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ABSTRACT. In this article, we will study the existence and uniqueness of optimal common fixed points for self-mappings in metric spaces with w-distance. We obtain generalizations of the Kocev and Rakočević fixed point theorems. The obtained results do not require the continuity or the condition $(C;k)$ of maps, but require the weaker condition (W). We also improve some of our results when the metric space is equipped with a w_0-distance. In this way, we get new existence results for non-cyclic quasi-contraction mappings of the Fisher type.

1. Introduction

In 1980, Fisher proved the following interesting result.

Theorem 1.1 ([4]). *Let $f, g : X \to X$ be two continuous mappings on a complete metric space (X, d) satisfying the inequality*

$$d\left(f^l x, g^q y\right) \leq \lambda \max\{d\left(f^r x, g^s y\right) : 0 \leq r \leq l, 0 \leq s \leq q\}, x, y \in X,$$

for some fixed $l, q \in \mathbb{N}$ and $\lambda \in [0, 1)$. Then f and g have a unique fixed point in X.

In 1996, Kada et al. [8] introduced and studied the concept of w-distance in fixed point theory. They gave examples of the w-distance and obtained some fixed point results in the framework of w-distance. See [1, 3, 5–7, 9, 12] and references therein for more recent information.

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Definition 1.2. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, +\infty)$ is called a w-distance on X if it satisfied the following conditions:

(i) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,
(ii) for any $x \in X$, $p(x, .) : X \to [0, +\infty)$ is lower semi-continuous,
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.

Definition 1.3. Let (X, d) be a metric space. A w-distance function p is called a w_0-distance on X if, additionally, for any $x \in X$, $p(., x) : X \to [0, +\infty)$ is lower semi-continuous.

Let us recall that a real-valued function f defined on a metric space X is said to be lower semi-continuous at a point $x_0 \in X$ if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$. It is clear that every symmetric w-distance is a w_0-distance.

Example 1.4 ([8])**.** Let (X, d) be a metric space and let $p : X \times X \to [0, +\infty)$ be a function. The following functions are examples of X:

(1) $p(x, y) = d(x, y)$;
(2) if X be a normed space, then $p(x, y) = \|x\| + \|y\|$;
(3) if X be a normed space, then $p(x, y) = \|y\|$.

The w-distances defined in each of these examples are, in fact, examples of the w_0-distance immediately. In [11, Example 1.10] Kostić et al. gave an example of which is not a lower semi-continuous function of the first variable. The following very useful lemma has been proved in [8].

Lemma 1.5. *Let (X, d) be a metric space and p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and suppose $x, y, z \in X$. Then the following hold*

(i) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(ii) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then y_n converges to z;*
(iii) *If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(iv) *If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Kocev and Rakočević [10] introduced the condition $(C; 1)$ and Cirić [2], extended it to the condition $(C; k)$. Then, they generalized Fisher's result for pairs of mappings on metric space to w-distance on complete metric space. Their results do not require the continuity of maps but require the condition $(C;k)$. The self map $T : X \to X$ on a metric space (X,d) obeys the condition $(C;k)$ if there is a constant $k \geq 0$ such that for every sequence $\{x_n\} \subseteq X$,

$$x_n \to x_0 \quad \Rightarrow \quad d(x_0, Tx_0) \leq k \limsup_{n\in\mathbb{N}} d(x_n, Tx_n).$$

For the convenience of readers, we recall the following main results of Kocev and Rakoćević [10, Theorems 2.3 and 2.4].

Theorem 1.6. *Let T and S be mappings of a complete metric space (X,d) into itself and let p be a w-distance. If T and S obey the condition $(C;k)$, and if for some fixed positive integers l and q and some $\lambda \in [0,1)$, the inequality*

$$\max\left\{p\left(T^l x, S^q y\right), p\left(S^q y, T^l x\right)\right\} \leq \lambda \max\{p(T^s x, S^r y), p(S^r y, T^s x) \\ : 0 \leq s \leq l, 0 \leq r \leq q\},$$

holds for every $x, y \in X$. Then S and T have a unique common fixed point $z^* \in X$. Moreover, $p(z^*, z^*) = 0$.

Theorem 1.7. *Let T and S be mappings of a complete metric space (X,d) into itself, assume that S obeys the condition $(C;k)$ and let p be a w-distance. If for some fixed positive integer q and some $\lambda \in [0,1)$*

$$\max\{p(Tx, S^q y), p(S^q y, Tx)\} \leq \lambda \max\{p(T^s x, S^r y), p(S^r y, T^s x) \\ : 0 \leq s \leq 1, 0 \leq r \leq q\},$$

holds for every $x, y \in X$. Then S and T have a unique common fixed point $z^* \in X$. Moreover, $p(z^*, z^*) = 0$.

