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Some Properties of Lebesgue Fuzzy Metric Spaces

Sugata Adhya[1*] and Atasi Deb Ray[2]

Abstract. In this paper, we establish a sequential characterisation of Lebesgue fuzzy metric and explore the relationship between Lebesgue, weak G-complete and compact fuzzy metric spaces. We also discuss the Lebesgue property of several well-known fuzzy metric spaces.

1. Introduction

The theory of fuzzy metric spaces, proposed by George and Veeramani [3], is widely accepted as a consistent notion for metric fuzziness. It is a slight modification of the one due to Kramosil and Michalek [13]. Throughout the paper, this is the only notion of fuzzy metric we will be working on. It should be noted that every fuzzy metric gives rise to a metrizable topology that allowed the researchers to adopt several concepts from metric spaces in this fuzzy setting. In particular, Gregori, Romaguera, and Sapena [7] introduced a notion similar to the Lebesgue number in the realm of fuzzy metric spaces.

In the theory of metric spaces, the Lebesgue number lemma states that every open cover $\mathcal{U}$ of a compact metric space (X, d) corresponds to a positive number δ such that any subset of X having diameter less than δ gets contained in some member of $\mathcal{U}$. This δ is called a Lebesgue number for $\mathcal{U}$. The property of having such positive real numbers for every open cover is called the Lebesgue property for metric spaces. It is important to note that one can find non-compact metric spaces (e.g. consider the set of positive integers endowed with discrete topology) that satisfy Lebesgue property. In fact, the study of metric spaces having Lebesgue property (precisely, Lebesgue metric spaces) is an interesting

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problem in the theory of metric spaces. For details one may refer to [2] and references therein.

In 2001, Gregori, Romaguera and Sapena [7] gave a satisfactory extension to the notion of Lebesgue property for fuzzy metric spaces and characterized it in terms of uniform continuity, equinormality and uniformity. They ensured the existence of a non-standard Lebesgue fuzzy metric that made Lebesgue property worth studying in the realm of fuzzy metric setting. Unfortunately, Lebesgue fuzzy metric spaces didn't get much attention of researchers, later on. Recently in [1], we discussed several new characterizations for Lebesgue fuzzy metric spaces and this paper is a continuation of that.

In this paper, we provide a sequential characterization for Lebesgue fuzzy metric and employ it to study the Lebesgue property of some well-known fuzzy metric spaces.

In 2018, Gregori, Miñana and Sapena introduced weak G-complete fuzzy metric spaces [9] which, alike Lebesgue fuzzy metric spaces, lie between compact and complete fuzzy metric spaces. It is then natural to ask the relationship between this class of fuzzy metric spaces with the class of Lebesgue fuzzy metric spaces. In this article, we address this question too.

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of real numbers and positive integers, respectively.

2. Preliminaries

In this section, we recall a series of definitions and some related results on fuzzy metric spaces that will be required subsequently. For undefined terms related to general topology, we refer to [18].

Definition 2.1 ([15])**.** Let $*$ be a binary operation on $I = [0,1]$ which is associative, commutative and continuous on $I \times I$. Then $*$ is said to be a continuous t-norm, if

a) $\forall\, a \in [0,1],\ a * 1 = a$;
b) $\forall\, a, b, c, d \in [0,1],\ a \leq b,\ c \leq d \Rightarrow a * c \leq b * d$.

Definition 2.2 ([3, 4])**.** Given a non-empty set X, a continuous t-norm $*$ and a mapping $M : X \times X \times (0, \infty) \to [0,1]$, the ordered pair $(M, *)$ is said to be a fuzzy metric on X if for all $x, y \in X$ and $t > 0$ the following conditions hold:

a) $M(x, y, t) > 0$;
b) $M(x, y, t) = 1 \Leftrightarrow x = y$;
c) $M(x, y, t) = M(y, x, t)$;
d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
e) $M(x, y, .) : (0, \infty) \to [0,1]$ is continuous.

In this case, $(X, M, *)$ is said to be a fuzzy metric space.

It is easy to note from the above axioms that given two elements x, y in a fuzzy metric space $(X, M, *)$, the mapping $t \mapsto M(x, y, t)$ is increasing on $(0, \infty)$.

