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Gabor Dual Frames with Characteristic Function Window

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Abstract. The duals of Gabor frames have an essential role in reconstruction of signals. In this paper we find a necessary and sufficient condition for two Gabor systems $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ to form dual frames for $L_2(\mathbb{R})$, where a and b are positive numbers and c_1, c_2, d_1 and d_2 are real numbers such that $c_1 < d_1$ and $c_2 < d_2$.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [6] in the study of nonharmonic Fourier series in 1952. Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal processing and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [5]. Given a separable Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A \left\| f \right\|^2 \leq \sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \leq B \left\| f \right\|^2 , \tag{1.1}$$

where A, B are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame. For more information concerning frames, we refer to [3–5, 12, 16, 18].

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For any $x, y \in \mathbb{R}$ the translation operator T_x and modulation operator E_y on $L_2(\mathbb{R})$ are defined by

$$(T_x g)(t) = g(t-x), \qquad (E_y g)(t) = e^{2\pi i y t} g(t).$$

A Gabor system (g, a, b) with window function $g \in L_2(\mathbb{R})$, time shift parameter $a > 0$ and frequency shift parameter $b > 0$ is the sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. A Gabor system (g, a, b) is called a Gabor frame if it is a frame for $L_2(\mathbb{R})$, i.e., if there exist constants $0 < A \leq B < \infty$ such that for all $f \in L_2(\mathbb{R})$,

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2, \tag{1.2}$$

where $\|.\|$ and $\langle ., .\rangle$ denote the standard norm and inner product of $L_2(\mathbb{R})$.

By the Ron-Shen theory [17] and [7] the Gabor system (g, a, b) is a Gabor frame with bounds $0 < A \leq B < \infty$ if and only if

$$bAI \leq M_g(t)\, M_g^*(t) \leq bBI, \qquad \text{a.e. } t \in \mathbb{R},$$

where I denote the identity operator on $\ell_2(\mathbb{Z})$ and $M_g(t))$ is the bi-infinite matrix defined by

$$M_g(t) = \left(g\left(t + na - \frac{k}{b}\right)\right)_{k,n\in\mathbb{Z}}, \qquad \text{a.e. } t \in \mathbb{R},$$

where k is the row index and n is the column index. The case when $g = \chi_{[0,c)}$, for some $c > 0$ has been studied by Janssen in [15].

By Ron-Shen theorem, $\left(\chi_{[0,c)}, a, b\right)$ is a Gabor frame with frame bounds $0 < A \leq B < \infty$ if and only if $\left(\chi_{[0,bc)}, ba, 1\right)$ is a Gabor frame with frame bounds bA and bB.

Two Gabor frames (g, a, b) and (h, a, b) form dual frames for $L_2(\mathbb{R})$ if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle\, E_{mb}T_{na}h,$$

for all $f \in L_2(\mathbb{R})$[5]. For more information concerning Gabor frames, we refer to [8, 9, 11].

The duals of Gabor frames have an essential role in the reconstruction of signals [1, 2, 10, 13]. In this paper, we find a simple duality condition for the case that g and h are characteristic functions on intervals $[c_1, d_1)$ and $[c_2, d_2)$, respectively, where c_1, c_2, d_1 and d_2 are real numbers such that $c_1 < d_1$ and $c_2 < d_2$ (Theorem 2.7).

2. Dual Gabor Frames

The duality condition for a pair of Gabor systems (g, a, b) and (h, a, b) is presented by Janssen as follows [14]:

Lemma 2.1. *Two Bessel sequences (g, a, b) and (h, a, b) form dual frames for $L_2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g\left(x - ka - \frac{n}{b}\right)} h(x - ka) = b\delta_{n,0}, \qquad a.e.\ x \in \mathbb{R}, \quad \forall n \in \mathbb{Z}.$$

Also

Lemma 2.2. *Two Bessel sequences (g, a, b) and (h, a, b) form dual frames for $L_2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g\left(x - ka - \frac{n}{b}\right)} h(x - ka) = b\delta_{n,0}, \qquad a.e.\ x \in [0, a], \quad \forall n \in \mathbb{Z}.$$

It is well known that if g is a bounded function with compact support, then the Gabor system (g, a, b) is a Bessel sequence and then every Gabor system $\left(\chi_{[c_1,d_1)}, a, b\right)$ is a Bessel sequence. In this section, we are going to find duals of a Bessel sequence $\left(\chi_{[c_1,d_1)}, a, b\right)$ having the form $\left(\chi_{[c_2,d_2)}, a, b\right)$.

