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Convex and Starlike Functions Defined on the Subclass of the Class of the Univalent Functions S with Order 2^{-r}

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Abstract. In this paper, some conditions have been improved so that the function $g(z)$ is defined as $g(z) = 1 + \sum_{k\geq 2}^{\infty} a_{n+k} z^{n+k}$, which is analytic in unit disk U, can be in more specific subclasses of the S class, which is the most fundamental type of univalent function. It is analyzed some characteristics of starlike and convex functions of order 2^{-r}.

1. Introduction

$\mathcal{U} = \{|z| < 1 : z \in \mathbb{C}\}$ is an open unit disk defined in the complex plane. $F(\mathcal{U})$ is taken the class of analytic functions defined as $g : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ analytic in $\mathcal{U}$. $g(z)$ is analytic in a complex plane $\mathbb{C}$. For any two points $z_1, z_2 \in \mathcal{U}$ such that $z_1 \neq z_2$ if $g(z_1) \neq g(z_2)$ then $g(z)$ is said to be univalent in $\mathcal{U}$ [2]. The theory of univalent functions is an old subject however it remains an active field of current research. It's progress has been especially rapid in recent years. This study is concerned with the class S of analytic functions and univalent in $\mathcal{U}$ and normalized in standard way [8].

Let $\mathcal{A}$ denote the class of analytic functions in $\mathcal{U}$ normalized by

$$g(z) = \sum_{n\geq 2}^{\infty} a_n z^n \quad \text{for} \quad g(0) = 1 - \left.\frac{dg(z)}{dz}\right|_{z=0} = 0$$

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which $\mathcal{A}$ is the set of functions $g(z)$ [9]. The set of univalent functions defined $g(z) \in \mathcal{A}$ is S with

$$S = \{g(z) \in \mathcal{A} : g \text{ is univalent in } \mathcal{U}\}.$$

On the other hand, the set of class starlike functions $g(z) \in \mathcal{A}$ of order δ is $S^{\otimes}$ such that

$$S^{\otimes} = \left\{ g(z) \in \mathcal{A} : \Re\left(\frac{zg'(z)}{g(z)}\right) > \delta \right\},$$

where $z \in \mathcal{U}$ and $0 \leq \delta < 1$ and similarly, the set of convex functions $g(z) \in \mathcal{A}$ of order δ is C such that

$$C = \left\{ f(z) \in \mathcal{A} : \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > \delta \right\},$$

where $z \in \mathcal{U}$ and $0 \leq \delta < 1$ [1, 4, 6, 7].

The class of convex functions $g(z)$ in class $\mathcal{A}$, which is a univalent function, is denoted by C. Here it should be noted that when $g(z) \in \mathcal{A}$ is a convex function, $g(\mathcal{U})$ is a convex region in $\mathbb{C}$. Also, $S^{\otimes}$ denotes the class of starlike functions $g(z)$ in class $\mathcal{A}$ which $g(z)$ is a univalent function. Also, it should be noted that when $g(z) \in \mathcal{A}$ is a starlike function, $g(\mathcal{U})$ is a starlike region relative to the origin in the complex plane.

The set of close-to-convex functions $g(z) \in \mathcal{A}$ of order δ is

$$C^{\otimes} = \left\{ g(z) \in \mathcal{A} : \Re\left(\frac{g'(z)}{f'(z)}\right) > \delta, f(z) \text{ is convex in } \mathcal{U} \right\},$$

[3, 5].

In this study, we analyze some characteristics of starlike and convex functions of order 2^{-r} where $r \in \{1, 2, 3, \dots\}$ analytically. According to this selection, we can give the above definitions as follows.

The set of starlike functions, convex functions and close-to convex functions $g(z) \in \mathcal{A}$ of order 2^{-r} $\left(0 < \frac{1}{2^r} \leq \frac{1}{2}\right)$ are given, respectively

$$S^{\otimes} = \left\{ g(z) \in \mathcal{A} : 0 < \Re\left(\frac{zg'(z)}{g(z)}\right) \leq 2^{-r}, r \in \{1, 2, \dots\}, z \in \mathcal{U} \right\},$$
$$C = \left\{ g(z) \in \mathcal{A} : 0 < \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) \leq 2^{-r}, r \in \{1, 2, \dots\}, z \in \mathcal{U} \right\},$$
$$C^{\otimes} = \left\{ g(z) \in \mathcal{A} : 0 < \Re\left(\frac{g'(z)}{f'(z)}\right) \leq 2^{-r}, r \in \{1, 2, \dots\}, z \in \mathcal{U} \right\},$$

where f is convex in $\mathcal{U}$.

