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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 4
Pages: 133-147

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.554936.1121

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 4 (2022), 133-147
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.554936.1121

On Some Properties of Log-Harmonic Functions Product

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Abstract. In this paper we define a new subclass $S_{LH}(k, \gamma; \varphi)$ of log-harmonic mappings, and then basic properties such as dilations, convexity on one direction and convexity of log functions of convex- exponent product of elements of that class are discussed. Also we find sufficient conditions on β such that $f \in S_{LH}(k, \gamma; \varphi)$ leads to $F(z) = f(z)|f(z)|^{2\beta} \in S_{LH}(k, \gamma, \varphi)$. Our results generalize the analogues of the earlier works in the combinations of harmonic functions.

1. Introduction and Preliminaries

Suppose $E = \{z \in C : |z| < 1\}$ and $\mathcal{H}(E)$ describe the linear space of all holomorphic functions defined in E. Let f be a 2-times continuously differentiable function, then f is harmonic if $\Delta f = 0$, and f is log-harmonic mapping if $\log f$ is harmonic, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Actually a log-harmonic mapping f is solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f}$$

where the second dilation function $a \in \mathcal{H}(E)$ is such that $|a(z)| < 1$ for all $z \in E$.

2020 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.
Key words and phrases. Univalent function, Log-harmonic function, Convex in the one direction, Sense-preserving functions
Received: 31 May 2022, Accepted: 01 October 2022.
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Furthermore Abdulhadi et al. in [1, 2] have showed that if f is a non-constant log-harmonic mapping that vanishes only at $z = 0$, then f should be in the form

$$f(z) = z^m |z|^{2m\beta} h(z)\overline{g(z)} \tag{1.1}$$

where m is a nonnegative integer, $Re\beta > -\frac{1}{2}$, while $h, g \in \mathcal{H}(E)$ satisfying $g(0) = 1$ and $h(0) \neq 0$. Note that β in (1.1) depends only on $a(0)$ and is given by

$$\beta = \frac{\overline{a(0)}\,(1 + a(0))}{1 - |a(0)|^2}.$$

Moreover, $f(0) \neq 0$ if and only if $m = 0$, and that a univalent log-harmonic mapping in E vanishes at the origin if and only if $m = 1$, that is, f is as follow

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)},$$

where $Re\beta > -\frac{1}{2}$ and $0 \notin hg(E)$. The similar of the harmonic functions the Jacobian of log-harmonic function f is taken by

$$J_f(z) = |f_z|^2 \left(1 - |a(z)|^2\right),$$

and is positive. So all non-constant log-harmonic mappings that we have discussed the above are sense-preserving in the unit disk E. Let B_0 describe the class of Schwarzian functions such that $a(0) = 0$. Also let S_{LH} be the class of all univalent and sense-preserving log-harmonic mappings in E with respect to $a \in B_0$. These mappings are in the form

$$f(z) = h(z)\overline{g(z)}$$
$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$
$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

The set of all function $f \in S_{LH}$ with $f_{\overline{z}}(0) = 0$ is denoted by S^0_{LH}.

In view of the definition $a(z)$ for the function $f(z) = h(z)\overline{g(z)}$, we observe that the second dilation $a(z)$ is

$$\begin{aligned} a(z) &= \frac{\overline{f_{\overline{z}}}(z).f(z)}{f_z(z).\overline{f}(z)} \\ &= \frac{g'(z)\overline{h}(z)h(z)\overline{g}(z)}{h'(z)\overline{g}(z)g(z)\overline{h}(z)} \\ &= \frac{g'(z)h(z)}{h'(z)g(z)}. \end{aligned}$$

If $f(z) = h(z)\overline{g(z)}$ be univalent and satisfies the condition

$$\left|\frac{g'(z)h(z)}{h'(z)g(z)}\right| \leq k < 1, \quad (z \in E),$$

we call it a log-harmonic K-quasi conformal mapping on E, where $K = \frac{k+1}{1-k}$.

Let $S_{LH}(k)$ be the subclass of S^0_{LH} consisting of log-harmonic K-quasi conformal mappings. We refer the reader for more information about log-harmonic mappings to [3, 6, 10].