Proposition 2.3. *The Gabor system $\left(\chi_{[c,d)}, a, b\right)$ is a Gabor frame with bounds A, B if and only if $\left(\chi_{[0,d-c)}, a, b\right)$ is a Gabor frame with bounds A and B, where c and d are real numbers such that $c < d$.*

Proof. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} M_{\chi_{[c,d)}}(t) &= \left(\chi_{[c,d)}\left(t + na - \frac{k}{b}\right)\right)_{k,n\in\mathbb{Z}} \\ &= \left(\chi_{[0,d-c)}\left(t - c + na - \frac{k}{b}\right)\right)_{k,n\in\mathbb{Z}} \\ &= M_{\chi_{[0,d-c)}}(t - c). \end{aligned}$$

Also $M^*_{\chi_{[c,d)}}(t) = M^*_{\chi_{[0,d-c)}}(t - c)$. Now the result is obtained from Ron-Shen theorem. □

For any $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \le x < n + 1$. In this case $\lfloor x \rfloor := n$ and $\lceil x \rceil := n + 1$.
A necessary condition for duality of two Bessel sequences $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ is given in the next lemma.

Lemma 2.4. *If $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$, then $b = \lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$.*

Proof. Let $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ be dual frames for $L_2(\mathbb{R})$. Then for almost all $x \in [0,a)$ we have

$$b = \sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)}(x-ka)\,\chi_{[c_2,d_2)}(x-ka),$$

by Lemma 2.2 and hence $b = \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \in \mathbb{N}$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$.

If $b < \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor - 1$, then for all $x \in [0,e) \subseteq [0,a)$, where $e = \min\{a, a\left\lfloor -\frac{c}{a} \right\rfloor - a\left\lfloor -\frac{d}{a} \right\rfloor - a - ab\}$ we have

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &\geq b+1, \end{aligned}$$

and this is a contradiction.

Also if $b = \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor - 1$, then for all $x \in [0,e) \subseteq [0,a)$, where $e = \min\left\{a, c + a\left\lfloor -\frac{c}{a} \right\rfloor + a, d + a\left\lfloor -\frac{d}{a} \right\rfloor + a\right\}$ we have

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor \\ &= b+1, \end{aligned}$$

and this is a contradiction.

If $b > \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor + 1$, then for all $x \in [e,a) \subseteq [0,a)$, where $e = \max\left\{0, a\left\lfloor -\frac{c}{a} \right\rfloor - a\left\lfloor -\frac{d}{a} \right\rfloor + 2a - ab\right\}$ we have

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &\leq b-1, \end{aligned}$$

and this is a contradiction.

Also if $b = \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor + 1$, then for all $x \in [0,e) \subseteq [0,a)$, where $e = \min\left\{a, c + a\left\lfloor -\frac{c}{a} \right\rfloor + a, d + a\left\lfloor -\frac{d}{a} \right\rfloor + a\right\}$ we have

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor \\ &= b-1, \end{aligned}$$

and this is a contradiction.

$\square$

Another necessary condition for duality of two Bessel sequences $(\chi_{[c_1,d_1)}, a, b)$ and $(\chi_{[c_2,d_2)}, a, b)$ is given in the next lemma.

Lemma 2.5. *If $(\chi_{[c_1,d_1)}, a, b)$ and $(\chi_{[c_2,d_2)}, a, b)$ are dual frames for $L_2(\mathbb{R})$, then $\lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor = \frac{d-c}{a}$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$.*

Proof. Let $(\chi_{[c_1,d_1)}, a, b)$ and $(\chi_{[c_2,d_2)}, a, b)$ be dual frames for $L_2(\mathbb{R})$. Then for almost all $x \in [0, a)$ we have

$$b = \sum_{k \in \mathbb{Z}} \chi_{[c_1,d_1)}(x - ka)\, \chi_{[c_2,d_2)}(x - ka),$$

by Lemma 2.2 and hence $b = \lfloor \frac{x-c}{a} \rfloor - \lceil \frac{x-d}{a} \rceil + 1$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$.

