathbb{R}^2$ which its Lebesgue measure is zero. We show that if a function $f: X \to Y$ between linear spaces X and Y, satisfies (1.1), then $f \equiv 0$ and f satisfies (1.2) if and only if f is additive. Therefore, some results in [2] need to be corrected with more accurate statements. In the investigation of Hyers–Ulam stability for functional equations, it is an interesting subject to consider the Hyers–Ulam stability on restricted

2020 *Mathematics Subject Classification.* 39B82.
Key words and phrases. Hyers-Ulam stability, Additive, Quadratic, Drygas, Functional equation, Lebesgue measure zero, Pexider equation.
Received: 13 May 2020, Accepted: 19 October 2020.
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domains or under restricted conditions [1–4, 11, 13, 14, 16–18, 20]. We also refer the interested reader to [5, 7–10, 12, 15].

In the whole paper, $\mathbb{R}$ is denoted by the set of real numbers, X and Y are indicated by a real normed space, and a real Banach space, respectively. In addition, we assume that $d > 0$ and $\varepsilon \geq 0$.

2. Abstract Approach

We improve [2, Lemma 2.1] as follows:

Lemma 2.1. *Let $\varphi : X \to Y$ fulfill*

$$2\varphi(x+y) + \varphi(x-y) = 3\varphi(x) + 3\varphi(y), \quad x, y \in X. \tag{2.1}$$

Then $\varphi \equiv 0$.

Proof. Letting $x = y = 0$ in (2.1), we get $\varphi(0) = 0$. If we put $x = 0$ in (2.1), we infer that φ is even. Replacing $y = x$ in (2.1) and using $\varphi(0) = 0$, we obtain

$$\varphi(2x) = 3\varphi(x), \quad x \in X. \tag{2.2}$$

Letting $y = -x$ in (2.1) and using evenness of φ, we obtain

$$\varphi(2x) = 6\varphi(x), \quad x \in X. \tag{2.3}$$

It follows from (2.2) and (2.3) that $\varphi(x) = 0$ for all $x \in X$. Hence the proof is complete. □

We improve [2, Lemma 2.2] as follows:

Lemma 2.2. *Let $\psi : X \to Y$ be a mapping. Then ψ fulfills*

$$2\psi(x+y) + \psi(x-y) = 3\psi(x) + 2\psi(y) + \psi(-y), \quad x, y \in X. \tag{2.4}$$

if and only if ψ is additive.

Proof. Let ψ satisfy (2.4). Putting $x = y = 0$ in (2.4), we get $\psi(0) = 0$. Letting $y = x$ and $y = -x$ in (2.4), respectively, we obtain

$$2\psi(2x) = 5\psi(x) + \psi(-x), \qquad \psi(2x) = 4\psi(x) + 2\psi(-x), \quad x \in X, \tag{2.5}$$

It follows from (2.5) that $\psi(x) + \psi(-x) = 0$ for all $x \in X$, i.e., ψ is odd. Hence (2.4) means

$$2\psi(x+y) + \psi(x-y) = 3\psi(x) + \psi(y), \quad x, y \in X \tag{2.6}$$

Replacing y by $-y$ in (2.6) and using oddness of ψ, we get

$$2\psi(x-y) + \psi(x+y) = 3\psi(x) - \psi(y), \quad x, y \in X. \tag{2.7}$$

Adding the equations (2.6) and (2.7), we obtain

$$\psi(x+y) + \psi(x-y) = 2\psi(x), \quad x, y \in X. \tag{2.8}$$

On the other hand, since ψ is odd, (2.5) implies $\psi(2x) = 2\psi(x)$ for all $x \in X$. In view of this result and (2.8) one can conclude that

$$\psi(x+y) + \psi(x-y) = \psi(2x), \quad x, y \in X. \tag{2.9}$$

Replacing x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2}$ in (2.9), respectively, we get $\psi(x) + \psi(y) = \psi(x+y)$ for all $x, y \in X$, i.e., ψ is additive. The converse part is straightforward. Hence the proof is complete. □

