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K-orthonormal and K-Riesz Bases

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Abstract. Let K be a bounded operator. K-frames are ordinary frames for the range K. These frames are a generalization of ordinary frames and are certainly different from these frames. This research introduces a new concept of bases for the range K. Here we define the K-orthonormal basis and the K-Riesz basis, and then we describe their properties. As might be expected, the K-bases differ from the ordinary ones mentioned in this article.

1. Introduction

Frames and bases play a very important role in linear algebra and approximation theory. Many up-to-date technologies, such as radio communications, cell phones, wireless networks, image processing, signal processing do not work well without applying these concepts.

Gabor [11] formulated a new method for signal decomposition and signal expansion based on preliminary signals. According to this study, Duffin and Shaeffer [9] provided frames for Hilbert spaces to solve the non-harmonic series. Until 1980, the importance of frames was not known. In this year, Daubechies, Grossmann, and Meyer [6] frames were re-introduced, and after that, a lot of research was done on frames.

Today, many researchers are researching and defining many generalizations for frames that can be referred to as G-frames [17], Fusion frames [4], Weaving frames [3], Coherent frames [2], etc.

One of the new branches of frame theory that has come to the attention of researchers today is K-frames, first introduced by L. Găvruta [12]. Assume that $\mathcal{H}$ is a separable Hilbert space and K is a bounded operator on it, then K-frames are ordinary frames for the range of K.

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If $K = I$, where I is the identity operator on $\mathcal{H}$, then ordinary frames which become I-frame.

Since ordinary frames are a special case of K-frames, so they have differences with these frames, for example, the corresponding K-frame synthesis operator is not onto. The alternative dual frames for K-frames also are not interchangeable.

Many researchs have been done in this area, such as weaving K-frames [7], controlled K-frames [14], K-frames and their duals [1], operator perturbation of a K-frame [13]. So far no one has talked about the bases of range K. In this manuscript, we define K-orthonormal and K-Riesz bases and describe their properties and differences with ordinary ones.

This article contains four sections with outlines below; In Section 2, we review important results related to frames, K-frames, orthonormal and Riesz bases for Hilbert spaces. In Section 3, we study K-orthonormal basis. This section explains the relationship between the K-orthonormal basis and an ordinary orthonormal basis. We will provide the necessary and sufficient condition for the K-dual orthonormal basis to be the K^*-orthonormal basis. Finally, in Section 4, we define K-Riesz basis and discuss about its properties.

2. Notaion and Preliminaries

This section contains concepts that will be used in later sections. We begin with some notation.

Below are the notations used throughout the article.
$\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces $B(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. An operator T on $\mathcal{H}$ is an isometry operator whenever $T^*T = I$ and is called co-isometry whenever $TT^* = I$ where I is the identity on $\mathcal{H}$. The range and the kernel of an operator T are denoted by $R(T)$ and $N(T)$, respectively.

Theorem 2.1 ([5]). *Let $T : \mathcal{H} \to \mathcal{K}$ be a bounded operator such that $R(T)$ is closed. Then there exists a bounded operator $T^\dagger : \mathcal{K} \to \mathcal{H}$ such that for any $f \in R(T)$, $TT^\dagger f = f$.*

The operator $T^\dagger$ in the above theorem, is called the pseudo-inverse of T. The following theorem, which is one of the most useful results in K-frame theory is known as Douglas Theorem:

Theorem 2.2 ([8]). *Assume that $S, T \in B(\mathcal{H})$. The following statements are equivalent:*

(i) $R(T) \subseteq R(S)$;
(ii) *there exists a constant c such that $TT^* \leq c^2 SS^*$;*
(iii) $T = SC$ *for some* $C \in B(\mathcal{H})$.