2. Main Results

Lemma 2.1. *Every convex function is a close-to-convex function [4, 6]. More generally, every starlike function is a close-to-convex function. Indeed, it can be written that each* $g(z) \in S^{\otimes}$ *has the form for some* $g(z) \in C^{\otimes}$ *such that*

$$\Re\left(\frac{g'(z)}{f'(z)}\right) = \Re\left(\frac{g'(z)}{g(z)}\right) > 0.$$

Lemma 2.2. $w(z)$ *is a regular in* $\mathcal{U}$ *with* $w(0) = 0$ *and* $|w(z)| < 1$ $(z \in \mathcal{U})$. *If* $|w(z)|$ *reaches its greatest value on* $\mathcal{U}$ *at* z_0, *then* $z_0 w'(z_0) = m w(z_0)$ *where* $m \geq 1$ *is a real number [4, 6].*

Theorem 2.3. *Suppose* $g(z) \in F(\mathcal{U})$ *is a function with* $g(0) = 0$. *Then,* $g(\mathcal{U})$ *is a starlike region if and only if*

$$g'(0) \neq 0 \quad \textit{and} \quad 0 < \Re\left(\frac{zg'(z)}{g(z)}\right) \leq 2^{-r}.$$

Proof. By applying Lemma (2.1) and (2.2), we can prove this theorem. If $g(z) = \dfrac{z(1 - w(z))}{1 + w(z)}$ is a function with $w(0) = 0$, then

$$\ln(g(z)) = \ln\left(\frac{z(1 - w(z))}{1 + w(z)}\right),$$

which implies that

$$\ln(g(z)) = \ln(z) + \ln(1 - w(z)) - \ln(1 + w(z)).$$

Thus

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= 1 - \frac{2zw'(z)}{(1 - w^2(z))} \\ &= 1 - \frac{2ke^{i\theta}}{1 - e^{2i\theta}} \\ &= 1 - \frac{2k(\cos\theta + i\sin\theta)}{(1 - \cos 2\theta - i\sin 2\theta)}, \end{aligned}$$

and

$$\Re\left(\frac{g'(z)}{g(z)}\right) = \Re\left(1 - \frac{k\cos\theta}{sin^2\theta}\right) \leq 2^{-r}, \quad 0 < 2^{-r} \leq 2^{-1}.$$

Let us define the class $\mathfrak{N}_{\mathcal{A}}$ which is a subclass of the class as follows

$$\begin{aligned} g(z) &= z + \sum_{n\geq 2}^{\infty} a_n z^n \\ &= z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + a_{n+1} z^{n+1} + \cdots \end{aligned}$$

$$\mathfrak{N}_{\mathcal{A}} = \left\{ F_k(\mathcal{U}) : g_k(z) = \sum_{k=1}^{\infty} a_{n+k} z^{n+k} \right\}$$

for n and k positive integers which $\mathfrak{N}_{\mathcal{A}} \subset \mathcal{A}$. □

Theorem 2.4. *Let p and q be real numbers, n be a positive integer and also $g(z) \in \mathfrak{N}_{\mathcal{A}}$. Therefore, it is said that the function $g(z)$ is in the class $\mathfrak{N}$ if the function $\mathfrak{H} : \mathcal{U} \to \mathbb{C}$ is defined as*

$$\frac{g(z).g'(z)}{z} \neq 0 \quad \text{and} \quad 1 - p + p\frac{zg'(z)}{g(z)} \neq 0.$$

Therefore, $g(z)$ is in the class $\mathfrak{N} \subset S^{\otimes}$ if the function $\mathfrak{H} : \mathcal{U} \to \mathbb{C}$ is specified as

$$f(z) = g(z) \left(\frac{zg'(z)}{g(z)} \right)^{p-pq} \left(2^{-r} \frac{zg'(z)}{g(z)} \right)^{q}$$

is a starlike function on $\mathcal{U}$.