In whole this section, Ω denotes a non-empty subset of $X \times X$ satisfying conditions $(C_1) - (C_3)$:

For every $a, b \in X$, there exists $x \in X$ such that

(C_1)

$$\Big\{(x,0),(0,a-2x),(-x,b+x),(0,-b-2x),$$
$$(a-x,x),(0,a-b-2x),(a-x,b+x)\Big\} \subseteq \Omega;$$

(C_2)

$$\Big\{(0,x),(x,b),(x,-b),(x,-x),(a,x),(-b,x),$$
$$(a-b,2x),(x,x),(a+x,-b+x),(a-x,b+x)\Big\} \subseteq \Omega;$$

(C_3)

$$\Big\{(b+x,-x),(-x,b+x),(a-x,x),(x,a-x),$$
$$(b+x,a-x),(a-x,b+x)\Big\} \subseteq \Omega.$$

The following theorem improves [2, Theorem 2.3].

Theorem 2.3. *Let $\varphi : X \to Y$ be an odd function and satisfy*

$$\|2\varphi(x+y) + \varphi(x-y) - 3\varphi(x) - 3\varphi(y)\| \le \varepsilon, \quad (x,y) \in \Omega. \tag{2.10}$$

Then φ is bounded on X. Indeed, $\|\varphi(x)\| \le \frac{5}{2}\varepsilon$ for all $x \in X$.

Proof. By using the same argument provided in the proof of [2, Theorem 2.3], we get $\|\varphi(x+y) - \varphi(x) - \varphi(y)\| \le \frac{5}{2}\varepsilon$ for all $x, y \in X$. By Hyers's theorem [6], there exists a unique additive function $A : X \to Y$ that satisfies

$$\|\varphi(x) - A(x)\| \le \frac{5}{2}\varepsilon, \quad x \in X. \tag{2.11}$$

Hence for each $(x,y) \in \Omega$, we have

$$\|2\varphi(x+y) - 2A(x+y)\| \le 5\varepsilon$$
$$\|\varphi(x-y) - A(x-y)\| \le \frac{5}{2}\varepsilon$$

$$\|3A(x) - 3\varphi(x)\| \leq \frac{15}{2}\varepsilon$$
$$\|3A(y) - 3\varphi(y)\| \leq \frac{15}{2}\varepsilon.$$

Since A is additive, by using the triangle inequality, we get

$$\|2\varphi(x+y) + \varphi(x-y) - 3\varphi(x) - 3\varphi(y) + 2A(y)\| \leq \frac{45}{2}\varepsilon, \tag{2.12}$$

for all $(x, y) \in \Omega$. Using the inequalities (2.10) and (2.12), we obtain

$$\|A(y)\| \leq \frac{47}{4}\varepsilon, \quad (x, y) \in \Omega. \tag{2.13}$$

Let $a \in X$ be an arbitrary element. Since Ω satisfies the condition (C_1), there exists $t \in X$ such that $(a-t, a+t), (a-t, t) \in \Omega$. Then (2.13) implies that

$$\|A(a+t)\| \leq \frac{47}{4}\varepsilon, \quad \|A(t)\| \leq \frac{47}{4}\varepsilon.$$

Since A is additive, we get $\|A(a)\| \leq \frac{47}{2}\varepsilon$. Consequently, A is bounded on X, and we conclude that $A \equiv 0$. Therefore, we infer from (2.11) that $\|\varphi(x)\| \leq \frac{5}{2}\varepsilon$ for all $x \in X$. □

The following theorem improves [2, Theorem 2.4].

Theorem 2.4. *Let $\varphi : X \to Y$ be an even function and satisfy*

$$\|2\varphi(x+y) + \varphi(x-y) - 3\varphi(x) - 3\varphi(y)\| \leq \varepsilon, \quad (x, y) \in \Omega. \tag{2.14}$$

Then φ is bounded on X. Indeed, $\|\varphi(x)\| \leq \frac{5}{3}\varepsilon$ for all $x \in X$.