Corollary 2.3 ([10]). *Assume that $S, T \in B(\mathcal{H})$. Then the following statements are equivalent*

(i) *$T = SC$ for som invertible operator C on $\mathcal{H}$,*
(ii) *$R(T) = R(S)$ and $N(T) = N(S)$.*

The sequence $\{Uu_n\}$ is called Riesz basis for $\mathcal{H}$, whenever $U \in B(\mathcal{H})$ is bijective and $\{u_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. We abbreviate orthonormal basis as ONB and " Riesz basis" as RB.

Definition 2.4. A sequence $\{h_j\}_{j=1}^{\infty}$ is called frame for $\mathcal{H}$ whenever there exist two constants $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \sum_{j=1}^{\infty} |\langle h, h_j \rangle|^2 \leq B\|h\|^2, \quad \forall h \in \mathcal{H}. \tag{2.1}$$

If only the right-hand inequality of (2.1) holds, $\{h_j\}_{j=1}^{\infty}$ is called a Bessel sequence. Now, two Bessel sequences $\{h_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ are called dual frames whenever the following equality holds

$$h = \sum_{i=1}^{\infty} \langle h, g_i \rangle \, h_i, \quad \forall h \in \mathcal{H}.$$

According to the definitions, both bases are frames and have unique dual. In addition dual necessarily has the properties of the primary bases.

2.1. K**-frames.** K-frames are an extension of the frames defined in Definition 2.4. In fact, a K-frame is an ordinary frame for $R(K)$. Here is a brief overview of the concept and theorems of K-frames and we refer to [1, 7, 13–16, 18] for more information.

Definition 2.5. Let $K \in B(\mathcal{H})$. A sequence $\{h_k\}_{k=1}^{\infty}$ is called a K-frame for $\mathcal{H}$ whenever for two constants $0 < A \leq B < \infty$ and for each $h \in \mathcal{H}$ the following inequality holds

$$A\|K^*h\|^2 \leq \sum_{k=1}^{\infty} |\langle h, h_k \rangle|^2 \leq B\|h\|^2. \tag{2.2}$$

Now if $\{h_k\}_{k=1}^{\infty}$ satisfies equation (2.3) for some $A > 0$, then it is called a tight K-frame.

$$A\|K^*h\|^2 = \sum_{k=1}^{\infty} |\langle h, h_k \rangle|^2, \quad \forall h \in \mathcal{H}. \tag{2.3}$$

If $A = 1$ in equality (2.3) then the sequence $\{h_k\}_{k=1}^{\infty}$ is called a Parseval K-frame for $\mathcal{H}$.

Theorem 2.6 ([12])**.** *Assume that $\{\delta_n\}_{n=1}^{\infty}$ is an ONB for l^2. The sequence $\{h_n\}_{n=1}^{\infty}$ is a K-frame iff there exists a bounded operator $L : l^2 \to \mathcal{H}$ such that $h_n = L\delta_n$ and $R(K) \subseteq R(L)$.*

Theorem 2.7 ([15])**.** *Assume that $K \in B(\mathcal{H})$ and $\{h_k\}_{k=1}^{\infty}$ is a K-frame. Let $T \in B(\mathcal{H})$ be a closed range operator such that $TK = KT$, then $\{Th_k\}_{k=1}^{\infty}$ is a K-frame.*

Some K-frames have the property that if any element is removed, they are not K-frames anymore. These K-frames are called exact K-frames. Also, a K-frame $\{h_i\}_{i=1}^{\infty}$ is called K-minimal frame, whenever for each $\{c_i\}_{i=1}^{\infty} \in l^2$ with $\sum_{i=1}^{\infty} c_i h_i = 0$, we have $c_i = 0$ for all $i \in \mathbb{N}$.

Proposition 2.8. *[1] Assume that $\{h_i\}_{i=1}^{\infty}$ is a K-frame. Then each of the following statements implies the other one.*

a) *the sequence $\{h_i\}_{i=1}^{\infty}$ has a unique K-dual,*
b) *the sequence $\{h_i\}_{i=1}^{\infty}$is a K-minimal frame.*

3. K-orthonormal Bases

Orthonormal bases are one of the applied topics in linear algebra and Hilbert spaces. With these bases, each member of a Hilbert space can be expanded in a unique way.