Result 2.3 ([3])**.** Let $(X, M, *)$ be a fuzzy metric space. Then $\{B_M(x, r, t) : x \in X, r \in (0, 1), t > 0\}$, where

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},$$

forms a base for some topology τ_M on X.

Definition 2.4. τ_M is called the topology induced by $(M, *)$.

Definition 2.5 ([3])**.** Let (X, d) be a metric space. If $M_d : X \times X \times (0, \infty) \to [0, 1]$ is defined for all $x, y \in X$ and $t > 0$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

then $(M_d, \cdot)$, '$\cdot$' being the usual multiplication on $[0, 1]$, defines a fuzzy metric on X. It is called the standard fuzzy metric induced by d.

Result 2.6 ([3])**.** If (X, d) is a metric space, then $\tau_{M_d} = \tau(d)$, where $\tau(d)$ denotes the topology induced by the metric d.

Definition 2.7 ([10])**.** A fuzzy metric space $(X, M, *)$ is said to be stationary if for all $x, y \in X$, $t \mapsto M(x, y, t)$ defines a constant mapping on $(0, \infty)$.

George and Veeramani [3] initiated the study of convergence of sequences for fuzzy metric spaces. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ converges to x (*resp.* clusters), if it does so in (X, τ_M).

Theorem 2.8 ([3])**.** *A sequence (x_n) in a fuzzy metric space $(X, M, *)$ converges to $x \in X$ if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1, \ \forall\ t > 0.$.*

Definition 2.9 ([3])**.** A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be Cauchy if for $\epsilon \in (0, 1)$ and $t > 0$, there exists $k \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \epsilon, \ \forall\ m, n \geq k$.

A fuzzy metric space, in which every Cauchy sequence converges, is said to be complete.

Definition 2.10. [11] A fuzzy metric space $(X, M, *)$ is said to be precompact if for $r \in (0, 1)$ and $t > 0$, there exists a finite subset A of X such that $X = \bigcup_{x \in A} B_M(x, r, t)$.

Proposition 2.11 ([14])**.** *A metric space (X, d) is precompact if and only if the standard fuzzy metric space $(X, M_d, \cdot)$ is precompact.*

Lemma 2.12 ([11])**.** *A fuzzy metric space $(X, M, *)$ is precompact if and only if every sequence in X has a Cauchy subsequence.*

In [11], Gregori and Romaguera introduced compactness for fuzzy metric spaces in the most obvious way: A fuzzy metric space $(X, M, *)$ is compact if so is (X, τ_M) as a topological space. They characterized compact fuzzy metric spaces as follows:

Theorem 2.13 ([11])**.** *A fuzzy metric space $(X, M, *)$ is compact if and only if it is precompact and complete.*

3. Sequential Characterization for Lebesgue Property

Definition 3.1 ([7])**.** A fuzzy metric space $(X, M, *)$ is said to have the Lebesgue property if given an open cover $\mathcal{G}$ of (X, τ_M), there exist $r \in (0, 1),\ t > 0$ such that $\{B_M(x, r, t) : x \in X\}$ refines $\mathcal{G}$. We call such fuzzy metric spaces *Lebesgue*.

Proposition 3.2 ([7])**.** *Let (X, d) be a metric space. Then (X, d) is Lebesgue if and only if $(X, M_d, \cdot)$ is Lebesgue.*

Definition 3.3 ([7])**.** A fuzzy metric space $(X, M, *)$ is said to be equinormal if for given nonempty, closed subsets B and C of (X, τ_M) with $B \cap C = \emptyset$, there exists $s > 0$ such that

$$\sup\{M(b, c, s) : b \in B, c \in C\} < 1.$$

Several characterizations of the Lebesgue property for fuzzy metric spaces have been discussed in [1] and [7]. In particular, it has been shown in [7] that a fuzzy metric space is Lebesgue if and only if it is equinormal. In what follows, we give a sequential characterization for Lebesgue fuzzy metric spaces.

To attain the requirement of our main result, we first extend the notion of pseudo-Cauchy sequences in fuzzy metric setting.