It is obvious that $\lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor = \frac{d-c}{a}$, if $\frac{c}{a} \in \mathbb{Z}$ and $\frac{d}{a} \in \mathbb{Z}$. Now let $\frac{c}{a} \notin \mathbb{Z}$ and hence $\lfloor -\frac{c}{a} \rfloor < -\frac{c}{a}$. If $\lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor > \frac{d-c}{a}$, then $c + a \lfloor -\frac{c}{a} \rfloor + a > d + a \lfloor -\frac{d}{a} \rfloor + a$ and hence

$$\begin{aligned}[0, a) = \left[0, d + a \left\lfloor -\frac{d}{a} \right\rfloor + a\right) \cup \left[d + a \left\lfloor -\frac{d}{a} \right\rfloor + a, c + a \left\lfloor -\frac{c}{a} \right\rfloor + a\right) \\ \cup \left[c + a \left\lfloor -\frac{c}{a} \right\rfloor + a, a\right).\end{aligned}$$

If $x \in \left[c + a \lfloor -\frac{c}{a} \rfloor + a, a\right) \subseteq [0, a)$, then $\lfloor \frac{x-c}{a} \rfloor = \lfloor -\frac{c}{a} \rfloor + 1$ and $\lceil \frac{x-d}{a} \rceil = \lfloor -\frac{d}{a} \rfloor + 2$ and hence

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor. \end{aligned}$$

If $x \in \left[d + a \lfloor -\frac{d}{a} \rfloor + a, c + a \lfloor -\frac{c}{a} \rfloor + a\right) \subseteq [0, a)$, then $\lfloor \frac{x-c}{a} \rfloor = \lfloor -\frac{c}{a} \rfloor$ and $\lceil \frac{x-d}{a} \rceil = \lfloor -\frac{d}{a} \rfloor + 2$. Thus

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor - 1 \\ &= b - 1, \end{aligned}$$

and this is a contradiction.

On the other hand if $\lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor < \frac{d-c}{a}$, then $0 < c + a \lfloor -\frac{c}{a} \rfloor + a < d + a \lfloor -\frac{d}{a} \rfloor + a$ and hence

$$[0, a) = \left[0, c + a \left\lfloor -\frac{c}{a} \right\rfloor + a\right) \cup \left[c + a \left\lfloor -\frac{c}{a} \right\rfloor + a, d + a \left\lfloor -\frac{d}{a} \right\rfloor + a\right)$$

$$\cup \left[d + a \left\lfloor -\frac{d}{a} \right\rfloor + a, a\right),$$

where the third interval may be empty.

If $x \in \left[c + a \left\lfloor -\frac{c}{a} \right\rfloor + a, d + a \left\lfloor -\frac{d}{a} \right\rfloor + a\right) \subseteq [0, a)$, then

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor + 1. \end{aligned}$$

If $x \in \left[0, c + a \left\lfloor -\frac{c}{a} \right\rfloor + a\right) \subseteq [0, a)$, then $\left\lfloor \frac{x-c}{a} \right\rfloor = \left\lfloor -\frac{c}{a} \right\rfloor$ and $\left\lceil \frac{x-d}{a} \right\rceil = \left\lfloor -\frac{d}{a} \right\rfloor + 1$. Thus

$$\begin{aligned} b &= \left\lfloor \frac{x-c}{a} \right\rfloor - \left\lceil \frac{x-d}{a} \right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a} \right\rfloor - \left\lfloor -\frac{d}{a} \right\rfloor \\ &= b - 1, \end{aligned}$$

and this is a contradiction. A similar argument proves the claim in the case that $\frac{d}{a} \notin \mathbb{Z}$. $\square$

Another necessary condition for duality of two Bessel sequences $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ is given in the next lemma.

Lemma 2.6. *If $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$, then $d_1 - c_2 \leq \frac{1}{b}$ and $d_2 - c_1 \leq \frac{1}{b}$.*

Proof. We first assume that $d_1 - c_2 > \frac{1}{b}$.