Let A and B be nonempty subsets of the metric space (X,d) and p be a w-distance on X. In this paper, in the setting of a complete metric space with w-distance, we establish the existence of an optimal common fixed point for the self maps $T : A \to A$ and $S : B \to B$ hold in a Fisher-type inequality. The obtained results do not require continuity or the condition $(C;k)$ of maps, but rather the weaker condition (W). We also improve some of our results when the metric space is equipped with a w_0-distance. As a result, we establish a fixed point theorem for non-cyclic quasi-contraction mappings of Fisher type in the framework of w-distance. Some of the results in [10, 13] are extended and improved in our study.

2. Main Results

Let C and D be nonempty subsets of the metric space (X,d) and p be a w-distance on X. Let $\delta^*[C,D] := \sup\{p^*(x,y) : x \in C, y \in D\}$ that $p^*(x,y) := \max\{p(x,y), p(y,x)\}$. It is easily that p^* holds in the triangular inequality and $p^*(x,y) = 0$ requires that $x = y$.

Definition 2.1. Let A and B be nonempty subsets of the metric space (X, d) and p be a w-distance on X. Let $T : A \to A$ and $S : B \to B$ be two self-mappings. We say that the self-mappings T and S hold in a Fisher-type inequality if for some $l, q \in \mathbb{N}$ and some $\lambda \in [0, 1)$ the inequality

$$p^*\left(T^l x, S^q y\right) \leq \lambda \delta^*\left[A_l^x, B_q^y\right], \tag{2.1}$$

is satisfied for all $x \in A$ and $y \in B$ where $A_n^x := \left\{x, Tx, T^2x, \ldots, T^n x\right\}$ and $B_n^y := \left\{y, Sy, S^2y, \ldots, S^n y\right\}$ for every $n \in \mathbb{N}$.

Definition 2.2. Let A and B be nonempty subsets of the metric space (X, d) and p be a w-distance on X. We say that $z^* \in A \cap B$ is an optimal common fixed point of self-mappings $T : A \to A$ and $S : B \to B$ provided that

$$Tz^* = z^*, \qquad Sz^* = z^* \qquad \text{and} \qquad p(z^*, z^*) = 0.$$

We begin with the following lemma which will be used later.

Lemma 2.3. *Let A and B be nonempty subsets of the metric space (X, d) and p be a w-distance on X. Let self-mappings $T : A \to A$ and $S : B \to B$ hold in a Fisher-type inequality. For $x_0 \in A$ and $y_0 \in B$, define $x_{n+1} := Tx_n$ and $y_{n+1} := Sy_n$ for each $n \geq 0$. Then for every $n, m \in \mathbb{N}$ with $m \geq n$ there exists $M_{x_0,y_0} \in \mathbb{R}^+$ such that*

$$p^*(x_n, y_m) \leq \lambda^{k_n} M_{x_0,y_0},$$

where $k_n = \min\left\{\left\lfloor \frac{n}{l} \right\rfloor, \left\lfloor \frac{n}{q} \right\rfloor\right\}$.

Proof. We first show that for each $n, m \in \mathbb{N}$, we have

$$\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = p^*\left(T^k x_0, S^{k'} y_0\right), \quad \text{where} \quad k < l \quad \text{or} \quad k' < q. \tag{2.2}$$

Suppose that $\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = p^*\left(T^i x_0, S^j y_0\right)$, where $l \leq i \leq n$ and $q \leq j \leq m$. From (2.1), we have

$$\begin{aligned} p^*\left(T^i x_0, S^j y_0\right) &= p^*\left(T^l T^{i-l} x_0, S^q S^{j-q} y_0\right) \\ &\leq \lambda \delta^*\left[A_l^{x_{i-l}}, B_q^{y_{j-q}}\right] \\ &\leq \lambda \delta^*\left[A_n^{x_0}, B_m^{y_0}\right]. \end{aligned} \tag{2.3}$$

Thus, we obtain $\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = 0$, then

$$\begin{aligned} 0 &\leq p^*(x_0, y_0) \\ &\leq \sup\left\{p^*(x, y) : x \in A_n^{x_0}, y \in B_m^{y_0}\right\} \\ &= \delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = 0. \end{aligned}$$

Hence $\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = p^*(x_0, y_0)$, so (2.2) holds.