Let $\Omega \subset \mathbb{C}$ be a domain. Then Ω is called a convex set in the direction $\gamma \in [0, \pi]$, if the set $\Omega \cap \{a + te^{i\gamma} : t \in \mathbb{C}\}$ is either connected or empty, for all $a \in \mathbb{C}$. Particularly, a domain such Ω is convex in the direction of real (imaginary) axis if every line parallel to the real (imaginary) axis has either an empty intersection or a connected intersection with the domain. A function is called a convex function in the direction γ if it maps E univalently on to a convex domain in the direction γ.

For, $k \in (0, 1]$, $\gamma \in [0, \pi]$ and $\nu \in \mathcal{H}(E)$, consider the following subclass $S_{LH}(k, \gamma, \nu)$ of S_{LH} defined by

$$\begin{aligned} &S_{LH}(k, \gamma, \nu) \\ &\quad := \left\{ f(z) = h(z)\overline{g(z)} \in S_{LH}(k) : \log h(z) - e^{2i\gamma} \log g(z) = \nu(z) \right\} \end{aligned}$$

For simplicity, we write

$$S_{LH}(k, 0, \nu) := S^-_{LH}(k, \nu), \qquad S_{LH}\left(k, \frac{\pi}{2}, \nu\right) := S^+_{LH}(k, \nu).$$

Recently the authors in [9–11], have been discussed on the linear combination of harmonic functions and in the [3, 4] the authors are considered the same problem for log-harmonic function. In this research, we define a new subclass of log-harmonic functions and discuss on the basic properties of the convex-exponent product of the elements of that class.

In Section 2 we discuss on the second dilation function of convex-exponent product of log-harmonic function and then prove that the convex-exponent product of elements of the class $S_{LH}(k, \gamma, \varphi)$ are belonging to this class and then by taking different functions of φ we solve this problem. Also, in Section 3, we consider other type of convex-exponent product of log-harmonic functions.

For achieving to our goals, we recall the following lemmas.

Lemma 1.1 ([5]). *A sense-preserving harmonic function $f = h + \bar{g}$ in E is a univalent mapping of E on to a domain convex in the direction γ with $0 \leq \gamma < \pi$ if and only if $e^{-i\gamma}h - e^{i\gamma}g$ is an analytic univalent mapping of E on to a domain convex in the direction real axis.*

Lemma 1.2 ([7])**.** *Assume that f be a holomorphic function in E with $f(0)=0$ and $f'(0)\neq 0$ and let for all, $(\theta\in\mathbb{R})$*

$$k(z)=\frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})}. \tag{1.2}$$

If

$$\Re\left(\frac{zf'(z)}{k(z)}\right)>0, \quad (z\in E),$$

then f is convex in the direction of the real axis.

Lemma 1.3 ([8])**.** *Let $\varphi(z)$ be a non-constant function analytic in E. The function $\varphi(z)$ maps E univalently on to a domain convex in the direction of imaginary axis, if and only if there are numbers ν and μ, $0\leq\nu<2\pi$ and $0\leq\mu<2\pi$ such that*

$$\Re\left(-ie^{i\mu}\left(1-2ze^{i\mu}cos\nu+z^2e^{-2i\mu}\right)\varphi'(z)\right)\geq 0, \quad \text{for } z\in E.$$

2. Convex-Exponent Product of Log-Harmonic Mappings

Lemma 2.1. *If $f_j\in S_{LH}(k,\gamma,\varphi)$ with $(j=1,2)$, then the dilation $a_3(z)$ of product $f_3(w)=f_1^t(w)f_2^{1-t}(w)$ with $(0\leq t\leq 1)$ satisfies*

$$\begin{aligned}|a_3(w)| &= \left|\frac{t\frac{g_1'(w)}{g_1(w)}+(1-t)\frac{g_2'(w)}{g_2(w)}}{t\frac{h_1'(w)}{h_1(w)}+(1-t)\frac{h_2'(w)}{h_2(w)}}\right|\\ &\leq k\\ &<1\end{aligned}$$