Proof. By the proof of [2, Theorem 2.4], there exists a unique quadratic function $Q : X \to Y$ such that

$$\|\varphi(x) - Q(x)\| \leq \frac{5}{3}\varepsilon, \quad x \in X. \tag{2.15}$$

Hence for each $(x, y) \in \Omega$, we obtain

$$\|2\varphi(x+y) - 2Q(x+y)\| \leq \frac{10}{3}\varepsilon$$
$$\|\varphi(x-y) - Q(x-y)\| \leq \frac{5}{3}\varepsilon$$
$$\|3Q(x) - 3\varphi(x)\| \leq 5\varepsilon$$
$$\|3Q(y) - 3\varphi(y)\| \leq 5\varepsilon.$$

Since Q is quadratic, by using the triangle inequality, we get

$$\|2\varphi(x+y) + \varphi(x-y) - 3\varphi(x) - 3\varphi(y) + Q(x) + Q(y) - Q(x+y)\| \leq 15\varepsilon, \tag{2.16}$$

for all $(x,y)\in\Omega$. Hence the inequalities (2.14) and (2.16) yield

$$\|Q(x+y)-Q(x)-Q(y)\|\le 16\varepsilon,\quad (x,y)\in\Omega. \tag{2.17}$$

Let $a\in X$ be an arbitrary element. Since Ω satisfies condition (C_2), there exists $t\in X$ such that $(t,t),(a-t,a+t)\in\Omega$. Since $Q(2a)=4Q(a)$ and $Q(2t)=4Q(t)$, inequality (2.17) implies that

$$\|4Q(a)-Q(a-t)-Q(y+t)\|\le 16\varepsilon,\qquad \|2Q(t)\|\le 16\varepsilon.$$

Since Q is quadratic, we infer from the last inequalities that $\|Q(y)\|\le 16\varepsilon$. Hence Q is bounded on X, and we conclude that $Q\equiv 0$. Therefore, (2.15) yields $\|\varphi(x)\|\le\frac{5}{3}\varepsilon$ for all $x\in X$. □

For a given $d>0$, it is clear that the set

$$\Delta:=\big\{(x,y)\in X^2:\|x\|+\|y\|\ge d\big\}$$

satisfies conditions $(C_1)-(C_3)$. Thus, as direct consequences of Theorem 2.3 and Theorem 2.4, one can obtain the following results which are improved versions of [2, Corollaries 2.7, 2.8].

Corollary 2.5. *For a given $d>0$, let $\varphi:X\to Y$ be an odd (even) function which fulfills*

$$\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-3\varphi(y)\|\le\varepsilon$$

for all $(x,y)\in\Delta:=\big\{(x,y)\in X^2:\ \|x\|+\|y\|\ge d\big\}$. Then φ is bounded with bounds $\frac{5}{2}\varepsilon$ (when φ is odd) and $\frac{5}{3}\varepsilon$ (when φ is even).

Corollary 2.6. *For a given $d>0$, let $\varphi:X\to Y$ be a function which fulfills*

$$\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-3\varphi(y)\|\le\varepsilon$$

for all $(x,y)\in\Delta:=\{(x,y)\in X^2:\ \|x\|+\|y\|\ge d\}$. Then φ is bounded with bound $\frac{25}{6}\varepsilon$.

Proof. Let φ_e and φ_o be the even and the odd parts of φ. Then by the assumptions we have

$$\|2\varphi_e(x+y)+\varphi_e(x-y)-3\varphi_e(x)-3\varphi_e(y)\|\le\varepsilon,$$
$$\|2\varphi_o(x+y)+\varphi_o(x-y)-3\varphi_o(x)-3\varphi_o(y)\|\le\varepsilon,$$

for all $x,y\in\Delta$. By Corollary 2.5, we get

$$\begin{aligned}\|\varphi(x)\|&\le\|\varphi_e(x)\|+\|\varphi_o(x)\|\\&\le\frac{5}{3}\varepsilon+\frac{5}{2}\varepsilon\\&=\frac{25}{6}\varepsilon,\quad x\in X.\end{aligned}$$

□

3. Stability of Functional Eqs. (1.1) and (1.2) on a Set of $\mathbb{R}^2$ with Lebesgue Measure-Zero

We start this section with the following key results.