Now, we show that for each $m, n \in \mathbb{N}$

$$\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] \leq M_{x_0,y_0}, \tag{2.4}$$

where

$$M_{x_0,y_0} = \frac{1}{1-\lambda} \max\left\{p^*\left(T^i x_0, S^j y_0\right), p\left(T^i x_0, T^j x_0\right), p\left(S^i y_0, S^j y_0\right) : 0 \leq i, j \leq \max\{l, q\}\right\}.$$

To prove the claim, note that from (2.2), we have $\delta^*\left[A_n^{x_0}, B_m^{y_0}\right] = p^*\left(T^k x_0, S^{k'} y_0\right)$, where $0 \leq k < l$ or $0 \leq k' < q$. If $k < l$ and $k' < q$, then (2.4) trivially holds. So, without loss of generality, we may assume that $0 \leq k < l$ and $q \leq k' \leq m$. Then, from (2.3) we get

$$\begin{aligned} \delta^*\left[A_n^{x_0}, B_m^{y_0}\right] &= p^*\left(T^k x_0, S^{k'} y_0\right) \\ &\leq p^*\left(T^k x_0, T^l x_0\right) + p^*\left(T^l x_0, S^{k'} y_0\right) \\ &\leq p^*\left(T^k x_0, T^l x_0\right) + \lambda \delta^*\left[A_n^{x_0}, B_m^{y_0}\right], \end{aligned}$$

so (2.4) holds.

Now, we prove that for each $r, s, m, n \in \mathbb{N} \cup \{0\}$ with $n, m \geq \max\{l, q\}$, we have

$$\delta^*\left[A_r^{x_n}, B_s^{y_m}\right] \leq \lambda \delta^*\left[A_{r+l}^{x_{n-l}}, B_{s+q}^{y_{m-q}}\right]. \tag{2.5}$$

From (2.3), for some $0 \leq r' \leq r$ and $0 \leq s' \leq s$, we have

$$\begin{aligned} \delta^*\left[A_r^{x_n}, B_s^{y_m}\right] &= p^*\left(T^{r'} x_n, S^{s'} y_m\right) \\ &= p^*\left(T^{l+r'} x_{n-l}, S^{q+s'} y_{m-q}\right) \\ &= p^*\left(T^l x_{n-l+r'}, S^q y_{m-q+s'}\right) \\ &\leq \lambda \delta^*\left[A_l^{x_{n-l+r'}}, B_q^{y_{m-q+s'}}\right] \\ &\leq \lambda \delta^*\left[A_{r'+l}^{x_{n-l}}, B_{s'+q}^{y_{m-q}}\right] \\ &\leq \lambda \delta^*\left[A_{r+l}^{x_{n-l}}, B_{s+q}^{y_{m-q}}\right]. \end{aligned}$$

Hence, (2.5) holds. Then, from (2.5) for $n, m \geq \max\{2l, 2q\}$, we have

$$\begin{aligned} p^*(x_n, y_m) &= \delta^*\left[A_0^{x_n}, B_0^{y_m}\right] \\ &\leq \lambda \delta^*\left[A_l^{x_{n-l}}, B_q^{y_{m-q}}\right] \\ &\leq \lambda^2 \delta^*\left[A_{2l}^{x_{n-2l}}, B_{2q}^{y_{m-2q}}\right]. \end{aligned}$$

By continuing this process and using (2.4), we obtain

$$\begin{aligned} p^*(x_n, y_m) &\leq \lambda^{k_{n,m}} \delta^* \left[A_{k_{n,m}l}^{x_{n-k_{n,m}l}}, B_{k_{n,m}q}^{y_{m-k_{n,m}q}}\right] \\ &\leq \lambda^{k_{n,m}} \delta^* \left[A_n^{x_0}, B_m^{y_0}\right] \\ &\leq \lambda^{k_{n,m}} M_{x_0,y_0}, \end{aligned}$$

where $k_{n,m} = \min\left\{\left\lfloor \frac{n}{l} \right\rfloor, \left\lfloor \frac{m}{q} \right\rfloor\right\}$. So when $m \geq n$ we can get

$$p^*(x_n, y_m) \leq \lambda^{k_n} M_{x_0,y_0},$$

where $k_n = \min\left\{\left\lfloor \frac{n}{l} \right\rfloor, \left\lfloor \frac{n}{q} \right\rfloor\right\}$. □

From Lemma 2.3, we get the following result immediately.

Corollary 2.4. *Let A and B be nonempty subsets of the metric space (X, d) and p be a w-distance on X. Let self-mappings $T : A \to A$ and $S : B \to B$ hold in a Fisher-type inequality. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then $\{x_n\}$ is Cauchy.*

Proof. Fix $y_0 \in B$ and define $y_{n+1} = Sy_n$ for each $n \geq 0$. For any $n, m \in \mathbb{N}$ with $m \geq n$, from Lemma 2.3 we can deduce