Proof. By definition of the class $S_{LH}(k,\gamma,\varphi)$, we have $\varphi(w)=\log\frac{h(w)}{g(w)^{e^{2i\gamma}}}$, for any $f=h\overline{g}\in S_{LH}(k,\gamma,\varphi)$. So by letting $f_j=h_j\overline{g_j}$ and $\varphi_j(w)=\log h_j(w)-e^{2i\gamma}\log g_j(w)$, for $j=1,2$

$$\varphi_j'(w)=\frac{h_j'(w)}{h_j(w)}-e^{2i\gamma}\frac{g_j'(w)}{g_j(w)}.$$

Also we will take the second dilations of the functions f_j with a_j $(j=1,2)$. By elementary calculations and taking partial differentiating of them we obtain

$$(f_j)_{\overline{z}}(w)=h_j(w)\overline{g_j'(w)}, \qquad (f_j)_z(w)=h_j'(w)\overline{g_j(w)},$$

and so in view of definition of a_j we have

$$\frac{\overline{h}_j(w)g_j'(w)}{\overline{h}_j(w)g_j(w)}=a_j\frac{h_j'(w)\overline{g}_j(w)}{h_j(w)\overline{g}_j(w)},$$

or

$$\frac{g_j'(w)}{g_j(w)}=a_j\frac{h_j'(w)}{h_j(w)}.$$

Also,

$$\varphi_j'(w) = \frac{h_j'(w)}{h_j(w)} - e^{2i\gamma} a_j \frac{h_j'(w)}{h_j(w)} = \frac{h_j'(w)}{h_j(w)} \left(1 - e^{2i\gamma} a_j\right)$$

or

$$\frac{h_j'(w)}{h_j(w)} = \frac{\varphi_j'(w)}{(1 - e^{2i\gamma} a_j)}, \quad (j = 1, 2).$$

By calculation $(f_3)_z$ and $(f_3)_{\overline{z}}$ and substituting them in the definition of second dilation of f_3 namely a_3 we have

$$\begin{aligned} a_3(w) &= \frac{(\overline{f_3})_{\overline{z}}(w).f_3(w)}{\overline{f}_3(w)(f_3)_z(w)} \\ &= \frac{t\frac{g_1'(w)}{g_1(w)} + (1-t)\frac{g_2'(w)}{g_2(w)}}{t\frac{h_1'(w)}{h_1(w)} + (1-t)\frac{h_2'(w)}{h_2(w)}} \\ &= \frac{t\frac{a_1\varphi'(w)}{(1-e^{2i\gamma}a_1)} + (1-t)\frac{a_2\varphi'(w)}{(1-e^{2i\gamma}a_2)}}{t\frac{\varphi'(w)}{(1-e^{2i\gamma}a_1)} + (1-t)\frac{\varphi'(w)}{(1-e^{2i\gamma}a_2)}} \\ &= \frac{t\frac{a_1}{(1-e^{2i\gamma}a_1)} + (1-t)\frac{a_2}{(1-e^{2i\gamma}a_2)}}{t\frac{1}{(1-e^{2i\gamma}a_1)} + (1-t)\frac{1}{(1-e^{2i\gamma}a_2)}}. \end{aligned}$$

But $|a_3| \leq k$ yields if

$$k^2 \left| \frac{t}{(1 - e^{2i\gamma} a_1)} + \frac{1-t}{(1 - e^{2i\gamma} a_1)} \right|^2 - \left| \frac{ta_1}{(1 - e^{2i\gamma} a_1)} + \frac{(1-t)a_2}{(1 - e^{2i\gamma} a_2)} \right|^2 \geq 0$$