Here, we define K-ONB and relate it to ONB and frames of a Hilbert space.

Definition 3.1. A sequence $\{u_k\}_{k=1}^{\infty}$ of $\mathcal{H}$ is called a K-ONB whenever the following statements hold:

(i) the sequence $\{u_k\}_{k=1}^{\infty}$ is an orthonormal system,
(ii)
$$\sum_{n\in\mathbb{N}} |\langle h, u_n\rangle|^2 = \|K^* h\|^2, \quad \forall h \in \mathcal{H}.$$

Obviously, $\{u_k\}_{k=1}^{\infty}$ is an ONB for Hilbert space $\mathcal{H}$ if $K = I_{\mathcal{H}}$. So the ordinary ONB is a special case of K-ONB.

Example 3.2. Assume that $\{u_i\}_{i=1}^{\infty}$ is an ONB in l^2. Define the operator K on l^2 by $Ku_i = u_{i+1}$. Obviously the sequence $\{Ku_i\}_{i=1}^{\infty}$ is a K-ONB for l^2.

The question here is whether a K-ONB is related to an ONB of the Hilbert space. If K is an isometry, the answer is yes. The following theorem illustrates the relationship between these bases.

Theorem 3.3. *Let K be an isometry operator. The K-ONBs are precisely the families $\{Tu_i\}_{i\in\mathbb{N}}$ where $\{u_i\}_{i=1}^{\infty}$ is an arbitrary ONB for $\mathcal{H}$ and T is an isometry operator of the same range K on $\mathcal{H}$.*

Proof. Suppose that $\{u_i\}_{i=1}^{\infty}$ is an arbitrary ONB for $\mathcal{H}$ and T is an isometry operator on $\mathcal{H}$ with a range similar to range K. Since K and T are isometries then $N(T) = N(K) = \{0\}$ and

$$\begin{aligned}\langle Tu_i, Tu_j\rangle &= \langle u_i, u_j\rangle \\ &= \delta_{i,j}.\end{aligned}$$

On the other hand by Corollary 2.3, $T = KU$ for some invertible operator U and we have

$$\begin{aligned}\langle h, h\rangle &= \langle Th, Th\rangle \\ &= \langle KUh, KUh\rangle \\ &= \langle Uh, Uh\rangle .\end{aligned} \tag{3.1}$$

Because U is an invertible operator, U is a unitary operator by (3.1). Thus

$$\begin{aligned}\sum_{j=1}^{\infty} |\langle h, Tu_j\rangle|^2 &= \|T^*h\|^2 \\ &= \|U^*K^*h\|^2 \\ &= \|K^*h\|^2, \quad \forall h \in \mathcal{H}.\end{aligned}$$

Conversely, assume that $\{u_i\}_{i=1}^{\infty}$ is an ONB for $\mathcal{H}$ and $\{u'_i\}_{i=1}^{\infty}$ is a K-ONB. Set $d_i = K^*u'_i$. Since K is an isometry then K on $R(K)$ is unitary and we have

$$\begin{aligned}\langle d_i, d_j\rangle &= \left\langle K^*u'_i, K^*u'_j\right\rangle \\ &= \left\langle KK^*u'_i, u'_j\right\rangle \\ &= \delta_{i,j},\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\sum_{i=1}^{\infty} \left|\left\langle h, K^*u'_i\right\rangle\right|^2 &= \|K^*Kh\|^2 \\ &= \|h\|^2, \quad \forall h \in \mathcal{H}.\end{aligned} \tag{3.3}$$

Thus equations (3.2) and (3.3) imply that $\{d_i\}_{i\in\mathbb{N}}$ is an ONB. By Theorem 2.3.7 of [5] $\{d_i\} = \{Uu_i\}$, where U is a unitary operator on $\mathcal{H}$. Set $T = KU$. Since K is assumed isometry and U is unitary, T is an isometry operator and $R(T) = R(K)$. □

The following theorem states that under what conditions on an operator $T \in B(\mathcal{H})$, a K-orthonormal basis becomes a T-orthonormal basis.