Case $c_1 \leq c_2$: If $d_1 - \frac{1}{b} \leq d_2$, then for all $x \in \left[c_2, d_1 - \frac{1}{b}\right)$ (the measure of this interval is $d_1 - c_2 - \frac{1}{b} > 0$), we have

$$\begin{aligned} \chi_{[c_2,d_2)}(x) &= \chi_{[c_1,d_1)}\left(x + \frac{1}{b})\right) \\ &= 1. \end{aligned}$$

Therefore for $n = -1$ we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)}\left(x - ka + \frac{1}{b}\right) \chi_{[c_2,d_2)}(x - ka) &\geq \chi_{[c_1,d_1)}\left(x + \frac{1}{b}\right) \chi_{[c_2,d_2)}(x) \\ &= 1. \end{aligned}$$

If $d_1 - \frac{1}{b} > d_2$, then for all $x \in [c_2, d_2)$ we have

$$\begin{aligned} \chi_{[c_2,d_2)}(x) &= \chi_{[c_1,d_1)}\left(x + \frac{1}{b}\right) \\ &= 1. \end{aligned}$$

Therefore for $n=-1$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\chi_{[c_1,d_1)}\left(x-ka+\frac{1}{b}\right)\chi_{[c_2,d_2)}\left(x-ka\right) &\geq \chi_{[c_1,d_1)}\left(x+\frac{1}{b}\right)\chi_{[c_2,d_2)}\left(x\right)\\ &=1.\end{aligned}$$

Thus $\left(\chi_{[c_1,d_1)},a,b\right)$ and $\left(\chi_{[c_2,d_2)},a,b\right)$ are not dual frames for $L_2\left(\mathbb{R}\right)$ by Lemma 2.1.

Case $c_1>c_2$ and $d_1\leq d_2$: If $c_1-c_2<\frac{1}{b}$, then for all $x\in\left[c_2,d_1-\frac{1}{b}\right)$ (the measure of this interval is $d_1-c_2-\frac{1}{b}>0$), we have

$$\begin{aligned}\chi_{[c_2,d_2)}\left(x\right) &= \chi_{[c_1,d_1)}\left(x+\frac{1}{b}\right)\\ &=1.\end{aligned}$$

Therefore for $n=-1$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\chi_{[c_1,d_1)}\left(x-ka+\frac{1}{b}\right)\chi_{[c_2,d_2)}\left(x-ka\right) &\geq \chi_{[c_1,d_1)}\left(x+\frac{1}{b}\right)\chi_{[c_2,d_2)}\left(x\right)\\ &=1.\end{aligned}$$

If $c_1-c_2\geq\frac{1}{b}$, then for all $x\in[c_1,d_1)$ we have

$$\begin{aligned}\chi_{[c_1,d_1)}\left(x\right) &= \chi_{[c_2,d_2))}\left(x-\frac{1}{b}\right)\\ &=1.\end{aligned}$$

Therefore for $n=1$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\chi_{[c_2,d_2)}\left(x-ka-\frac{1}{b}\right)\chi_{[c_1,d_1)}\left(x-ka\right) &\geq \chi_{[c_2,d_2)}\left(x-\frac{1}{b}\right)\chi_{[c_1,d_1)}\left(x\right)\\ &=1.\end{aligned}$$

Thus $\left(\chi_{[c_1,d_1)},a,b\right)$ and $\left(\chi_{[c_2,d_2)},a,b\right)$ are not dual frames for $L_2\left(\mathbb{R}\right)$ by Lemma 2.1.