Let $a_j = \rho_j e^{i\theta_j}$, $(0 \leq \rho_j < 1, \theta_j \in \mathbb{R}; j = i, 2)$ and

$$\phi := \frac{2t(1-t)}{|1 - e^{2i\gamma} a_1|^2 |1 - e^{2i\gamma} a_2|^2} \geq 0$$

then we have

$$\begin{aligned} k^2 &\left| \frac{u}{(1 - e^{2i\gamma} a_1)} + \frac{1-u}{(1 - e^{2i\gamma} a_1)} \right|^2 - \left| \frac{ua_1}{(1 - e^{2i\gamma} a_1)} + \frac{(1-u)a_2}{(1 - e^{2i\gamma} a_2)} \right|^2 \\ &= \frac{u^2(k^2 - |a_1|^2)}{|1 - e^{2i\gamma} a_1|^2} + \frac{(1-u)^2(k^2 - |a_2|^2)}{|1 - e^{2i\gamma} a_2|^2} \\ &\quad + 2u(1-u)\Re\left(\frac{k^2 - a_1\overline{a}_2}{(1 - e^{2i\gamma} a_1)\,(1 - e^{-2i\gamma} \overline{a}_2)} \right) \\ &\geq \frac{2u(1-u)}{|\,(1 - e^{2i\gamma} a_1)\,|^2 |\,(1 - e^{2i\gamma} a_2)\,|^2} \end{aligned}$$

$$\begin{aligned}
&\times \Re((k^2 - a_1\overline{a}_2)(1 - e^{-2i\gamma}\overline{a}_1)(1 - e^{-2i\gamma}a_2)) \\
&= \phi\Big(\left(k^2 - \rho_1^2\rho_2^2\right) + \rho_1\left(\rho_2^2 - k^2\right)\cos\left(2\gamma + \theta_1\right) \\
&\quad + \rho_2\left(\rho_1^2 - k^2\right)\cos\left(2\gamma + \theta_2\right) + \rho_1\rho_2(k^2 - 1)\cos\left(\theta_2 - \theta_1\right)\Big) \\
&\geq \phi\left(\left(k^2 - \rho_1^2\rho_2^2\right) - \rho_1\left(k^2 - \rho_2^2\right) - \rho_2\left(k^2 - \rho_1^2\right) - \rho_1\rho_2\left(1 - k^2\right)\right) \\
&= \phi\left(k^2 - \rho_1\rho_2\right)\left(1 - \rho_1\right)\left(1 - \rho_2\right) \\
&\geq 0.
\end{aligned}$$

Thus the proof is completed. □

Corollary 2.2. *Let $f_\mu = h_\mu \overline{g}_\mu \in S_{LH}(k, \gamma, \varphi)$, $(\mu = 1, 2, \ldots, n)$ be a log-harmonic univalent mapping in E. Then, dilation of the product of $F = f_1^{t_1} f_2^{t_2} \ldots f_n^{t_n}$ satisfies*

$$\begin{aligned}
|a| &= \left| \frac{t_1\frac{g_1'}{g_1} + t_2\frac{g_2'}{g_2} + \cdots + t_n\frac{g_n'}{g_n}}{t_1\frac{h_1'}{h_1} + t_2\frac{h_2'}{h_2} + \cdots + t_n\frac{h_n'}{h_n}} \right| \\
&\leq k \\
&< 1
\end{aligned}$$

where $0 \leq t_\mu \leq 1$ $(\mu = 1, 2, \ldots, n)$, and $t_1 + t_2 + \cdots + t_n = 1$.

Theorem 2.3. *Let $f_\mu(w) = h_\mu(w)\overline{g}_\mu(w)$, $\mu = 1, 2$. Then*

$$f(w) = f_1^t(w) f_2^{1-t}(w) \in S_{LH}(k, \gamma, \varphi), \quad (0 \leq t \leq 1).$$

Proof. By considering Lemma 2.1, we know that the dilation of f satisfies

$$|a| = \left| \frac{u\frac{g_1'(w)}{g_1(w)} + (1-u)\frac{g_2'(w)}{g_2(w)}}{u\frac{h_1'(w)}{h_1(w)} + (1-u)\frac{h_2'(w)}{h_2(w)}} \right| \leq k < 1,$$

and

$$\begin{aligned}
&\log h_j(w) - e^{2i\gamma}\log g_j(w) \\
&\quad = \log\left(h_1^u(w)h_2^{1-u}(w)\right) - e^{2i\gamma}\log\left(g_1^u(w)g_2^{1-u}(w)\right) \\
&\quad = \left(u\log h_1(w) + (1-u)\log h_2(w)\right) \\
&\qquad - e^{2i\gamma}\left(u\log g_1(w) + (1-u)\log g_2(w)\right) \\
&\quad = u\left(\log h_1(w) - e^{2i\gamma}\log g_1(w)\right) + (1-u)\left(\log h_2(w) - e^{2i\gamma}\log g_2(w)\right) \\
&\quad = u\varphi + (1-u)\varphi \\
&\quad = \varphi.
\end{aligned}$$