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, y_m) + p(y_m, x_m) \\ &\leq p^*(x_n, y_m) + p^*(x_m, y_m) \\ &\leq \left(\lambda^{k_n} + \lambda^{k_m}\right) M_{x_0,y_0} \\ &\leq 2\lambda^{k_n} M_{x_0,y_0}, \end{aligned} \tag{2.6}$$

so from Lemma 1.5 (iii), we get $\{x_n\}$ is Cauchy. □

Definition 2.5. Let A and B be nonempty and complete subsets of the metric space (X, d) and p be a w-distance on X. Let self-mappings $T : A \to A$ and $S : B \to B$ hold in a Fisher-type inequality. We say that T obeys the condition (W) if

$$(W) \quad for\ all\ z \in A\ with\ Tz \neq z\ we\ have\ \max\{\alpha(z), \beta(z)\} > 0,$$

that

$$\alpha(z) = \inf\left\{p(y, z) + p(y, y) + \sum_{i=1}^{q} p\left(y, S^i y\right) : y \in B\right\}$$

and

$$\beta(z) = \inf\left\{p(x, z) + p(x, x) + \sum_{i=1}^{l} p\left(x, T^i x\right) : x \in A\right\}.$$

Also, we say that T obeys the condition (W^*) if

(W^*) *for all* $z \in A$ *with* $Tz \neq z$ *we have* $\max\{\alpha(z), \beta(z)\} > 0$,

that

$$\alpha(z) = \inf\left\{p^*(y,z) + p^*(y,y) + \sum_{i=1}^{q} p^*\left(y, S^i y\right) : y \in B\right\}$$

and

$$\beta(z) = \inf\left\{p^*(x,z) + p^*(x,x) + \sum_{i=1}^{l} p^*\left(x, T^i x\right) : x \in A\right\}.$$

Now, we are ready to prove our main result in this paper.

Theorem 2.6. *Let A and B be nonempty and complete subsets of the metric space (X,d) and p be a w-distance on X. Let self-mappings $T : A \to A$ and $S : B \to B$ hold in a Fisher-type inequality and T obeys the condition (W). Then T has a unique fixed point $z^* \in A \cap B$ such that the sequences $\{T^n x_0\}$ and $\{S^n y_0\}$ converge to z^* for every $x_0 \in A$ and $y_0 \in B$.*

Proof. Fix $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. From Corollary 2.4, $\{x_n\}$ is a Cauchy sequence and by the completeness of A, $\{x_n\}$ converges to some $z^* \in A$. Fix $y_0 \in B$, define $y_{n+1} = Sy_n$ for each $n \geq 0$. From (2.6), we have

$$p(x_n, z^*) \leq \liminf_m p(x_n, x_m) \leq 2\lambda^{k_n} M_{x_0,y_0}, \tag{2.7}$$

also

$$p(x_n, y_n) \leq \lambda^{k_n} M_{x_0,y_0},$$

so from Lemma 1.5 (ii) we get $\{y_n\}$ converges to z^*. Therefore z^* is unique and because B is complete, thus $z^* \in B$.

Suppose that $Tz^* \neq z^*$. So $\max\{\alpha(z^*), \beta(z^*)\} > 0$. Now, if $\alpha(z^*) > 0$, from Lemma 2.3 and relation (2.7), we have

$$\begin{aligned} p(y_n, z^*) &\leq p(y_n, x_n) + p(x_n, z^*) \\ &\leq p^*(x_n, y_n) + p(x_n, z^*) \\ &\leq 3\lambda^{k_n} M_{x_0,y_0}. \end{aligned}$$

Also, for any $n, m \in \mathbb{N}$ with $m \geq n$, from Lemma 2.3 we can deduce

$$\begin{aligned} p(y_n, y_m) &\leq p(y_n, x_n) + p(x_n, y_m) \\ &\leq p^*(x_n, y_n) + p^*(x_n, y_m) \\ &\leq 2\lambda^{k_n} M_{x_0,y_0}. \end{aligned}$$

Then we get

$$\begin{aligned}
0 &< \alpha(z^*) \\
&= \inf\left\{p(y,z^*)+p(y,y)+\sum_{i=1}^{q}p\left(y,S^iy\right):y\in B\right\} \\
&\le \inf\left\{p(y_n,z^*)+p(y_n,y_n)+\sum_{i=1}^{q}p\left(y_n,S^iy_n\right):n\in\mathbb{N}\right\} \\
&= \inf\left\{p(y_n,z^*)+p(y_n,y_n)+\sum_{i=1}^{q}p(y_n,y_{n+i}):n\in\mathbb{N}\right\} \\
&\le (2q+5)M_{x_0,y_0}\inf\left\{\lambda^{k_n}:n\in\mathbb{N}\right\} \\
&= 0,
\end{aligned}$$

leads to a contradiction. And if $\beta(z^*)>0$, then from relations (2.6) and (2.7), we have