Theorem 3.4. *Assume that $\{u_k\}_{k=1}^{\infty}$ is a K-ONB, and let $T \in B(\mathcal{H})$ be an isometry operator. Assume that K is a co-isometry operator, then $\{Tu_k\}_{k=1}^{\infty}$ is a T-ONB.*

Proof. Assuming T is an isometry operator on $\mathcal{H}$, we have:

$$\begin{aligned}\langle Tu_i, Tu_j\rangle &= \langle T^*Tu_i, u_j\rangle \\ &= \delta_{i,j}.\end{aligned}$$

Thus $\{Tu_k\}_{k\in\mathbb{N}}$ is an orthonormal system. Also, since K is a co-isometry operator then

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle h, Tu_i\rangle|^2 &= \sum_{i=1}^{\infty} |\langle T^*h, u_i\rangle|^2 \\ &= \|K^*T^*h\|^2 \\ &= \|T^*h\|^2, \quad \forall h \in \mathcal{H},\end{aligned}$$

which shows that $\{Tu_k\}_{k=1}^{\infty}$ is a T-ONB, as desired. □

Theorem 3.5. *Let $\{u_k\}_{k=1}^{\infty}$ be a K-ONB and $T \in B(\mathcal{H})$ such that $TK = KT$. Then, $\{Tu_k\}_{k=1}^{\infty}$ is a K-ONB if and only if T is unitary.*

Proof. Assume that $\{u_k\}_{k=1}^{\infty}$ is a K-ONB and $h \in \mathcal{H}$. These assumptions imply that

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle h, Tu_k\rangle|^2 &= \sum_{k=1}^{\infty} |\langle T^*h, u_k\rangle|^2 \\ &= \|K^*T^*h\|^2 \\ &= \|T^*K^*h\|^2 \\ &= \|K^*h\|^2.\end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned}\langle Tu_i, Tu_j\rangle &= \langle T^*Tu_i, u_j\rangle \\ &= \delta_{i,j}.\end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) are satisfied if and only if T is unitary.

□

According to K-ONB's definition, one will naturally find a difference between this basis and the ordinary ONB.

One of the most important ONB properties is this: if the Parseval equation holds for the normalized sequence $\{h_n\}_{n=1}^{\infty}$ then the sequence $\{h_n\}_{n=1}^{\infty}$ will be ONB [5]. But this is not true for the normalized K-sequence.

Example 3.6. Assume that $\mathcal{H}^n$ denotes a Hilbert space with $dim(\mathcal{H}^n) = n$, where $n = 2k$ for some $k \in \mathbb{N}$. Also let $\{u_i\}_{i=1}^{n}$ be an ONB for $\mathcal{H}^n$.

Define

$$Ku_1 = u_1, \quad Ku_2 = u_1, \quad \dots \quad Ku_{2k-1} = u_k, \quad Ku_{2k} = u_k.$$

Then the sequence $\{u_1, u_1, u_2, u_2, \dots, u_k, u_k\}$ is a normalized sequence which K-Parseval equality holds for it but it is not K-ONB.

A K-ONB with an ordinary ONB also has other differences that are related to their duals. In the follownig proposition, we state this difference.

Lemma 3.7. *Let $\{u_k\}_{k=1}^{\infty}$ be a K-ONB. Then $\{h_k\}_{k=1}^{\infty} = \{K^* u_k\}_{k=1}^{\infty}$ is a K-dual ONB.*

Proof. Let $h \in \mathcal{H}$ and $\{g_k\}_{k=1}^{\infty}$ be K-dual for $\{u_k\}_{k=1}^{\infty}$. According to the definition $\{u_k\}_{k=1}^{\infty}$ is a K-Parseval frame. By Theorem 3 of [12], there exists a sequence $\{g_k\}_{k=1}^{\infty}$, that satisfies the following equation.