Thus, $f(w) = f_1^u(w) f_2^{1-u}(w) \in S_{LH}(k, \gamma, \varphi)$. □

Corollary 2.4. *Let* $f_j(z) = h_j(z)\overline{g}_j(z)$, $j = 1, 2, \dots n$. *Then*

$$f(z) = f_1^{t_1}(z) f_2^{t_2}(z) \dots f_n^{t_n}(z) \in S_{LH}(k, \gamma, \varphi), \quad \left(0 \le t \le 1, \sum_{j=i}^{n} t_j = 1\right).$$

Corollary 2.5. *Let* $k_j(z) = h_j(z)\overline{g}_j(z) \in S_{LH}(k, \gamma, \zeta), (j = 1, 2)$ *with*

$$\zeta(z) = \int_0^z \frac{e^{i\gamma} d\rho}{(1 + \rho e^{i\theta})(1 + \rho e^{-i\theta})}, \quad (\theta \in \mathbb{R})$$

then for $(0 \le u \le 1)$, $f(w) = k_1^u(w) k_2^{1-u}(w) := h(w)\overline{g}(w) \in S_{LH}(k, \gamma, \zeta)$, *and the function* $\log f$ *is convex in the position* γ.

Proof. Let $k(w)$ be the function defined by (1.2). Now

$$\begin{aligned}
&\Re\left(\frac{we^{-i\gamma}(\frac{h'(w)}{h(w)} - e^{2i\gamma}\frac{g'(w)}{g(w)})}{k(w)}\right) \\
&\quad = \Re\left(\frac{we^{-i\gamma}}{k(w)}\left[u\frac{h_1'(w)}{h_1(w)} + (1-u)\frac{h_2'(w)}{h_2(w)}\right]\right) \\
&\quad\quad - \Re\left(\frac{we^{-i\gamma}}{k(w)}\left[e^{2i\gamma}\left(u\frac{g_1'(w)}{g_1(w)} - (1-u)\frac{g_2'(w)}{g_2(w)}\right)\right]\right) \\
&\quad = u.\Re\left(\frac{we^{-i\gamma}\varphi'(w)}{k(w)}\right) + (1-u).\Re\left(\frac{we^{-i\gamma}\varphi'(w)}{k(w)}\right) \\
&\quad = u + (1-u) \\
&\quad = 1 \\
&\quad > 0,
\end{aligned}$$

and so by using Lemma 1.2 we deduce $e^{-i\gamma}(\log h - e^{2i\gamma}\log g)$ is convex in the position of the real axis, and by Lemma 1.1 the function $\log f$ is convex in the position γ. On the other hand, according to the Theorem 2.3 we know $f(w) = k_1^u(w)k_2^{1-u}(w) \in S_{LH}(k, \gamma, \zeta)$ and the proof is complete. □

Corollary 2.6. *Suppose that* $\alpha \in [-1, 1]$, $\theta \in (0, \pi)$ *and* $a, b \ge 0, a + b \ne 0$. *Let* $f_j(w) = h_j(w)g_j(w) \in S_{LH}^{+}(k, \delta)$, $(j = 1, 2)$, *where*

$$\delta(w) = a\frac{w(1 - \alpha w)}{1 - w^2} + b\frac{1}{2i\sin\theta}\log\left(\frac{1 + we^{i\theta}}{1 + we^{-i\theta}}\right)$$

then $f(w) = f_1^t(w) f_2^{1-t}(w) \in S_{LH}^{+}(k, \delta)$, $(0 \le t \le 1)$, *and* $\log f$ *is convex in the position of the imaginary axis.*