Theorem 3.1 ([19, Theorem 1.6]). *The set of real numbers can be decomposed as $\mathbb{R} = A \cup B$, where A and B are disjoint nonempty sets such that A is of first category and B is of Lebesgue measure zero.*

Proposition 3.2 ([4, Lemma 2.4]). *Let A and B be countable subsets of $\mathbb{R}$ such that $0 \notin B$, and $c > 0$ be a constant. If $\mathcal{C}$ is a nonempty subset of $\mathbb{R}$ with Lebesgue measure zero and $\mathbb{R} \setminus \mathcal{C}$ is of first category, then there exists $\lambda \geq c$ such that*

$$\begin{aligned} A + \lambda B &= \{a + \lambda b : \ a \in A, \ b \in B\} \\ &\subseteq \mathcal{C}. \end{aligned}$$

Corollary 3.3. *Let $\delta > 0$ and $\mathcal{C}$ be a nonempty subset of $\mathbb{R}$ with Lebesgue measure zero such that $\mathbb{R} \setminus \mathcal{C}$ is of first category. Then $\Delta_\delta := \{(x, y) \in \mathcal{C} \times \mathcal{C} : \ |x|, |y| \geq \delta\}$ satisfies the condition (C_3).*

Proof. Let $a, b \in \mathbb{R}$, $A = \{a, b, 0\}$ and $B = \{-1, 1\}$. For $c = \delta + |a| + |b|$, by Proposition 3.2, there exists $x \geq c$ such that

$$\begin{aligned} A + xB &= \{a + x, a - x, b + x, b - x, x, -x\} \\ &\subseteq \mathcal{C}. \end{aligned}$$

Since $A + xB \subseteq \{t \in \mathbb{R} : |t| \geq \delta\}$, we deduce that Δ_δ fulfills (C_3). □

As a direct consequence of the Theorem 3.1, each subset of real numbers can be represented as the union of a symmetric nullset and a set of first category. Let $\mathcal{C}$ be a nonempty symmetric ($\mathcal{C} = -\mathcal{C}$) subset of $\mathbb{R}$ with Lebesgue measure zero and $\mathbb{R} \setminus \mathcal{C}$ be of first category, then $\Gamma := e^{i\alpha}(\mathcal{C} \times \mathcal{C})$ has Lebesgue measure zero and satisfies $(C_1) - (C_3)$ for some $\alpha \in [0, 2\pi)$ (see [4]). Hence for such a Γ and a given $\delta > 0$, the set $\Gamma_\delta := \{(x, y) \in \Gamma : \ |x| + |y| \geq \delta\}$ is of Lebesgue measure zero satisfying $(C_1) - (C_3)$ (see [4]).

With the above assumptions, the following theorem improves [2, Theorem 3.5, Theorem 3.6]. In the following results Y denotes a Banach space.

Theorem 3.4. *Let $\varphi : \mathbb{R} \to Y$ be a function satisfying*

$$\|2\varphi(x + y) + \varphi(x - y) - 3\varphi(x) - 3\varphi(y)\| \leq \varepsilon,$$

for all $(x, y) \in \Gamma_\delta$ where $\delta > 0$ is fixed. Then φ is bounded, and

$$\sup_{x \in \mathbb{R}} \|\varphi(x)\| \leq \frac{25}{6}\varepsilon.$$

As an immediate consequence of Theorem 3.4, we acquire the following asymptotic behavior of $\varphi : \mathbb{R} \to Y$ satisfying

$$\limsup_{|x|+|y|\to+\infty} \|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-3\varphi(y)-z\| = 0, \tag{3.1}$$

only for $(x, y) \in \Gamma$, where $z \in Y$ is fixed. The following result as a generalization improves [2, Corollary 3.9, Corollary 3.10].