Recall that, a sequence (x_n) in a metric space (X, d) is pseudo-Cauchy if given $\epsilon > 0$ and $k \in \mathbb{N}$, there exist $j, n\ (> k) \in \mathbb{N}$ with $j \neq n$ such that $d(x_j, x_n) < \epsilon$. We propose the notion of fuzzy pseudo-Cauchy sequence as follows:

Definition 3.4. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be fuzzy pseudo-Cauchy if given $\epsilon \in (0, 1),\ t > 0$ and $k \in \mathbb{N}$, there exist $j,\ n\ (> k) \in \mathbb{N}$ with $j \neq n$ such that $M(x_j, x_n, t) > 1 - \epsilon$.

Clearly, a Cauchy sequence in a (fuzzy) metric space is (fuzzy) pseudo-Cauchy, however, the converse may fail.

Proposition 3.5. *Let (X, d) be a metric space. A sequence (x_n) in $(X, M_d, \cdot)$ is fuzzy pseudo-Cauchy if and only if (x_n) is pseudo-Cauchy in (X, d).*

Proof. Consider a fuzzy pseudo-Cauchy sequence (x_n) in $(X, M_d, \cdot)$.

Choose $\epsilon \in (0,1)$ and $k \in \mathbb{N}$. Then there exist $j, n\ (> k) \in \mathbb{N}$ with $j \neq n$ such that $M_d(x_j, x_n, 1-\epsilon) > 1-\epsilon$, i.e., $d(x_j, x_n) < \epsilon$. Thus (x_n) is pseudo-Cauchy in (X, d).

Conversely, let (x_n) be a pseudo-Cauchy sequence in (X, d). Choose $\epsilon \in (0,1),\ t > 0$ and $k \in \mathbb{N}$. Then there exist $j, n\ (> k) \in \mathbb{N}$ with $j \neq n$ such that $d(x_j, x_n) < \frac{t\epsilon}{1-\epsilon}$. Consequently,

$$\frac{t}{t + d(x_j, x_n)} > 1 - \epsilon \quad \Rightarrow \quad M_d(x_j, x_n, t) > 1 - \epsilon.$$

Thus (x_n) is fuzzy pseudo-Cauchy in $(X, M_d, \cdot)$. □

Example 3.6. Consider the non-standard fuzzy metric space $(\mathbb{N}, M, *)$ (see [7]) where $a * b = ab,\ \forall\ a, b \in [0,1]$ and for $x, y \in \mathbb{N},\ t > 0$,

$$M(x,y,t) = \begin{cases} 1, & \text{if } x = y, \\ \frac{1}{xy}, & \text{otherwise,} \end{cases}$$

Then $(1, 2, 1, 3, 1, 4, \ldots)$ is a fuzzy pseudo-Cauchy sequence in $(\mathbb{N}, M, *)$ which is not Cauchy.

We are now at a stage to discuss the main result of this section.

Theorem 3.7. *Let $(X, M, *)$ be a fuzzy metric space. Then $(X, M, *)$ is Lebesgue if and only if every fuzzy pseudo-Cauchy sequence in $(X, M, *)$ having distinct terms has a cluster point in (X, τ_M).*

Proof. Let $(X, M, *)$ be Lebesgue. Choose a fuzzy pseudo-Cauchy sequence (x_n) having distinct terms in X. Then there exists a strictly increasing sequence (r_n) of natural numbers such that

$$M\left(x_{r_{2n-1}}, x_{r_{2n}}, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1}, \quad \forall n \in \mathbb{N}.$$

If possible, let none of $(x_{r_{2n-1}})$ and $(x_{r_{2n}})$ has cluster point in (X, τ_M). Then, $B = \{x_{r_{2n-1}} : n \in \mathbb{N}\}$ and $C = \{x_{r_{2n}} : n \in \mathbb{N}\}$ are disjoint, closed subsets of (X, τ_M).

Since $(X, M, *)$ is equinormal, being Lebesgue, there exists $s > 0$ such that

$$\sup\{M(b,c,s) : b \in B, c \in C\} = p \cdots, \tag{3.1}$$

where $p < 1$.

Choose, $k \in \mathbb{N}$ such that $\frac{1}{k} < \min\{s, 1-p\}$.

Then

$$M\left(x_{r_{2n-1}}, x_{r_{2n}}, s\right) \geq M\left(x_{r_{2n-1}}, x_{r_{2n}}, \frac{1}{n+1}\right)$$

$$\begin{aligned} &> 1 - \frac{1}{n+1} \\ &> p, \quad \forall n \geq k, \end{aligned}$$

a contradiction to (3.1).