Case $c_1>c_2$ and $d_1>d_2$: In this case if $d_2\leq c_1$, then the intersection of two intervals $[c_1,d_1)$ and $[c_2,d_2)$ is empty ($[c_1,d_1)\cap[c_2,d_2)=\emptyset$) and hence

$$\sum_{k\in\mathbb{Z}}\chi_{[c_2,d_2)}\left(x-ka\right)\chi_{[c_1,d_1)}\left(x-ka\right)=0.$$

Thus $\left(\chi_{[c_1,d_1)},a,b\right)$ and $\left(\chi_{[c_2,d_2)},a,b\right)$ are not dual frames for $L_2\left(\mathbb{R}\right)$ by Lemma 2.1. Therefore we can assume that $d_2-c_1>0$. Now if $c_1-c_2\geq\frac{1}{b}$, then for all $x\in[c_1,d_2)$ we have

$$\chi_{[c_1,d_1)}\left(x\right)=\chi_{[c_2,d_2)}\left(x-\frac{1}{b}\right)$$

$$= 1.$$

Therefore for $n = 1$ we have

$$\sum_{k\in\mathbb{Z}} \chi_{[c_2,d_2)}\left(x - ka - \frac{1}{b}\right)\chi_{[c_1,d_1)}(x - ka) \geq \chi_{[c_2,d_2)}\left(x - \frac{1}{b}\right)\chi_{[c_1,d_1)}(x)$$
$$= 1.$$

If $c_1 - c_2 < \frac{1}{b}$ and $d_1 - d_2 \leq \frac{1}{b}$, then for all $x \in \left[c_2 + \frac{1}{b}, d_1\right)$ we have

$$\chi_{[c_1,d_1)}(x) = \chi_{[c_2,d_2)}\left(x - \frac{1}{b}\right)$$
$$= 1.$$

Therefore for $n = 1$ we have

$$\sum_{k\in\mathbb{Z}} \chi_{[c_2,d_2)}\left(x - ka - \frac{1}{b}\right)\chi_{[c_1,d_1)}(x - ka) \geq \chi_{[c_2,d_2)}\left(x - \frac{1}{b}\right)\chi_{[c_1,d_1)}(x)$$
$$= 1.$$

Also if $c_1 - c_2 < \frac{1}{b}$ and $d_1 - d_2 > \frac{1}{b}$, then for all $x \in [c_2, d_2)$ we have

$$\chi_{[c_2,d_2)}(x) = \chi_{[c_1,d_1)}\left(x + \frac{1}{b}\right)$$
$$= 1.$$

Therefore for $n = 1$ we have

$$\sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)}\left(x - ka + \frac{1}{b}\right)\chi_{[c_2,d_2)}(x - ka) \geq \chi_{[c_1,d_1)}\left(x + \frac{1}{b}\right)\chi_{[c_2,d_2)}(x)$$
$$= 1.$$

Thus $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are not dual frames for $L_2(\mathbb{R})$ by Lemma 2.1. A similar argument proves the claim in the case that $d_2 - c_1 > \frac{1}{b}$. □

Finally the necessary and sufficient condition for duality of two Bessel sequences $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ is given in the next theorem.

Theorem 2.7. *Let a and b be positive numbers and c_1, c_2, d_1 and d_2 be real numbers such that $c_1 < d_1$ and $c_2 < d_2$. Then two Gabor systems $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$ if and only if $d_1 - c_2 \leq \frac{1}{b}$, $d_2 - c_1 \leq \frac{1}{b}$ and $b = \left\lfloor -\frac{c}{a}\right\rfloor - \left\lfloor -\frac{d}{a}\right\rfloor = \frac{d-c}{a}$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$.*

Proof. If $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$, then $d_1 - c_2 \leq \frac{1}{b}$, $d_2 - c_1 \leq \frac{1}{b}$ and $b = \left\lfloor -\frac{c}{a}\right\rfloor - \left\lfloor -\frac{d}{a}\right\rfloor = \frac{d-c}{a}$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$, by Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Conversely let $d_1 - c_2 \le \frac{1}{b}$, $d_2 - c_1 \le \frac{1}{b}$ and $b = \lfloor -\frac{c}{a} \rfloor - \lfloor -\frac{d}{a} \rfloor = \frac{d-c}{a}$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$. Then

$$c + a\left\lfloor -\frac{c}{a}\right\rfloor + a = d + a\left\lfloor -\frac{d}{a}\right\rfloor + a.$$

Let $x \in [0, a) = \left[0, c + a\lfloor -\frac{c}{a}\rfloor + a\right) \cup \left[c + a\lfloor -\frac{c}{a}\rfloor + a, a\right)$ (in the case that $\frac{c}{a} \in \mathbb{Z}$, the second interval is empty).