Corollary 3.5. *Let $\varphi : \mathbb{R} \to Y$ be a function which fulfills* (3.1)*. Then φ is a constant function.*

Proof. Define $g : \mathbb{R} \to Y$ by $g(x) = \varphi(x) + \frac{1}{3}z$. Then

$$\limsup_{|x|+|y|\to+\infty} \|2g(x+y)+g(x-y)-3g(x)-3g(y)\| = 0, \tag{3.2}$$

for $(x, y) \in \Gamma$. By the assumption (3.2), there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\|2g(x+y)+g(x-y)-3g(x)-3g(y)\| \le \frac{1}{n}, \quad (x, y) \in \Gamma_{\delta_n}.$$

By Theorem 3.4, we get

$$\|g(x)\| \le \frac{25}{6n}, \quad x \in \mathbb{R}.$$

Therefore $g(x) = 0$ for all $x \in \mathbb{R}$. This means $\varphi(x) = -\frac{1}{3}z$ for all $x \in \mathbb{R}$ which completes the proof. □

Theorem 3.6. *Let $\delta > 0$ and $\varphi : X \to Y$ be a function such that*

$$\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-\varphi(y)\| \le \varepsilon, \tag{3.3}$$

for some $\varepsilon \ge 0$ and all $x, y \in X$ with $\|x\| \ge \delta$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|A(x)-\varphi(x)+\varphi(0)\| \le 4\varepsilon, \quad x \in X.$$

Proof. Letting $y = x$ in (3.3), we acquire

$$\left\|\varphi(2x)-2\varphi(x)+\frac{1}{2}\varphi(0)\right\| \le \frac{\varepsilon}{2}, \quad \|x\| \ge \delta.$$

Then

$$\left\|\frac{\varphi(2^{n+1}x)}{2^{n+1}}-\frac{\varphi(2^m x)}{2^m}+\sum_{k=m}^{n}\frac{1}{2^{k+2}}\varphi(0)\right\| \le \sum_{k=m}^{n}\frac{\varepsilon}{2^{k+2}}, \quad \|x\| \ge \delta. \tag{3.4}$$

Hence $\left\{\frac{\varphi(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for all $x \in X$, and one can define

$$A : X \to Y, \qquad A(x) := \lim_{n\to\infty}\frac{\varphi(2^n x)}{2^n}.$$

By (3.4), we obtain

$$\left\|A(x)-\varphi(x)+\frac{1}{2}\varphi(0)\right\|\le\frac{\varepsilon}{2},\quad \|x\|\ge\delta. \tag{3.5}$$

Additivity of A follows from (3.3). Let $a, b \in X$ such that $\|a\|, \|b\|, \|a+b\| \ge \delta$. By (3.3) and using (3.5) for $x = a, b, a+b$, we conclude

$$\|A(a-b)-\varphi(a-b)+\varphi(0)\|\le 4\varepsilon. \tag{3.6}$$

Let x be a nonzero element of X and choose $a=\frac{\delta+2\|x\|}{\|x\|}x$ and $b=\frac{\delta+\|x\|}{\|x\|}x$. Then $\|a\|, \|b\|, \|a+b\| \ge \delta$ and $x = a-b$. By (3.6) one obtains

$$\|A(x)-\varphi(x)+\varphi(0)\|\le 4\varepsilon. \tag{3.7}$$

It is obvious that (3.7) holds in the case $x = 0$. The uniqueness of A which satisfies assertions of theorem, follows immediately from (3.7). □

Remark 3.7. The result above holds if we replace the condition $\|x\| \ge \delta$ by $\|y\| \ge \delta$ or $\|x\| + \|y\| \ge \delta$.