$$\begin{aligned}
0 &< \beta(z^*) \\
&= \inf\left\{p(x,z^*)+p(x,x)+\sum_{i=1}^{l}p\left(x,T^ix\right):x\in A\right\} \\
&\le \inf\left\{p(x_n,z^*)+p(x_n,x_n)+\sum_{i=1}^{i}p\left(x_n,T^ix_n\right):n\in\mathbb{N}\right\} \\
&= \inf\left\{p(x_n,z^*)+p(x_n,x_n)+\sum_{i=1}^{i}p(x_n,x_{n+i}):n\in\mathbb{N}\right\} \\
&\le 2(l+2)M_{x_0,y_0}\inf\left\{\lambda^{k_n}:n\in\mathbb{N}\right\} \\
&= 0,
\end{aligned}$$

leads to a contradiction. Hence $Tz^*=z^*$. □

Theorem 2.7. *Let A and B be nonempty and complete subsets of the metric space (X,d) and p be a w-distance on X. Assume that the self-mappings $T:A\to A$ and $S:B\to B$ hold in a Fisher-type inequality such that T and S obey the condition (W). Then T and S have a unique optimal common fixed point $z^*\in A\cap B$.*

Proof. From Theorem 2.6, there exists a unique $z^*\in A\cap B$ such that $Tz^*=Sz^*=z^*$. Moreover,

$$\begin{aligned}
p(z^*,z^*) &= p\left(T^lz^*,S^qz^*\right) \\
&\le \lambda\delta^*\left[A_l^{z^*},B_q^{z^*}\right]
\end{aligned}$$

$$= \lambda p^*(z^*, z^*)$$
$$= \lambda p(z^*, z^*),$$

therefore, $p(z^*, z^*) = 0$. □

Remark 2.8. In the same way, we can prove that the statement of Theorems 2.6 and 2.7 remains valid if we replace 'p be a w-distance' with 'p be a w_0-distance' and 'T and S obey the condition (W)' with 'T and S obey the condition (W^*)'.

The following example illustrates Theorems 2.6 and 2.7.

Example 2.9. Let $A = [-1, 0]$ and $B = [0, 1]$ be subsets of $X = \mathbb{R}$ endowed with the standard metric $d : X \times X \to [0, \infty)$, that is, $d(x, y) = |x - y|$, for all $x, y \in X$. Let w-distance $p : X \times X \to [0, +\infty)$ defined by $p(x, y) = |x| + |y|$ for every $x, y \in X$. Suppose that $T : A \to A$ and $S : B \to B$ are defined by

$$Tx = \begin{cases} x + \frac{1}{2} & \text{if } x \in \left[-1, -\frac{1}{2}\right), \\ 0 & \text{if } x \in \left[-\frac{1}{2}, 0\right]. \end{cases} \quad \text{and} \quad Sy = \begin{cases} 0 & \text{if } y \in \left[0, \frac{1}{2}\right), \\ y - \frac{1}{2} & \text{if } y \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, for each $x \in A$ and $y \in B$, we have

$$\begin{aligned} p^*\left(T^2x, S^2y\right) &= \left|T^2x\right| + \left|S^2y\right| \\ &= 0 \\ &\leq \frac{1}{2}\delta^*\left[A_2^x, B_2^y\right]. \end{aligned}$$

Also, for all $z \in [-1, 0)$ we have $Tz \neq z$ and

$$\begin{aligned} \beta(z) &= \inf\left\{p(x, z) + p(x, x) + p\left(x, Tx\right) + p\left(x, T^2x\right) : x \in [-1, 0]\right\} \\ &= \inf\left\{5\left|x\right| + \left|z\right| + \left|Tx\right| + \left|T^2x\right| : x \in [-1, 0]\right\} \\ &\geq |z| \\ &> 0. \end{aligned}$$

So, A, B and T satisfy all conditions of Theorem 2.6 and T has a unique fixed point $z^* = 0 \in A \cap B$. Also, all conditions of Theorem 2.7 hold and $z^* = 0$ is a unique optimal common fixed point T and S such that $p(z^*, z^*) = 0$.

The next proposition requires that if $T : A \to A$ obeys the condition $(C; k)$, then for all $z \in A$ with $Tz \neq z$, $\beta(z) > 0$.

Proposition 2.10. *Let (X, d) be a complete metric space and p be a w-distance on X. Let $T : A \to A$ obeys the condition $(C; k)$. Let $z \in A$ with $Tz \neq z$, then we have*

$$\inf\left\{p(x, z) + p(x, x) + p\left(x, Tx\right) : x \in A\right\} > 0.$$

Proof. Let $z \in A$ with $Tz \neq z$ and

$$\inf \{p(x,z) + p(x,x) + p(x,Tx) : x \in A\} = 0.$$

Then there exists a sequence $\{x_n\} \subseteq A$ such that for every $n \in \mathbb{N}$ we have

$$p(x_n,z) + p(x_n,x_n) + p(x_n,Tx_n) \leq \frac{1}{n}.$$

Hence for every $n \in \mathbb{N}$ we get

$$p(x_n,z) \leq \frac{1}{n}, \qquad p(x_n,x_n) \leq \frac{1}{n}, \qquad p(x_n,Tx_n) \leq \frac{1}{n}.$$