$$Kh = \sum_{n=1}^{\infty} \langle h, g_k \rangle u_k.$$

Because $\{u_k\}_{k=1}^{\infty}$ is an orthonormal system we have

$$\begin{aligned}\langle Kh, u_j \rangle &= \left\langle \sum_{n=1}^{\infty} \langle h, g_k \rangle u_k, u_j \right\rangle \\ &= \langle h, g_j \rangle .\end{aligned}$$

By using Proposition 2.8 and the fact that $\{u_k\}_{k=1}^{\infty}$ is K-minimal we conclude that $g_j = K^* u_j$ is a K-dual for $\{u_k\}_{k=1}^{\infty}$. □

Proposition 3.8. *Suppose that $\{u_k\}_{k=1}^{\infty}$ is a K-ONB. The K-dual ONB $\{g_k\}_{k=1}^{\infty} = \{K^* u_k\}_{k=1}^{\infty}$ is a K^*-ONB if and only if K is co-isometry.*

Proof. Assume that $\{u_k\}_{k=1}^{\infty}$ is a K-ONB. By using Lemma 3.7, $\{g_k\}_{k=1}^{\infty} = \{K^* u_k\}_{k=1}^{\infty}$ is a K-dual ONB for $\{u_k\}_{k=1}^{\infty}$.

$$\begin{aligned}\langle g_k, g_j \rangle &= \langle K^* u_k, K^* u_j \rangle \\ &= \langle KK^* u_k, u_j \rangle \\ &= \langle u_k, u_j \rangle .\end{aligned} \tag{3.6}$$

Also, for all $h \in \mathcal{H}$

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle h, g_i \rangle|^2 &= \sum_{i=1}^{\infty} |\langle Kh, u_i \rangle|^2 \\ &= \|K^* Kh\|^2 \\ &= \|Kh\|^2.\end{aligned} \tag{3.7}$$

Equations (3.6) and (3.7) hold if and only if K is co-isometry. □

Proposition 3.9. *Assume that $\{\delta_k\}_{k=1}^{\infty}$ is the canonical ONB for l^2 and K is an isometry. The sequence $\{u_k\}_{k=1}^{\infty}$ is a K-ONB if and only if there exists an isometry operator $D : l^2 \to \mathcal{H}$ such that $\{u_k\}_{k=1}^{\infty} = \{D\delta_k\}_{k=1}^{\infty}$ and for all $h \in \mathcal{H}$, $\|K^*h\| = \|D^*h\|$.*

Proof. First, let $\{u_k\}_{k=1}^{\infty}$ be a K-ONB, then

$$\sum_{k=1}^{\infty} |\langle h, u_k\rangle|^2 = \|K^*h\|^2, \quad \forall h \in \mathcal{H}.$$

Define the operator $L : \mathcal{H} \to l^2$ by

$$Lu_k = \delta_k, \quad \forall k \in \mathbb{N}.$$

Via the definition of L we have

$$Lx = \sum_{k=1}^{\infty} \langle Lx, u_k\rangle \, \delta_k,$$

and

$$\begin{aligned} \langle L^*\delta_k, h\rangle &= \langle \delta_k, Lh\rangle \\ &= \langle u_k, h\rangle . \end{aligned}$$

Considering $D = L^*$, the proof is completed. So, for any $h \in \mathcal{H}$ we have $\|K^*h\|^2 = \|D^*h\|^2$.

Conversely, let $D : l^2 \to \mathcal{H}$ be defined by $\{u_k\}_{k=1}^{\infty} = \{D\delta_k\}_{k=1}^{\infty}$, which is an isometry operator and for any $h \in \mathcal{H}$, $\|K^*h\|^2 = \|D^*h\|^2$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle h, u_k\rangle|^2 &= \sum_{k=1}^{\infty} |\langle h, D\delta_k\rangle|^2 \\ &= \sum_{k=1}^{\infty} |\langle D^*h, \delta_k\rangle|^2 \\ &= \|D^*h\|^2 \\ &= \|K^*h\|^2, \end{aligned}$$

this means that $\{u_k\}_{k=1}^{\infty} = \{D\delta_k\}_{k=1}^{\infty}$ satisfies K-Parseval equation. On the other hand,

$$\begin{aligned} \langle u_n, u_m\rangle &= \langle D\delta_n, D\delta_m\rangle \\ &= \langle D^*D\delta_n, \delta_m\rangle \\ &= \delta_{n,m}. \end{aligned}$$