Thus, at least one of $(x_{r_{2n-1}})$ or $(x_{r_{2n}})$ has a cluster point in (X, τ_m), which establishes the fact that (x_n) has a cluster point in (X, τ_m).

Conversely, let the condition hold. If possible, let $(X, M, *)$ be not a Lebesgue fuzzy metric space. Then there exists an open cover $\mathcal{G} = \{U_\lambda : \lambda \in \Lambda\}$ of (X, τ_M) such that for no $r \in (0,1)$ and $s > 0$, $\{B(x, r, s) : x \in X\}$ refines $\mathcal{G}$. Thus for each $n \geq 1$, there exists $x_{2n-1} \in X$ such that

$$B\left(x_{2n-1}, \frac{1}{n+1}, \frac{1}{n+1}\right) \not\subset U_\lambda, \quad \forall \lambda \in \Lambda.$$

Since $\mathcal{G}$ covers (X, τ_M), $x_{2n-1} \in U_{\lambda_n}$ for some $\lambda_n \in \Lambda$. Thus $\exists\ x_{2n} \in B\left(x_{2n-1}, \frac{1}{n+1}, \frac{1}{n+1}\right) \setminus U_{\lambda_n}$ and consequently,

$$M\left(x_{2n-1}, x_{2n}, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1}, \quad \forall n \geq 1. \tag{3.2}$$

We first show that (x_n) is a fuzzy pseudo-Cauchy sequence. Let $\epsilon \in (0,1), t > 0$ and $k > 1$. Choose $q > k$ such that $\frac{1}{q} < \min\{\epsilon, t\}$. Then $2q-1, 2q > k$ and

$$\begin{aligned} M\left(x_{2q-1}, x_{2q}, t\right) &\geq M\left(x_{2q-1}, x_{2q}, \frac{1}{q+1}\right) \\ &> 1 - \frac{1}{q+1} \\ &> 1 - \epsilon. \end{aligned}$$

Thus (x_n) is fuzzy pseudo-Cauchy.

We now show that (x_n) has a fuzzy pseudo-Cauchy subsequence (x_{r_n}) of distinct terms.

Case I: Suppose (x_n) does not have any constant subsequence. We proceed by induction.

Set $x_{r_1} = x_1$ and $x_{r_2} = x_2$. For chosen

$$\left\{x_{r_1}, x_{r_2}, x_{r_3}, x_{r_4}, \ldots, x_{r_{2k-1}}, x_{r_{2k}}\right\}$$

find $p > r_{2k}$ such that $x_{2p-1}, x_{2p} \notin \left\{x_{r_1}, x_{r_2}, x_{r_3}, x_{r_4}, \ldots, x_{r_{2k-1}}, x_{r_{2k}}\right\}$ and set $x_{r_{2k+1}} = x_{2p-1},\ x_{r_{2k+2}} = x_{2p}$. Thus we obtain a subsequence (x_{r_n}) of (x_n) having distinct terms.

Choose $t > 0$, $\epsilon \in (0,1)$ and $k \in \mathbb{N}$. Find $q > k$ such that $\frac{1}{q} < \min\{\epsilon, t\}$. Then

$$M\left(x_{2r_q-1}, x_{2r_q}, t\right) \geq M\left(x_{2r_q-1}, x_{2r_q}, \frac{1}{q+1}\right)$$

$$
\begin{aligned}
&\geq M\left(x_{2r_q-1}, x_{2r_q}, \frac{1}{r_q+1}\right)\\
&> 1-\frac{1}{r_q+1}\\
&\geq 1-\frac{1}{q+1}\\
&> 1-\epsilon.
\end{aligned}
$$

Consequently (x_{r_n}) is fuzzy pseudo-Cauchy.

Case II: Suppose (x_n) has a constant subsequence (x_{r_n}), where

$$x_{r_n} = a, \quad \forall\ n \geq 1.$$

By setting

$$m_n = \begin{cases} r_n - 1, & \text{if } r_n \text{ is even,}\\ r_n + 1, & \text{if } r_n \text{ is odd,}\end{cases}$$

we see that for chosen $n \in \mathbb{N}$, $\exists\ k \in \mathbb{N}$ such that $\{r_n, m_n\} = \{2k-1, 2k\}$. Since (r_n) defines a strictly increasing sequence of natural numbers, so does (m_n). Thus (x_{m_n}) forms a subsequence of (x_n).