Therefore, by Lemma 1.5 (ii), the sequences $\{x_n\}$ and $\{Tx_n\}$ converges to z. Since T obeys the condition $(C;k)$ for some ≥ 0 we get

$$\begin{aligned} d(z,Tz) &\leq k \limsup_n d(x_n,Tx_n) \\ &= 0, \end{aligned}$$

leads to a contradiction. □

As a result of Theorem 2.7 and Proposition 2.10, we get the following common fixed point result which is an extension of Theorem 1.6.

Corollary 2.11. *Let A and B be nonempty and complete subsets of the metric space (X,d) and p be a w-distance on X. If $T : A \to A$ and $S : B \to B$ obey the condition $(C;k)$, and if for some fixed positive integers l and q and some $\lambda \in [0,1)$ the inequality*

$$p^*\left(T^l x, S^q y\right) \leq \lambda \max \{p^*(T^s x, S^r y) : 0 \leq s \leq l, 0 \leq r \leq q\},$$

hold for every $x \in A$ and $y \in B$. Then S and T have a unique optimal common fixed point $z^* \in A \cap B$.

From Theorem 2.6 and Proposition 2.10, we get the following extension of Theorem 1.7.

Corollary 2.12. *Let A and B be nonempty and complete subsets of the metric space (X,d) and p be a w-distance on X. Let $T : A \to A$ and $S : B \to B$ such that S obeys the condition $(C;k)$. If for some fixed positive integer q and some $\lambda \in [0,1)$ the inequality*

$$p^*(Tx, S^q y) \leq \lambda \max \{p^*(Tx, S^r y) : 0 \leq s \leq 1, 0 \leq r \leq q\},$$

hold for every $x \in A$ and $y \in B$. Then S and T have a unique optimal common fixed point $z^* \in A \cap B$.

Proof. Fix $x_0 \in A$ and $y_0 \in B$ and let $x_{n+1} := Tx_n$ and $y_{n+1} := Sy_n$. From Theorem 2.6 and proposition 2.10 S has a unique fixed point $z^* \in A \cap B$ such that the sequences $\{x_n\}$ and $\{y_n\}$ converge to z^*.

Furthermore, because T and S hold in a Fisher-type inequality with $l = 1$ and $z^* \in A \cap B$ is the fixed point of S we get

$$\begin{aligned} p^*(y_{n+q}, Tz^*) = p^*(Tz^*, S^q y_n) &\leq \lambda \max\left\{p^*\left(z^*, S^i y_n\right), p^*\left(Tz^*, S^i y_n\right)\right. \\ &\quad \left. : 0 \leq i \leq q\right\}, \\ &\leq \lambda \max\{p^*(S^n z^*, x_n) + p^*(x_n, y_{n+i}) \\ &\quad , p^*(Tz^*, z^*) + p^*(S^n z^*, x_n) + p^*(x_n, y_{n+i}) : 0 \leq i \leq q\}. \end{aligned}$$

By taking the limit and using Lemma 2.3, we have

$$\lim_{n\to\infty} p^*(y_{n+q}, Tz^*) \leq \lambda p^*(z^*, Tz^*). \tag{2.8}$$

Now, because

$$p^*(z^*, Tz^*) \leq p^*(S^n z^*, x_n) + p^*(x_n, y_{n+q}) + p^*(y_{n+q}, Tz^*).$$

By reusing Lemma 2.3 we get $p^*(z^*, Tz^*) \leq \lambda p^*(z^*, Tz^*)$, so $p^*(z^*, Tz^*) = 0$, so we get $Tz^* = z^*$. Since S obeys the condition $(C; k)$ for some $k \geq 0$, we have

$$\begin{aligned} p(z^*, z^*) &= \lim_{n\to\infty} p(T^n z^*, S^n z^*) \\ &= 0, \end{aligned}$$

therefor $p(z^*, z^*) = 0$. □

Let us recall that $T : A \cup B \to A \cup B$ is said to be non-cyclic mapping provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. In the following, we get a fixed point result for a non-cyclic mapping $T : A \cup B \to A \cup B$ in the framework of w-distance.

Corollary 2.13. *Let A and B be nonempty and complete subsets of the metric space (X, d) and p be a w-distance on X. Let $T : A \cup B \to A \cup B$ be a non-cyclic quasi-contraction mapping of Fisher type, that is there exists some fixed positive integers l and q and some $\lambda \in [0, 1)$ such that the inequality*

$$p^*\left(T^l x, T^q y\right) \leq \lambda \max\{p^*(T^r x, T^s y) : 0 \leq r \leq l, 0 \leq s \leq q\},$$

hold for every $x \in A$ and $y \in B$. If in addition $T_{|A}$ and $T_{|B}$ obey the condition $(C; k)$, then T has a unique fixed point $z^* \in A \cap B$ such that $p^*(z^*, z^*) = 0$.