Proof. If we take

$$\delta(w) = a\frac{w(1-\alpha w)}{1-w^2} + b\frac{1}{2i\sin\theta}\log\left(\frac{1+we^{i\theta}}{1+we^{-i\theta}}\right),$$

then it is proved in [10] that δ is convex in the position of the imaginary axis and so $\log f$ is convex in the position of the imaginary axis. Also by using Theorem 2.3 we have $f(w) = f_1^t(w)f_2^{1-t}(w) \in S_{LH}^+(k,\delta)$, $(0 \le t \le 1)$ and the proof is complete. □

Corollary 2.7. *Suppose that* $c \in [-2,2]$, $\theta \in (0,\pi)$ *and* $a, b \ge 0, a+b \ne 0$. *Let* $f_j(w) = h_j(w)g_j(w) \in S_{LH}^-(k,\eta)$, $(j=1,2)$, *where*

$$\eta(w) = a\log\left(\frac{1+w}{1-w}\right) + b\left(\frac{w}{1+cw+w^2}\right) \tag{2.1}$$

then $f_j(w) = k_1^u(w)k_2^{1-u}(w) \in S_{LH}^-(k,\eta)$, $(0 \le u \le 1)$ *and* $\log f$ *is convex in the position of the real axis.*

Proof. It has showed in [10] the function defined by (2.1) is convex in the position of the real axis and so $\log f$ is convex in the position of the real axis. Also by using Theorem 2.3 we have $f(w) = k_1^u(w)f_2^{1-u}(w) \in S_{LH}^-(k,\eta)$, $(0 \le u \le 1)$ and the proof is complete. □

Theorem 2.8. *Let* $f_1(z) = h_1(z)\overline{g}_1(z) \in S_{LH}(k,\gamma,\psi)$ *and* $f_2(z) = h_2(z)\overline{g}_2(z) \in S_{LH}(k,\gamma,\psi)$. *Suppose that*

$$\Re\left(k^2\frac{h_1'\overline{h}_2'}{h_1\overline{h}_2} - \frac{g_1'\overline{g}_2'}{g_1\overline{g}_2}\right) \ge 0,$$

and $u\varphi + (1-u)\psi$ *is convex in the direction* γ.

Then $f(w) = f_1^u(w)f_2^{1-u}(w) \in S_{LH}(k)$, $(0 \le u \le 1)$ *and* $\log f$ *is convex in the direction* γ.

Proof. By considering a as second dilation of f and a_1, a_2 second dilations of f_1 and f_2, respectively, we have

$$\begin{aligned}|a| &= \left|\frac{u\frac{g_1'(w)}{g_1(w)} + (1-u)\frac{g_2'(w)}{g_2(w)}}{u\frac{h_1'(w)}{h_1(w)} + (1-u)\frac{h_2'(w)}{h_2(w)}}\right| \\ &= \left|\frac{ua_1\frac{h_1'(w)}{h_1(w)} + (1-u)a_2\frac{h_2'(w)}{h_2(w)}}{u\frac{h_1'(w)}{h_1(w)} + (1-u)\frac{h_2'(w)}{h_2(w)}}\right|.\end{aligned}$$

Now for proving $|a| \le k$ it is sufficient to show that

$$k^2\left|u\frac{h_1'(w)}{h_1(w)} + (1-u)\frac{h_2'(w)}{h_2(w)}\right|^2 - \left|ua_1\frac{h_1'(w)}{h_1(w)} + (1-u)a_2\frac{h_2'(w)}{h_2(w)}\right|^2 \ge 0.$$