We first show that, $(x_{m_{2n}})$ is fuzzy pseudo-Cauchy.

Using equation (3.2) we see that, $\forall n \in \mathbb{N}$,

$$M\left(x_{r_{2n}}, x_{m_{2n}}, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1},$$

that is, $M\left(a, x_{m_{2n}}, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1}$. Thus $(x_{m_{2n}})$ is convergent and hence, is fuzzy pseudo-Cauchy.

Since $\lim\limits_{n\to\infty} x_{m_{2n}} = a$, and $x_{m_{2n}} \neq x_{r_{2n}}$, $\forall\ n \in \mathbb{N}$, it follows that $(x_{m_{2n}})$ has no constant subsequence. Thus, in view of Case I, it must have a fuzzy pseudo-Cauchy subsequence of distinct terms.

Consequently, in any case, (x_n) has a fuzzy pseudo-Cauchy subsequence of distinct terms. Thus, in view of the hypothesis, (x_n) must have a cluster point z in (X, τ_M).

Clearly $z \in U_\lambda$ for some $\lambda \in \Lambda$. Since U_λ is open, there exist $r \in (0,1)$ and $s > 0$ such that $B_M(z, r, s) \subset U_\lambda$.

Since $*$ is continuous, there exists $r' \in (0,1)$ with $r' < r$ such that

$$(1-r') * (1-r') * (1-r') > 1 - r.$$

Also z being a cluster point of (x_n), there is a natural number p satisfying $\frac{1}{p} < \min\{r', \frac{s}{3}\}$ such that at least one of x_{2p} and x_{2p-1} belongs to $B_M(z, r', \frac{s}{3})$. Set y to be a point among x_{2p} and x_{2p-1} such that it lies in $B_M(z, r', \frac{s}{3})$.

Note that for $w \in B_M\left(x_{2p-1}, \frac{1}{p+1}, \frac{1}{p+1}\right)$, we have

$$\begin{aligned}
M(w,z,s) &\geq M\left(w, x_{2p-1}, \frac{s}{3}\right) * M\left(x_{2p-1}, y, \frac{s}{3}\right) * M\left(y, z, \frac{s}{3}\right) \\
&\geq M\left(w, x_{2p-1}, \frac{1}{p+1}\right) * M\left(x_{2p-1}, y, \frac{1}{p+1}\right) * M\left(y, z, \frac{s}{3}\right) \\
&\geq \left(1 - \frac{1}{p+1}\right) * \left(1 - \frac{1}{p+1}\right) * (1 - r') \\
&\geq (1 - r') * (1 - r') * (1 - r') \\
&> 1 - r,
\end{aligned}$$

that is, $w \in B_M(z,r,s)$. Thus

$$\begin{aligned}
B_M\left(x_{2p-1}, \frac{1}{p+1}, \frac{1}{p+1}\right) &\subset B_M(z,r,s) \\
&\subset U_\lambda,
\end{aligned}$$

a contradiction.

So $(X, M, *)$ is Lebesgue. □

Example 3.8. It is worth recalling, at this stage, that $(\mathbb{N}, M, *)$, defined in Example 3.6, forms a non-standard, Lebesgue fuzzy metric space [7].

In fact, Theorem 3.7 can be employed to realize that $(\mathbb{N}, M, *)$ is Lebesgue: Choose $\epsilon = \frac{1}{2}$ and $t > 0$. Then for no $x, y\ (x \neq y) \in \mathbb{N}$ we can have $M(x,y,t) > 1 - \epsilon$. So, there is no fuzzy pseudo-Cauchy sequence in $(\mathbb{N}, M, *)$. Consequently, $(\mathbb{N}, M, *)$ is Lebesgue.

Note 3.9. Example 3.8 establishes the generality of Theorem 3.7 with respect to its classical counterpart [17].

Before proceeding further, we note from pseudo-Cauchy characterization of the Lebesgue property that the class of Lebesgue fuzzy metric spaces resides strictly in-between the classes of compact and complete fuzzy metric spaces.