We want to prove that when p is a w_0-distance on X, we can relax the $(C; k)$ condition of S in Corollary 2.12. For this purpose, we need the following proposition.

Proposition 2.14. *Let A and B be nonempty and complete subsets of the metric space (X, d) and p be a w-distance on X. Assume that the* *self-mappings* $T : A \to A$ *and* $S : B \to B$ *hold in a Fisher-type inequality with* $l = 1$. *Then*

$$\text{for all } z \in A \text{ with } Tz \neq z \text{ we have } \alpha^*(z) > 0.$$

Proof. Let $z \in A$ with $Tz \neq z$ and $\alpha^*(z) = 0$ Then there exists a sequence $\{y_n\} \subseteq B$ such that

$$\begin{aligned} \lim_{n\to\infty} p^*(y_n, z) &= \lim_{n\to\infty} p^*\left(y_n, Sy_n\right) \\ &= \lim_{n\to\infty} p^*\left(y_n, S^2 y_n\right) \\ &\ \ \vdots \\ &= \lim_{n\to\infty} p^*\left(y_n, S^q y_n\right) = 0. \end{aligned} \tag{2.9}$$

Because T and S hold in a Fisher-type inequality with $l = 1$ we get

$$\begin{aligned} p^*\left(Tz, S^q y_n\right) \leq{}& \lambda \max\left\{p^*(z, y_n), p^*\left(z, S^i y_n\right), p^*\left(Tz, y_n\right), p^*\left(Tz, S^i y_n\right) \right. \\ & \left. : 1 \leq i \leq q\right\}, \\ \leq{}& \lambda \max\left\{p^*(z, y_n), p^*(z, y_n) + p^*\left(y_n, S^i y_n\right) \right. \\ & , p^*\left(Tz, z\right) + p^*(z, y_n), p^*\left(Tz, z\right) + p^*(z, y_n) + p^*\left(y_n, S^i y_n\right) \\ & \left. : 1 \leq i \leq q\right\}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} p^*\left(Tz, S^q y_n\right) \leq \lambda p^*\left(z, Tz\right). \tag{2.10}$$

Now, because

$$p^*\left(z, Tz\right) \leq p^*(z, y_n) + p^*\left(y_n, S^q y_n\right) + p^*\left(S^q y_n, Tz\right).$$

From (2.9) and (2.10) we get $p^*\left(z, Tz\right) \leq \lambda p^*\left(z, Tz\right)$, so $p^*\left(z, Tz\right) = 0$ and $Tz = z$, a contradiction. □

From Remark 2.8 and Proposition 2.14, we obtain the following common fixed point results immediately.

Corollary 2.15. *Let* A *and* B *be nonempty and complete subsets of the metric space* (X, d) *and* p *be a* w_0*-distance on* X. *Suppose that for self-mappings* $T : A \to A$ *and* $S : B \to B$ *there exists* $\lambda \in [0, 1)$ *such that*

$$p\left(Tx, S^q y\right) \leq \lambda \max\left\{p\left(T^r x, S^s y\right) : 0 \leq r \leq 1, 0 \leq s \leq q\right\},$$

for every $x \in A$ *and* $y \in B$. *Then* T *has a unique fixed point* $z^* \in A \cap B$ *such that the sequences* $\{T^n x_0\}$ *and* $\{S^n y_0\}$ *converge to it for every* $x_0 \in A$ *and* $y_0 \in B$.

The next corollary is a generalization of Corollary 2.11 in [13].

Corollary 2.16. *Let* (X, d) *be a complete metric space and* p *be a symmetric w-distance on* X*. Let* $T : X \to X$ *and* $S : X \to X$ *be two mappings satisfying*

$$p(Tx, Sy) \leq \lambda \max \{p(x, y), p(x, Sy), p(Tx, y)\},$$

for all $x, y \in X$ *where* $\lambda \in [0, 1)$*. Then* S *and* T *have a unique optimal common fixed point in* X*.*

Corollary 2.17. *Let* T *be a mapping of a complete metric space* (X, d) *into itself and let* p *be a symmetric w-distance on* X*. Suppose that*

$$p(Tx, Ty) \leq \lambda \max \{p(x, y), p(x, Ty), p(Tx, y)\},$$

for all $x, y \in X$ *where* $\lambda \in [0, 1)$*. Then* T *has a unique fixed point* $z^* \in X$*.*