Corollary 3.8. *A function $\varphi : X \to Y$ is affine if one of the following conditions holds:*

(*i*) $\limsup_{\|x\|\to\infty}\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-\varphi(y)\|=0$;
(*ii*) $\limsup_{\|y\|\to\infty}\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-\varphi(y)\|=0$;
(*iii*) $\limsup_{\|x\|+\|y\|\to\infty}\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-\varphi(y)\|=0$.

4. A Pexider Case

Theorem 4.1. *Let $\varphi, f, g : X \to Y$ be functions satisfying*

$$\|2\varphi(x+y)+\varphi(x-y)-f(x)-g(y)\|\le\varepsilon, \tag{4.1}$$

for some $\varepsilon \ge 0$ and all $(x,y) \in \Delta := \{(x,y)\in X^2 : \|x\|+\|y\|\ge\delta\}$, where $\delta > 0$ is fixed. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\left\|\varphi(x)-A(x)-\frac{f(0)+g(0)}{3}\right\|\le 27\varepsilon, \tag{4.2}$$

$$\|3A(x)-f(x)+f(0)\|\le 20\varepsilon, \tag{4.3}$$

$$\|A(x)-g(x)+g(0)\|\le 20\varepsilon, \tag{4.4}$$

for all $x \in X$.

Proof. Setting $x = 0$ and $y = 0$ separately in (4.1), we obtain

$$\|2\varphi(y)+\varphi(-y)-f(0)-g(y)\|\le\varepsilon,\quad \|y\|\ge\delta; \tag{4.5}$$

$$\|3\varphi(x)-f(x)-g(0)\|\le\varepsilon,\quad \|x\|\ge\delta. \tag{4.6}$$

By using the triangle inequality in (4.1), (4.5) and (4.6), we achieve

$$\|2\varphi(x+y)+\varphi(x-y)-3\varphi(x)-2\varphi(y)-\varphi(-y)+f(0)+g(0)\|\le 3\varepsilon, \tag{4.7}$$

for all $x, y \in X$ with $\|x\|, \|y\| \geq \delta$. Replacing $y = x$ and $y = -x$ separately in (4.7), we get

$$\|2\varphi(2x) - 5\varphi(x) - \varphi(-x) + \varphi(0) + f(0) + g(0)\| \leq 3\varepsilon; \tag{4.8}$$

$$\|\varphi(2x) - 4\varphi(x) - 2\varphi(-x) + 2\varphi(0) + f(0) + g(0)\| \leq 3\varepsilon, \tag{4.9}$$

for all $x \in X$ with $\|x\| \geq \delta$. By applying the triangle inequality for (4.8) and (4.9), we obtain

$$\left\|\varphi(2x) - 2\varphi(x) + \frac{f(0) + g(0)}{3}\right\| \leq 3\varepsilon, \quad \|x\| \geq \delta. \tag{4.10}$$

Replacing x by $2^n x$ in the equation above and dividing both sides of the resultant inequality by 2^{n+1}, we get

$$\left\|\frac{\varphi(2^{n+1}x)}{2^{n+1}} - \frac{\varphi(2^n x)}{2^n} + \frac{f(0) + g(0)}{3 \times 2^{n+1}}\right\| \leq \frac{3}{2^{n+1}}\varepsilon, \quad \|x\| \geq \delta \tag{4.11}$$

for all nonnegative integers n. Therefore,

$$\begin{aligned} &\left\|\frac{\varphi(2^{n+1}x)}{2^{n+1}} - \frac{\varphi(2^m x)}{2^m} + \sum_{k=m}^{n} \frac{f(0) + g(0)}{3 \times 2^{k+1}}\right\| \\ &\leq \sum_{k=m}^{n} \left\|\frac{\varphi(2^{k+1}x)}{2^{k+1}} - \frac{\varphi(2^k x)}{2^k} + \frac{f(0) + g(0)}{3 \times 2^{k+1}}\right\| \\ &\leq \sum_{k=m}^{n} \frac{3\varepsilon}{2^{k+1}}, \quad \|x\| \geq \delta \end{aligned} \tag{4.12}$$