Proof. Let's do this by taking $0 < 2^{-r} \leq 2^{-1}$ instead of a δ $(0 \leq \delta < 1)$ in the proof of the theorem. According to the values that the function $f(z)$, we can make the theorem in four cases as follows:

(i) If $q = 0$, then

$$f(z) = g(z) \left(\frac{zg'(z)}{g(z)} \right)^{p}. \tag{2.1}$$

If the logarithmic derivative is taken from both sides of the equation and the necessary operations are performed

$$\ln(f(z)) = \ln(g(z)) + p \left(\ln(z) + \ln(g'(z)) - \ln(g(z)) \right)$$

and

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{g'(z)}{g(z)} + p \left(\frac{1}{z} + \frac{g''(z)}{g'(z)} - \frac{g'(z)}{g(z)} \right) \\ &= (1-p)\frac{zg'(z)}{g(z)} + p \left(1 + \frac{zg''(z)}{g'(z)} \right), \end{aligned}$$

then,

$$\begin{aligned} \Re \left(\frac{zf'(z)}{f(z)} \right) &= \Re \left\{ (1-p)\frac{zg'(z)}{g(z)} + p \left(1 + \frac{zg''(z)}{g'(z)} \right) \right\} \\ &\leq \Re \left((1-p)\frac{zg'(z)}{g(z)} \right) + \Re \left(p \left(1 + \frac{zg''(z)}{g'(z)} \right) \right) \\ &\leq (1-p)2^{-r} + p2^{-r} \\ &= 2^{-r} \\ &< 1, \end{aligned}$$

where $0 < \Re\left(\frac{zg'(z)}{g(z)}\right) \leq 2^{-r}$ from starlike function and the function $f(z) \in \mathfrak{N}$ is the starlike function with $\Re\left(\frac{f'(z)}{f(z)}\right) \leq 2^{-r} < 1$.

(ii) If $p = 0$, then

$$f(z) = g(z)\left(2^{-r}\frac{zg'(z)}{g(z)}\right)^q.$$

Thus,

$$\begin{aligned}\ln(f(z)) &= \ln(g(z)) + q\ln(2^{-r}zg'(z)) - q\ln(g(z)) \\ &= \ln(g(z)) + q\ln(2^{-r}z) + q\ln(2^{-r}g'(z)) - q\ln(g(z))\end{aligned}$$

and

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{g'(z)}{g(z)} + q\frac{1}{z} + q\frac{g''(z)}{g'(z)} - q\frac{g'(z)}{g(z)} \\ \frac{zf'(z)}{f(z)} &= (1-q)\frac{zg'(z)}{g(z)} + q\left(1 + \frac{zg''(z)}{g'(z)}\right) \\ \frac{zf'(z)}{f(z)} &= (1-q)2^{-r} + q2^{-r} \\ &= 2^{-r} \\ &< 1.\end{aligned}$$

In this case, $g(z) \in \mathfrak{N}$ is the starlike function with

$$0 < \Re\left(\frac{f'(z)}{f(z)}\right) \leq 2^{-r} < 1.$$

(iii) If $q = 1$, then

$$f(z) = g(z)\left(2^{-r}\frac{zg'(z)}{g(z)}\right).$$

Thus, $\ln(f(z)) = \ln(z) + \ln(g'(z))$ and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) \leq 2^{-r} < 1.$$

Thus $f(z) \in \mathfrak{N}$ is the convex function with

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) \leq 2^{-r} < 1.$$

(iv) If $p = 1$, then

$$\begin{aligned} f(z) &= g(z)\left(\frac{zg'(z)}{g(z)}\right)^{1-q}\left(2^{-r}\frac{zg'(z)}{g(z)}\right)^{q} \\ &= 2^{-rq}zg'(z). \end{aligned}$$

Additionally,

$$\ln(g(z)) = \ln(z) + \ln(g'(z)) \quad \text{and} \quad \frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{g''(z)}{g'(z)}.$$

Thus

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(1 + \frac{g''(z)}{g'(z)}\right) \le 2^{-r} < 1.$$

In this case, the function $f(z) \in \mathfrak{N}$ is the convex function with

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \le 2^{-r} < 1.$$

□

Theorem 2.5. *For a function $f(z) \in \mathfrak{N}$ defined on the class $\mathfrak{N}$ by real number 2^{-r} with $\mathfrak{N}$ is subclass of $S^{\otimes}$.*