Therefore, $\{u_k\}_{k=1}^{\infty} = \{D\delta_k\}_{k=1}^{\infty}$ is a K-ONB, as desired. □

We know that every frames are preciesly $\{Uu_k\}_{k=1}^{\infty}$, where U is a bounded surjective operator on $\mathcal{H}$ and $\{u_k\}_{k=1}^{\infty}$ is an ONB for $\mathcal{H}$ [5]. But, this does not hold for K-frames. In the following theorem, an interesting relationship is presented between the K-frames and the K-orthonormal bases.

Theorem 3.10. *Let $\{u_k\}_{k=1}^{\infty}$ be a K-ONB and let $T : \mathcal{H} \to \mathcal{H}$ be a closed range operator which commutes with operator K, i.e. $TK = KT$ and $R(K) \subset R(T)$. Then $\{Tu_k\}_{k=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Moreover, if $\{h_k\}_{k=1}^{\infty}$ is a K-frame for $\mathcal{H}$ then there exists an operator $T : \mathcal{H} \to \mathcal{H}$ such that $R(T)$ is closed, $R(K) \subset R(T)$ and $\{h_k\}_{k\in\mathbb{N}} = \{Tu_k\}_{k=1}^{\infty}$.*

Proof. Let $\{h_k\}_{k=1}^{\infty} = \{Tu_k\}_{k=1}^{\infty}$. By the assumptions we have

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle h, h_k\rangle|^2 &= \sum_{k=1}^{\infty} |\langle T^*h, u_k\rangle|^2 \\ &= \|K^*T^*h\|^2 \\ &\leq \|K^*\|^2\|T^*\|^2\|h\|^2. \end{aligned} \tag{3.8}$$

Also, for $h \in \mathcal{H}$

$$\begin{aligned} \|K^*h\|^2 &= \|(T^\dagger)^*T^*K^*h\|^2 \\ &\leq \|(T^\dagger)^*\|^2\|K^*T^*h\|^2 \\ &= \sum_{k=1}^{\infty} |\langle h, h_k\rangle|^2 . \end{aligned} \tag{3.9}$$

Therefore

$$\sum_{k=1}^{\infty} |\langle h, h_k\rangle|^2 \geq \|(T^\dagger)^*\|^{-2}\|K^*h\|^2.$$

Equations (3.8) and (3.9) imply that $\{h_k\}_{k=1}^{\infty} = \{Tu_k\}_{k=1}^{\infty}$ is a K-frame.

For the second part, suppose that $\{h_k\}_{k=1}^{\infty}$ is a K-frame. By Theorem 2.6, there exists a closed range operator $D : l^2 \to \mathcal{H}$ such that $R(K) \subset R(D)$ and $D\delta_k = h_k$. Also, by using Proposition 3.9, there exists an isometry operator $L : l^2 \to \mathcal{H}$ such that $\|K^*h\| = \|D^*h\|$ *for all* $h \in \mathcal{H}$ and $L\delta_k = u_k$. So, $h_k = DL^{-1}u_k$. Set $T = DL^{-1}$. Since $R(K) \subset R(D)$ and D is a closed range, so T has the same properties. □

4. K-Riesz Bases

The Riesz bases are obtained from the operation of a bounded bijective operator on an ONB. They have the same frame conditions, in addition, they are linearly independent.

In this section, we define such a base for the range K. These bases are not necessarily K-frames. Next, we consider the conditions where a K-RB is a K-frame.