for all nonnegative integers $n \geq m \geq 0$. This proves the sequence $\left\{\frac{\varphi(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for each $\|x\| \geq \delta$. Therefore, it is clear that the sequence $\{\frac{\varphi(2^n x)}{2^n}\}_n$ is a Cauchy sequence for each $x \in X$. Since Y is a Banach space, we define

$$A : X \to Y, \quad A(x) := \lim_{n\to\infty} \frac{\varphi(2^n x)}{2^n}.$$

Using the definition of A and (4.7), we conclude that

$$2A(x + y) + A(x - y) = 3A(x) + 2A(y) + A(-y), \quad x, y \in X.$$

By Lemma 2.2, we infer that A is additive. Putting $m = 0$ and taking the limit as n tends to $+\infty$, one obtains

$$\left\|A(x) - \varphi(x) + \frac{f(0) + g(0)}{3}\right\| \leq 3\varepsilon, \quad \|x\| \geq \delta. \tag{4.13}$$

Let $a, b \in X$ such that $\|a\|, \|b\|, \|a + b\| \geq \delta$. Then (4.13) holds for $x \in \{a, \pm b, a + b\}$. Therefore adding inequalities (4.7) (by letting $x =$

$a, y = b$) and (4.13) for $x \in \{a, \pm b, a+b\}$, and then applying the triangle inequality, we conclude

$$\left\|\varphi(a-b) + 2A(a+b) - 3A(a) - 2A(b) - A(-b) - \frac{f(0)+g(0)}{3}\right\| \leq 27\varepsilon.$$

Since A is additive, we get

$$\left\|\varphi(a-b) - A(a-b) - \frac{f(0)+g(0)}{3}\right\| \leq 27\varepsilon. \tag{4.14}$$

Let x be a nonzero element of X and choose $a = \frac{\delta+2\|x\|}{\|x\|}x$ and $b = \frac{\delta+\|x\|}{\|x\|}x$. Then it is clear that $\|a\|, \|b\|, \|a+b\| \geq \delta$ and $x = a - b$. By inequality (4.14) one obtains

$$\left\|\varphi(x) - A(x) - \frac{f(0)+g(0)}{3}\right\| \leq 27\varepsilon. \tag{4.15}$$

Also, inequalities (4.8) and (4.13) yield that inequality (4.15) holds for $x = 0$. This proves (4.2).

From (4.6) and (4.13) it follows that

$$\|3A(x) - f(x) + f(0)\| \leq 10\varepsilon, \quad \|x\| \geq \delta. \tag{4.16}$$

Let $y \in X$ be an arbitrary element and choose $x \in X$ with $\|x\| \geq \delta + \|y\|$. Hence because of the additivity of A we obtain from (4.1), (4.13) and (4.16)

$$\|A(y) - g(y) + g(0)\| \leq 20\varepsilon,$$

and this proves (4.4). Similarly, (4.5) and (4.13) yield

$$\|A(y) - g(y) + g(0)\| \leq 10\varepsilon, \quad \|y\| \geq \delta. \tag{4.17}$$

Let $x \in X$ be arbitrary and choose $y \in X$ such that $\|y\| \geq \delta + \|x\|$. Using the additivity of A we obtain from (4.1), (4.13) and (4.17)

$$\|3A(x) - f(x) + f(0)\| \leq 20\varepsilon.$$

Hence we get (4.3).

The uniqueness of an additive function which satisfies assertions of theorem follows immediately from (4.2). □

Remark 4.2. The result above holds if we replace Δ by the set $\{(x, y) \in X^2 : \|x\| \geq \delta \text{ or } \|y\| \geq \delta\}$.