Proof. For the proof of this theorem, the function $f(z)$ is a starlike function relative to the class $S^{\otimes}$ is sufficient. For this, let's define

$$f(z) = zg'(z)\left((2^{-r} - 1) + \frac{zg'(z)}{g(z)}\right).$$

Let's define the function for convenience in the operations to be performed like

$$h(z) = (2^{-r} - 1) + \frac{zg'(z)}{g(z)}.$$

Thus

$$\begin{aligned} &zg'(z) = (1 - 2^{-r})g(z) + g(z)h(z) \\ &g'(z) + zg''(z) = (1 - 2^{-r})g'(z) + g'(z)h(z) + g(z)h'(z). \end{aligned}$$

Let's multiply both sides of this equality $g'(0) \neq 0$

$$1 + \frac{zg''(z)}{g'(z)} = (1 - 2^{-r}) + h(z) + \frac{g(z)}{g'(z)}h'(z). \tag{2.2}$$

By taking logarithmic derivative

$$f(z) = zf'(z)h(z) \quad \Rightarrow \quad \ln(f(z)) = \ln(z) + \ln(f'(z)) + \ln(h(z))$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zg''(z)}{g'(z)} + \frac{zh'(z)}{h(z)}. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\frac{zf'(z)}{f(z)} = (1-2^{-r}) + h(z) + \frac{g(z)}{g'(z)}h'(z) + \frac{zh'(z)}{h(z)} \tag{2.4}$$

and

$$\frac{g(z)}{g'(z)} = \frac{z}{(1-2^{-r}) + h(z)}. \tag{2.5}$$

From (2.4) and (2.5), it is clear that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \leq 2^{-r}.$$

The function $f(z)$ is a starlike function in class $\mathfrak{N}$ and we get $\mathfrak{N}(2^{-r}) \subset S^{\otimes}$. □

Theorem 2.6. *Let $f(z) = 1 + \sum_{n\geq 1}^{\infty} A_n z^n$ be analytic defined on $\mathcal{U} = \{|z| < 1 : z \in \mathbb{C}\}$ and just suppose that $f(z) \prec \frac{1-2^{-r}z}{1+3^{-r}z}$ in $\mathcal{U}$ and where $0 < \frac{1}{2^r} \leq \frac{1}{2}$ and $0 < \frac{1}{3^r} \leq \frac{1}{3}$. Then, it is said that $f(z) \in \mathfrak{N}$ is a starlike function and $\frac{1-2^{-r}}{1+3^{-r}} < \Re(f(z)) < \frac{1+2^{-r}}{1-3^{-r}}$.*

Proof. Suppose that $f(z) = zg'(z)\frac{1-2^{-r}z}{1+3^{-r}z}$ where $zg'(z) \in S^{\otimes}$ in $\mathcal{U}$. Then, logarithmic derivative is taken and necessary operations are performed

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{g''(z)}{g'(z)} - \frac{(2^{-r}+3^{-r})}{(1-2^{-r}z)(1+3^{-r}z))}$$

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zg''(z)}{g'(z)} - \frac{(2^{-r}+3^{-r})z}{(1-2^{-r}z)(1+3^{-r}z)}$$

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left\{1 + \frac{zg''(z)}{g'(z)} - \frac{(2^{-r}+3^{-r})z}{(1-2^{-r}z)(1+3^{-r}z)}\right\}.$$

Thus $\Re\left(\frac{zg'(z)}{g(z)}\right) < 2^{-r}$ is in $S^{\otimes}$ with $z \in \mathcal{U}$. For the second part of the proof

$$\frac{1-2^{-r}}{1+3^{-r}} = \frac{3^r(2^r-1)}{2^r(3^r+1)}$$

and

$$\frac{1+2^{-r}}{1-3^{-r}} = \frac{3^r(2^r+1)}{2^r(3^r+1)}.$$

Then

$$\frac{1-2^{-r}}{1+3^{-r}} < \Re(f(z)) < \frac{1+2^{-r}}{1-3^{-r}}.$$

Thus, the proof of the theorem is completed. □

3. Conclusion

Considering all the results together, we can say that every starlike function defined on $\mathfrak{N}$ will be a close-to-convex function; it is said that the function $f(z) \in \mathfrak{N}$ is also a close-to-convex function.

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