Example 3.10. For $X = (0, \infty)$, define $M : X^2 \times (0,\infty) \to [0,1]$ by $M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}},\ \forall\ x, y \in X,\ t > 0$. It has been shown in [12] that, $(X, M, \cdot)$ forms a complete fuzzy metric space which is not compact.

We now show that, $(X, M, \cdot)$ is not even Lebesgue.

Set $a_n = n,\ \forall\ n \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} (a_n, a_{n+1}, t) &= \lim_{n\to\infty} \frac{n}{n+1} \\
&= 1, \quad \forall t > 0.
\end{aligned}$$

So, given $\epsilon \in (0,1), t > 0$ and $k \in \mathbb{N}$, $\exists$ p, q $(p \neq q) > n$ such that $M(x_p, x_q, t) > 1 - \epsilon$. Thus, (a_n) is a fuzzy pseudo-Cauchy sequence of distinct terms in X.

If possible, let c be a cluster point of (a_n). Then there exists a subsequence (a_{r_n}) of (a_n) that converges to c with respect to the topology τ_M. Note that, $\exists$ $k \in \mathbb{N}$ such that $a_{r_n} > c$, $\forall$ $n \geq k$, whence

$$\begin{aligned}\lim_{n\to\infty} M(a_{r_n}, c, t) &= \lim_{n\to\infty} \frac{c}{r_n}\\ &= 0, \quad \forall\ t > 0.\end{aligned}$$

Thus (a_{r_n}) cannot converge to c, a contradiction. Consequently, (a_n) has no cluster point.

So, in view of Theorem 3.7, $(X, M, \cdot)$ is not Lebesgue.

Example 3.11. For $X = [0, \infty)$, define $M : X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \quad \forall\ x, y \in X,\ t > 0.$$

Then $(X, M, \cdot)$ forms a complete fuzzy metric space [12]. Arguing as Example 3.10, it can be shown that $(X, M, \cdot)$ is not Lebesgue.

Example 3.12. Let $\phi : (0, \infty) \to (0, 1]$ be a function such that $\phi(t) = t$, $t \leq 1$ and $\phi(t) = 1$ otherwise. For $X = (0, \infty)$, define $M : X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ \frac{\min\{x,y\}}{\max\{x,y\}}.\phi(t), & x \neq y. \end{cases}$$

It has been shown in [6] that, $(X, M, \cdot)$ forms a complete fuzzy metric space.

We now show that $(X, M, \cdot)$ is, in fact, Lebesgue.

Choose a fuzzy pseudo-Cauchy sequence (a_n) of distinct terms in $(X, M, \cdot)$. Then there exists a strictly increasing sequence (r_n) of natural numbers such that

$$M\left(a_{r_{2n-1}}, a_{r_{2n}}, \frac{1}{n+1}\right) > 1 - \frac{1}{n+1}, \quad \forall\ n \in \mathbb{N}.$$

Note that for chosen $t > 0$ and $\epsilon \in (0, 1)$, we can find $p \in \mathbb{N}$ such that $\frac{1}{p} < \min\{\epsilon,\ t\}$. Then

$$\begin{aligned}M\left(a_{r_{2n-1}}, a_{r_{2n}}, t\right) &\geq M\left(a_{r_{2n-1}}, a_{r_{2n}}, \frac{1}{n+1}\right)\\ &> 1 - \frac{1}{n+1}\\ &> 1 - \epsilon, \quad \forall\ n \geq p,\end{aligned}$$

then $\lim\limits_{n\to\infty} M(a_{r_{2n-1}}, a_{r_{2n}}, t) = 1,\ \forall\ t > 0$.

However

$$\begin{aligned}\lim_{n\to\infty} M\left(a_{r_{2n-1}}, a_{r_{2n}}, \frac{1}{2}\right) &= \frac{1}{2} \times \lim_{n\to\infty} \frac{\min\{a_{r_{2n-1}}, a_{r_{2n}}\}}{\max\{a_{r_{2n-1}}, a_{r_{2n}}\}} \\ &= \frac{1}{2} \times \lim_{n\to\infty} M\left(a_{r_{2n-1}}, a_{r_{2n}}, 1\right),\end{aligned}$$

a contradiction.