Definition 4.1. A sequence $\{h_k\}_{k=1}^{\infty} = \{Vu_k\}_{k=1}^{\infty}$ is called K-RB, where V is an injective operator on $\mathcal{H}$ such that $V : R(K) \to R(K)$ is a bounded bijective operator and $\{u_k\}_{k=1}^{\infty}$ is a K- ONB.

According to the above definition, it is easy to see that if $K = I_{\mathcal{H}}$ then K-RB is the same as ordinary RB.

Example 4.2. Let $\{u_i\}_{i=1}^{\infty}$ be an ONB in l^2. Consider the operator K on l^2 defined as:

$$Ku_i = \begin{cases} u_{i+1}, & i \text{ is odd}, \\ 2u_{i+1}, & i \text{ is even}. \end{cases}$$

It is easy to see that $\{Ku_i\}_{i=1}^{\infty}$ is a K-RB for l^2.

Another difference between frames and ordinary K-frames is that: not every K-RB is necessarily a K-frame. Under the following conditions, K-RB is also a K-frame.

Proposition 4.3. *Let $\{h_k\}_{k=1}^{\infty} = \{Vu_k\}_{k=1}^{\infty}$ be a K-RB, as stated in Definition 4.1 and $KV = VK$. Then $\{h_k\}$ is a K-frame with the frame bounds $\|K\|^2\|V\|^2$ and $\|V^{-1}\|^{-2}$.*

Proof. Let $h \in \mathcal{H}$. Since $\{h_k\}_{k=1}^{\infty}$ is a K-RB, so we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle h, h_k\rangle|^2 &= \sum_{i=1}^{\infty} |\langle h, Vu_k\rangle|^2 \\ &= \sum_{i=1}^{\infty} |\langle V^*h, u_k\rangle|^2 \\ &= \|K^*V^*h\|^2 \\ &= \|V^*K^*h\|^2 \\ &\leq \|V^*\|^2\|K^*\|^2\|h\|^2.\end{aligned}$$

To obtain lower bound we use the invertibility of V on $R(K)$

$$\begin{aligned}\|K^*h\|^2 &= \|(V^{-1})^*V^*K^*h\|^2 \\ &\leq \|(V^{-1})^*\|^2\|V^*K^*h\|^2 \\ &= \|(V^{-1})^*\|^2 \sum_{i=1}^{\infty} |\langle V^*h, u_k\rangle|^2 .\end{aligned}$$

Therefore,

$$\|(V^{-1})^*\|^{-2}\|K^*h\|^2 \leq \sum_{i=1}^{\infty} |\langle h, h_k\rangle|^2 \leq \|V^*\|^2\|K^*\|^2\|h\|^2.$$

This shows that $\{h_k\}_{k=1}^{\infty} = \{Vu_k\}_{k=1}^{\infty}$ is a K-frame. □

The similarity of K-RB to ordinary RB is that they are both minimal. In other words, a K-RB is exactly a K-minimal frame.

Proposition 4.4. *Let $\{h_i\}_{i=1}^{\infty}$ be K-frame. Then each of the following statements implies the other one*

a) *the sequence $\{h_i\}_{i=1}^{\infty}$ is K-RB,*
b) *the sequence $\{h_i\}_{i=1}^{\infty}$ is K-minimal.*

Proof. Let $\{h_k\}_{i=1}^{\infty}$ be a K-minimal frame. By Proposition 3.9 there exists a bounded closed range operator $L : l^2 \to \mathcal{H}$ such that $R(K) \subset R(L)$ and for any i, $L\delta_i = h_i$. Furthermore, by Proposition 3.9 there exists an isometry operator D on $\mathcal{H}$ such that $D\delta_k = u_k$. Now, set $Vu_k = LD^{-1}u_k = h_k$. Since $\{h_k\}$ is K -minimal, so

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} c_i h_i \\ &= \sum_{i=1}^{\infty} c_i V u_i \\ &= V\left(\sum_{i=1}^{\infty} c_i u_i\right). \end{aligned}$$

This implies that V is a bounded injective operator such that V is bounded bijective on $R(K)$.