Corollary 2.18. *Let* A *and* B *be nonempty and complete subsets of the metric space* (X, d)*. Let* $T : A \cup B \to A \cup B$ *be a non-cyclic quasi-contraction mapping of Fisher type with* $l = 1$*, that is there exists* $\lambda \in [0, 1)$ *such that*

$$d(Tx, T^q y) \leq \lambda \max \{d(T^r x, T^s y) : 0 \leq r \leq 1, 0 \leq s \leq q\},$$

for every $x \in A$ *and* $y \in B$*. Then* T *has a unique fixed point* $z^* \in A \cap B$ *such that* $\{T^n z_0\}$ *is convergent to it for every* $z_0 \in A \cup B$*.*

Note that condition (W) is essential for Theorem 2.6. However, condition (W^*) alone is not sufficient to establish the fixed point of T, even if $l = q = 1$. This follows from the next example. Note that in this example p is not a w_0-distance.

Example 2.19. Let $X = [0, 1] \subseteq \mathbb{R}$ equipped with the standard metric $d : X \times X \to [0, \infty)$, that is, $d(x, y) = |x - y|$, for all $x, y \in X$. Define also the mapping $p : X \times X \to [0, \infty)$ such that

$$p(x, y) = \begin{cases} 9 & \text{if } x = 0, \\ y - x & \text{if } 0 < x \leq y, \\ 3x - 3y & \text{if } x > y, \end{cases}$$

which is a w-distance on X (Example 3.3 of [14]). Let us define $T : X \to X$ by

$$Tx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{x}{10} & \text{if } x \neq 0. \end{cases}$$

It is not difficult to check that

$$p^*(x, y) = \begin{cases} 9 & \text{if } x = 0 \quad or \quad y = 0, \\ |3x - 3y| & \text{if } x, y \neq 0. \end{cases}$$

We will check that for some $\lambda \in [0,1)$, T satisfies

$$p^*(Tx,Ty) \leq \lambda \max\{p^*(x,y), p^*(x,Ty), p^*(Tx,y)\}, \tag{2.11}$$

for all $x, y \in X$. We will have the following cases.

Case 1. $x = 0$, $y = 0$. Then $p^*(Tx,Ty) = p^*(1,1) = 0$ and $p^*(x,y) = p^*(0,0) = 9$.
Case 2. $x = 0$, $y \neq 0$. Then $p^*(Tx,Ty) = p^*\left(1, \frac{y}{10}\right) = 3 - \frac{3y}{10}$ and $p^*(x,y) = p^*(0,y) = 9$.
Case 3. $x \neq 0$, $y = 0$. Then $p^*(Tx,Ty) = p^*\left(\frac{x}{10}, 1\right) = 3 - \frac{3x}{10}$ and $p^*(x,y) = p^*(x,0) = 9$.
Case 4. $x \neq 0$, $y \neq 0$. Then $p^*(Tx,Ty) = p^*\left(\frac{x}{10}, \frac{y}{10}\right) = \left|\frac{3x}{10} - \frac{3y}{10}\right|$ and $p^*(x,y) = |3x - 3y|$.

According to the four cases, we get

$$p^*(Tx,Ty) \leq \frac{1}{3} p^*(x,y).$$

So (2.11) holds for $\lambda = \frac{1}{3}$. Note that for all $z \in [0,1]$ that $Tz \neq z$ we have

$$\begin{aligned}
\beta^*(z) &= \inf\{p^*(x,z) + p^*(x,x) + p^*(x,Tx) : x \in [0,1]\} \\
&= \inf\Big\{p^*(0,z) + p^*(0,0) + p^*(0,1), p^*(x,z) + p^*(x,x) + p^*\left(x, \frac{x}{10}\right) \\
&\quad : x \in (0,1]\Big\} \\
&= \begin{cases} \inf\left\{27, 9 + 0 + \left|3x - \frac{3x}{10}\right| : x \in (0,1]\right\} & \text{if } z = 0 \\ \inf\left\{27, |3x - 3z| + 0 + \left|3x - \frac{3x}{10}\right| : x \in (0,1]\right\} & \text{if } z \neq 0 \end{cases} \\
&> 0,
\end{aligned}$$

but T does not have any fixed point. Furthermore, although for $z = 0$ we have $T0 \neq 0$, but

$$\begin{aligned}
\alpha(0) &= \beta(0) \\
&= \inf\{p(x,0) + p(x,x) + p(x,Tx) : x \in [0,1]\} \\
&= \inf\Big\{p(0,0) + p(0,0) + p(0,1), p(x,0) + p(x,x) + p\left(x, \frac{x}{10}\right) \\
&\quad : x \in (0,1]\Big\} \\
&= \inf\left\{27, 3x + 0 + 3x - 3\frac{x}{10} : x \in (0,1]\right\} \\
&= 0.
\end{aligned}$$

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