Thus no such fuzzy pseudo-Cauchy sequence (a_n) exists in $(X, M, \cdot)$. So, in view of Theorem 3.7, $(X, M, \cdot)$ is Lebesgue.

Example 3.13. For $X = (0, 1)$, define $M : X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ xy.\phi(t), & x \neq y, \end{cases}$$

where ϕ is defined in Example 3.12.

It has been shown in [6] that $(X, M, \cdot)$ forms a complete fuzzy metric space . Arguing as Example 3.12, we can see that $(X, M, \cdot)$ is, *in fact*, Lebesgue.

4. Weak G-Completeness Versus Lebesgue Property

In this section, we investigate the relationship between weak G-compl-eteness and Lebesgue property for (fuzzy) metric spaces. We start by recalling the following weaker notion than Cauchy sequences, due to M. Grabiec [5].

Definition 4.1. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be G-Cauchy if for each $t > 0$ and $p \in \mathbb{N}$, $\lim\limits_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, or equivalently, $\lim\limits_{n\to\infty} M(x_n, x_{n+1}, t) = 1,\ \forall\ t > 0$.

Tirado, in [16], proposed the notion of G-Cauchyness for metric spaces:

Definition 4.2. A sequence (x_n) in a metric space (X, d) is said to be G-Cauchy if for each $p \in \mathbb{N}$, $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$, or equivalently, $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0,\ \forall\ t > 0$.

Definition 4.3 ([9])**.** A (fuzzy) metric space X is said to be

i) weak G-complete if every G-Cauchy sequence in X has a cluster point in it;

ii) G-complete if every G-Cauchy sequence in X converges in it.

Clearly,

$$G\text{-completeness} \to \text{weak } G\text{-completeness} \to \text{completeness}$$

though the implications cannot be reversed as is shown in [9].

Note 4.4. The space $(X, M, \cdot)$, in Example 3.10, is not weak G-complete, since (n) is a G-Cauchy sequence in X without any cluster point.

Observation 4.5. It is known that $(X, M, \cdot)$, where $X = [0, 1]$ and $M(x, y, t) = \frac{\min\{x,y\}+t}{\min\{x,y\}+t}$, $\forall\ x, y \in X, t > 0$, defines a compact, non-G-complete fuzzy metric space [9]. Thus a Lebesgue fuzzy metric space may not be G-complete.

Let us recall the following results before proceeding further:

Proposition 4.6 ([9])**.** *Let (X, d) be a metric space. Then (X, d) is weak G-complete if and only if the standard fuzzy metric space $(X, M_d, \cdot)$ is weak G-complete.*

It is observed in [9] that every compact metric space is weak G-complete. A stronger result can be realized from the succeeding discussion.

Theorem 4.7. *A Lebesgue fuzzy metric space is weak G-complete.*

Proof. Let $(X, M, *)$ be a Lebesgue fuzzy metric space and (x_n) be a G-Cauchy sequence in X.

If (x_n) has a constant subsequence, then it must have a cluster point in X. So we assume that, (x_n) has no constant subsequence. We proceed by induction. Choose x_{r_1}, x_{r_2} from the sequence such that $r_1 < r_2$, $x_{r_1} \neq x_{r_2}$ and $M\left(x_{r_1}, x_{r_2}, \frac{1}{2}\right) > 1 - \frac{1}{2}$.

Next, for chosen $x_{r_1}, x_{r_2}, \ldots, x_{r_{2k-1}}, x_{r_{2k}}$, find $x_{r_{2k+1}}, x_{r_{2k+2}} \notin \{x_{r_1}, x_{r_2}, \ldots, x_{r_{2k-1}}, x_{r_{2k}}\}$ such that $r_{2k} < r_{2k+1} < r_{2k+2}$, $x_{r_{2k+1}} \neq x_{r_{2k+2}}$ and

$$M\left(x_{r_{2k+1}}, x_{r_{2k+2}}, \frac{1}{2k+2}\right) > 1 - \frac{1}{2k+2}.$$

Clearly (x_{r_n}) defines a subsequence of distinct terms.