Conversely, let $\{h_i\}_{i=1}^{\infty} = \{Vu_i\}_{i=1}^{\infty}$ be a K-RB, where $V : R(K) \to R(K)$ is a bounded bijective operator and $\{u_i\}_{i=1}^{\infty}$ is a K-ONB. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence such that

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} c_i h_i \\ &= \sum_{i=1}^{\infty} c_i V u_i \\ &= V\left(\sum_{i=1}^{\infty} c_i u_i\right). \end{aligned}$$

Since, V is injective then for any $i \in \mathbb{N}$, $c_i = 0$. Therefore, $\{h_i\}_{i=1}^{\infty}$ is a K-minimal frame. □

Corollary 4.5. *Let $\{h_j\}_{j=1}^{\infty} = \{Vu_j\}_{j=1}^{\infty}$ be an ordinary RB, where V is a bounded bijective operator on $\mathcal{H}$ and $\{u_j\}_{j=1}^{\infty}$ is an ONB for $\mathcal{H}$. If K is an isometry operator such that $VK = KV$ then $\{Kh_j\}_{j=1}^{\infty}$ is K-RB for $\mathcal{H}$.*

Proof. The assumptions imply that

$$\begin{aligned}\{Kh_j\}_{j=1}^{\infty} &= \{KVu_j\}_{j=1}^{\infty}\\ &= \{VKu_j\}_{j=1}^{\infty}.\end{aligned}$$

By Theorem 3.3, $\{Ku_j\}_{j\in\mathbb{N}}$ is a K-ONB. Thus $\{Kh_j\}_{j=1}^{\infty}$ is K-RB. □

Example 4.6. Let $\mathcal{H} = \mathbb{C}^3$ and $\{u_1, u_2, u_3\}$ be canonical basis for $\mathcal{H}$. Now define $K \in L(\mathcal{H})$ as follows

$$K : \mathcal{H} \to \mathcal{H}, \quad Ku_1 = u_1, \quad Ku_2 = u_1, \quad Ku_3 = u_2.$$

Obviously, K is not an isometry also $\{Ku_j\}_{j=1}^{\infty}$ is not K-RB.

The next proposition gives a condition in which a K-RB has K-dual. By Proposition 4.4 and Proposition 2.8 the K-dual RB is unique. But, the K-Riesz dual dose not have the same property.

Proposition 4.7. *Assume that $\{h_j\}_{j=1}^{\infty} = \{Vu_j\}_{j=1}^{\infty}$ is a K-RB such that $VK = KV$. Then $\{g_j\}_{j=1}^{\infty} = \left\{K^*(V^{-1})^*u_j\right\}_{j=1}^{\infty}$ is a K-dual frame for $\{h_j\}_{j\in\mathbb{N}} = \{Vu_j\}_{j=1}^{\infty}$. The sequence $\{g_j\}_{j=1}^{\infty}$ is a K^*-RB whenever K is injective.*

Proof. By Theorem 2.6, $\{h_j\}_{j=1}^{\infty}$ is a K-frame. Thus there exists a Bessel sequence $\{g_j\}_{j=1}^{\infty}$ such that

$$Kh = \sum_{j=1}^{\infty} \langle h, g_j\rangle h_j, \quad \forall h \in \mathcal{H}.$$

On the other hand, since V is an invertible operator on $R(K)$, so

$$\begin{aligned}Kh &= VV^{-1}Kh\\ &= VKV^{-1}h\\ &= V\sum_{j=1}^{\infty}\left\langle V^{-1}h, K^*u_j\right\rangle u_j\\ &= \sum_{j=1}^{\infty}\left\langle h, K^*(V^{-1})^*u_j\right\rangle h_j.\end{aligned}$$

Assuming $\{h_j\}_{j=1}^{\infty}$ is a K-RB we can conclude that it is K-minimal, so the K-dual is unique. Therefore $g_j = K^*(V^{-1})^*u_j$. □

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