Corollary 4.3. *If functions $\varphi, f, g : X \to Y$ satisfy*

$$\limsup_{\|x\|+\|y\|\to\infty} [2\varphi(x+y) + \varphi(x-y) - f(x) - g(y)] = 0, \tag{4.18}$$

then φ, f and g are affine.

Proof. By the assumption (4.18), there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\|2\varphi(x+y)+\varphi(x-y)-f(x)-g(y)\| \le \frac{1}{n}, \quad \|x\|+\|y\| \ge \delta_n.$$

By Theorem 4.1, there exists a unique additive function $A_n : X \to Y$ satisfying

$$\left\|\varphi(x)-A_n(x)-\frac{f(0)+g(0)}{3}\right\| \le \frac{27}{n}, \tag{4.19}$$

$$\|3A_n(x)-f(x)+f(0)\| \le \frac{20}{n}, \tag{4.20}$$

$$\|A_n(x)-g(x)+g(0)\| \le \frac{20}{n}, \tag{4.21}$$

for all $x \in X$ and all $n \ge 1$. Since A_{n+1} satisfies in the above-mentioned inequalities, we infer that $A_{n+1} = A_n$ for all $n \ge 1$. So $A_n = A_1$ for all $n \ge 1$. Taking the limit as $n \to \infty$ in (4.19), (4.20) and (4.21), the result follows. □

Acknowledgment. We wish to thank the referees for their detailed and very helpful comments.

References

1. B. Batko, *Stability of an alternative functional equation*, J. Math. Anal. Appl., 339 (2008), pp. 303-311.
2. C. Choi and B. Lee, *Stability of Mixed Additive-Quadratic and Additive-Drygas Functional Equations*, Results Math., 75 (2020), Paper No. 38.
3. J. Chung, *Stability of conditional Cauchy functional equations*, Aequat. Math., 83 (2012), pp. 313-320.
4. J. Chung and J.M. Rassias, *Quadratic functional equations in a set of Lebesgue measure zero*, J. Math. Anal. Appl., 419 (2014), pp. 1065-1075.
5. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publ. Co., 2002.
6. D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, 27 (1941), pp. 222-224.
7. D.H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.
8. S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, 2011.
9. Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, New York, 2009.

10. P. Kaskasem, A. Janchada and C. Klin-eam, *On approximate solutions of the generalized radical cubic functional equation in quasi β- Banach spaces*, Sahand Commun. Math. Anal., 17 (2020), pp. 69–90.
11. B. Khosravi, M.B. Moghimi and A. Najati, *Asymptotic aspect of Drygas, quadratic and Jensen functional equations in metric abelian groups*, Acta Math. Hungar., 155 (2018), pp. 248-265.
12. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Publ. and Silesian Univ. Press, Warsaw, 1985.
13. Y.-H. Lee, S.-M. Jung and M.Th. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Appl. Math. Comput., 228 (2014), pp. 13-16.
14. Y.-H. Lee, S.-M. Jung and M.Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal., 12 (2018), pp. 43-61.
15. M. Maysami Sadr, *Stability of additive functional equation on discrete quantum semigroups*, Sahand Commun. Math. Anal., 8 (2017), pp. 73–81.
16. D. Molaei and A. Najati, *Hyperstability of the general linear equation on restricted domains*, Acta Math. Hungar., 149 (2016), pp. 238-253.
17. A. Najati and Soon-Mo Jung, *Approximately quadratic mappings on restricted domains*, J. Inequal. Appl., (2010), Art. ID 503458, 10 pages.
18. A. Najati and Th.M. Rassias, *Stability of the Pexiderized Cauchy and Jensen's equations on restricted domains*, Sahand Commun. Math. Anal., 8 (2010), pp. 125-135.
19. J. C. Oxtoby, *Measure and Category*, Springer, New York, 1980.
20. J. Senasukh and S. Saejung, *On the hyperstability of the Drygas functional equation on a restricted domain*, Bull. Aust. Math. Soc., 102 (2020), pp. 126-137.

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