If $x \in \left[0, c + a\lfloor -\frac{c}{a}\rfloor + a\right)$, then we have

$$\begin{aligned} c + a\left\lfloor -\frac{c}{a}\right\rfloor &= d + a\left\lfloor -\frac{d}{a}\right\rfloor \\ &\le x \\ &< c + a\left\lfloor -\frac{c}{a}\right\rfloor + a \\ &= d + a\left\lfloor -\frac{d}{a}\right\rfloor + a, \end{aligned}$$

and hence $\lfloor \frac{x-c}{a} \rfloor = \lfloor -\frac{c}{a} \rfloor$ and $\lceil \frac{x-d}{a} \rceil = \lfloor -\frac{d}{a} \rfloor + 1$. Therefore

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)}(x - ka)\,\chi_{[c_2,d_2)}(x - ka) &= \left\lfloor \frac{x-c}{a}\right\rfloor - \left\lceil \frac{x-d}{a}\right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a}\right\rfloor - \left\lfloor -\frac{d}{a}\right\rfloor \\ &= b. \end{aligned}$$

If $x \in \left[c + a\lfloor -\frac{c}{a}\rfloor + a, a\right)$, then we have

$$\begin{aligned} c + a\left\lfloor -\frac{c}{a}\right\rfloor + a &= d + a\left\lfloor -\frac{d}{a}\right\rfloor + a \\ &\le x \\ &< c + a\left\lfloor -\frac{c}{a}\right\rfloor + 2a \\ &= d + a\left\lfloor -\frac{d}{a}\right\rfloor + 2a, \end{aligned}$$

and hence $\lfloor \frac{x-c}{a} \rfloor = \lfloor -\frac{c}{a} \rfloor + 1$ and $\lceil \frac{x-d}{a} \rceil = \lfloor -\frac{d}{a} \rfloor + 2$. Therefore

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)}(x - ka)\,\chi_{[c_2,d_2)}(x - ka) &= \left\lfloor \frac{x-c}{a}\right\rfloor - \left\lceil \frac{x-d}{a}\right\rceil + 1 \\ &= \left\lfloor -\frac{c}{a}\right\rfloor - \left\lfloor -\frac{d}{a}\right\rfloor \\ &= b. \end{aligned}$$

Also for any $x \in [0,a)$, if $c_2 \leq x - ka < d_2$, then

$$\begin{aligned} x - ka - \frac{n}{b} &\leq x - ka - \frac{1}{b} \\ &< d_2 - \frac{1}{b} \\ &\leq c_1, \end{aligned}$$

for all positive integer n and

$$\begin{aligned} x - ka - \frac{n}{b} &\geq x - ka + \frac{1}{b} \\ &> c_2 + \frac{1}{b} \\ &\geq d_1, \end{aligned}$$

for all negative integer n. Thus for all $x \in [0,a)$ and for all $n \in \mathbb{Z} - \{0\}$ we have

$$\sum_{k\in\mathbb{Z}} \chi_{[c_1,d_1)} \left(x - ka - \frac{n}{b}\right) \chi_{[c_2,d_2)} \left(x - ka\right) = 0.$$

Therefore $\left(\chi_{[c_1,d_1)}, a, b\right)$ and $\left(\chi_{[c_2,d_2)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$ by Lemma 2.2. □

Corollary 2.8. *Let a, b, c and d be positive numbers. Then two Gabor systems $\left(\chi_{[0,c)}, a, b\right)$ and $\left(\chi_{[0,d)}, a, b\right)$ are dual frames for $L_2(\mathbb{R})$ if and only if $b \in \mathbb{N}$, $c \leq \frac{1}{b}$, $d \leq \frac{1}{b}$ and $ab = \min\{c,d\}$.*

Corollary 2.9. *Let a, c and d be positive numbers. Then two Gabor systems $\left(\chi_{[0,c)}, a, 1\right)$ and $\left(\chi_{[0,d)}, a, 1\right)$ are dual frames for $L_2(\mathbb{R})$ if and only if $c \leq 1$, $d \leq 1$ and $a = \min\{c,d\}$.*

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