Choose $\epsilon \in (0, 1)$ and $t > 0$. Then for any $k \in \mathbb{N}$ satisfying $\frac{1}{k} < \min\{\epsilon, t\}$, we have

$$\begin{aligned} M\left(x_{r_{2k+1}}, x_{r_{2k+2}}, t\right) &\geq M\left(x_{r_{2k+1}}, x_{r_{2k+2}}, \frac{1}{k}\right) \\ &\geq M\left(x_{r_{2k+1}}, x_{r_{2k+2}}, \frac{1}{2k+2}\right) \\ &> 1 - \frac{1}{2k+2} \\ &> 1 - \epsilon. \end{aligned}$$

Consequently, (x_{r_n}) is a fuzzy pseudo-Cauchy sequence. So by hypothesis, (x_{r_n}), and hence (x_n), has a cluster point in X. Hence the result follows. $\square$

The following corollary is immediate from Proposition 3.2 and Proposition 4.6.

Corollary 4.8. *A Lebesgue metric space is weak G-complete.*

In view of Theorem 2.13, it is now clear that the class of Lebesgue fuzzy metric spaces $\mathcal{L}$ lies in-between the classes of compact fuzzy metric spaces $\mathcal{K}$ and weak G-complete fuzzy metric spaces $\mathcal{G}$. In what follows, we show that $\mathcal{K} \subsetneq \mathcal{L} \subsetneq \mathcal{G}$.

Example 4.9. (A weak G-complete, non-Lebesgue metric space) Let $X = \{n : n \in \mathbb{N}\} \cup \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$ and d be the usual metric on $\mathbb{R}$ restricted to $X \times X$.

The metric space (X, d) is not Lebesgue: Clearly, τ_d is the discrete topology on X. Thus $\{\{x\} : x \in X\}$ is an open cover of X without any Lebesgue number. Consequently (X, d) is not Lebesgue.

The metric space (X, d) is weak G-complete: It suffices to show that the only sequences which are G-Cauchy are those that contain a constant subsequence.

Assume on the contrary, there exists a G-Cauchy sequence (x_n) in X which does not have a constant subsequence. Then there exists a subsequence (x_{r_n}) of (x_n) having distinct terms such that $d(x_{r_{n+1}}, x_{r_n}) < \frac{1}{3}$, $\forall\ n \in \mathbb{N}$, a contradiction. Hence (X, d) is weak G-complete.

In view of the last example, the following observation is immediate from Proposition 3.2 and Proposition 4.6:

Observation 4.10. $\mathcal{L} \subsetneq \mathcal{G}$.

Example 4.11. (A non-compact, Lebesgue fuzzy metric space) Let $X = \left\{\frac{1}{2^n} : n \geq 2\right\} \cup \left[\frac{1}{2}, 1\right]$. It has been shown in [9] that the stationary fuzzy metric space $(X, M, \cdot)$, where $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$, $\forall\ x, y \in X$ and $t > 0$, is a non-compact, weak G-complete fuzzy metric space.

We now show that $(X, M, \cdot)$ is, *in fact*, Lebesgue. Choose a fuzzy pseudo-Cauchy sequence (x_n) of distinct terms in $(X, M, \cdot)$.

Clearly, (x_n) can not be eventually in $\left\{\frac{1}{2^n} : n \geq 2\right\}$. Otherwise, there exists $k \in \mathbb{N}$ such that $M(x_m, x_n, t) \leq \frac{1}{2}$, $\forall\ n \geq k, t > 0$, which is a contradiction since (x_n) is pseudo-Cauchy.

So there exists a subsequence (x_{r_n}) of (x_n) such that $x_{r_n} \in \left[\frac{1}{2}, 1\right], \forall\, n \in \mathbb{N}$.

Since τ_M defines the usual topology of $\mathbb{R}$ restricted to X [12], so $[\frac{1}{2}, 1]$ is a compact subset of $(X, M, \cdot)$. Consequently, in view of Theorem 2.13, (x_{r_n}) (and hence (x_n)) has a cluster point in X.

Thus $(X, M, \cdot)$ is Lebesgue.

Observation 4.12. $\mathcal{K} \subsetneq \mathcal{L}$.

The following result is an immediate consequence of Lemma 2.12:

Theorem 4.13. *A precompact, weak G-complete fuzzy metric space is Lebesgue.*

In view of the last theorem, we have the next corollary from Proposition 2.11, Proposition 4.6 and Proposition 3.2:

Corollary 4.14. *A precompact, weak G-complete metric space is Lebesgue.*

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