But by assumption, it follows that

$$
\begin{aligned}
&k^2\left|u\frac{h_1'(w)}{h_1(w)}+(1-u)\frac{h_2'(w)}{h_2(w)}\right|^2-\left|ua_1\frac{h_1'(w)}{h_1(w)}+(1-u)a_2\frac{h_2'(w)}{h_2(w)}\right|^2\\
&=k^2\left(u\frac{h_1'(w)}{h_1(w)}+(1-u)\frac{h_2'(w)}{h_2(w)}\right)\left(u\frac{\overline{h}_1'(w)}{\overline{h}_1(w)}+(1-u)\frac{\overline{h}_2'(w)}{\overline{h}_2(w)}\right)\\
&\quad-\left(ua_1\frac{h_1'(w)}{h_1(w)}+(1-u)a_2\frac{h_2'(w)}{h_2(w)}\right)\left(ua_1\frac{\overline{h}_1'(w)}{\overline{h}_1(w)}+(1-u)a_2\frac{\overline{h}_2'(w)}{\overline{h}_2(w)}\right)\\
&=u^2\left|\frac{h_1'(w)}{h_1(w)}\right|^2\left(k^2-|a_1|^2\right)+(1-u)^2\left|\frac{h_2'(w)}{h_2(w)}\right|^2\left(k^2-|a_2|^2\right)\\
&\quad+2u(1-u)\Re\left(k^2-a_1\overline{a}_2)\frac{h_1'(w)}{h_1(w)}\frac{\overline{h}_2'(w)}{\overline{h}_2(w)}\right)\\
&\geq 2u(1-u)\Re\left(k^2\frac{h_1'(w)}{h_1(w)}\frac{\overline{h}_2'(w)}{\overline{h}_2(w)}-\frac{g_1'(w)}{g_1(w)}\frac{\overline{g}_2'(w)}{\overline{g}_2(w)}\right)\\
&\geq 0,
\end{aligned}
$$

so $|a|\leq k<1$. Since

$$\log h_1(w)-e^{2i\gamma}\log g_1(w)=\varphi(w)$$

and

$$\log h_2(w)-e^{2i\gamma}\log g_2(w)=\psi(w),$$

we have

$$
\begin{aligned}
&\log h(w)-e^{2i\gamma}\log g(w)\\
&\quad=\log\left(h_1^u(w)h_2^{1-u}(w)\right)-e^{2i\gamma}\log\left(g_1^u(w)g_2^{1-u}(w)\right)\\
&\quad=u\log h_1(w)+(1-u)\log h_2(w)\\
&\qquad-e^{2i\gamma}\left(u\log g_1(w)+(1-u)log g_2(w)\right)\\
&\quad=u\log h_1(w)-e^{2i\gamma}\log g_1(w)+(1-u)\left(\log h_2(w)-e^{2i\gamma}\log g_2(w)\right)\\
&\quad=u\varphi+(1-u)\psi
\end{aligned}
$$

which is convex in the position γ by the assumption. Thus, $f(w)=f_1^u(w)f_2^{1-u}(w)\in S_{LH}(k)$, $(0\leq u\leq 1)$ and $\log f$ convex in the position γ. □

Theorem 2.9. *Let* $k_1(w)=h_1(w)\overline{g}_1(w)\in S_{LH}(k,\gamma,\vartheta)$ *and* $k_2(w)=h_2(w)\overline{g}_2(w)\in S_{LH}\left(k,\gamma+\frac{\pi}{2},\vartheta\right)$ *where*

$$\vartheta(w)=\int_0^w\frac{e^{i\gamma}d\xi}{\left(1+\xi e^{i\theta}\right)\left(1+\xi e^{-i\theta}\right)},\quad(\theta\in\mathbb{R}).$$

Suppose that

$$\Re\left(k^2\frac{h_1'\overline{h}_2'}{h_1\overline{h}_2}-\frac{g_1'\overline{g}_2'}{g_1\overline{g}_2}\right)\geq 0,$$

then $f(w)=k_1^u(w)k_2^{1-u}(w)\in S_{LH}(k)$, $(0\leq u\leq 1)$, *and* $\log f$ *is convex in the direction* γ.

Proof. By using similar argument in Theorem 2.8, the dilation $a(z)$ of $f(w)=k_1^u(w)k_2^{1-u}(z)$ satisfies $|a|\leq k<1$. Now we show that $\log f$ is convex in the direction γ. First we note that

$$\begin{aligned}\frac{h_2'(w)}{h_2(w)}-e^{2i\gamma}\frac{g_2'(w)}{g_2(w)}&=\left(\frac{h_2'(w)}{h_2(w)}+e^{2i\gamma}\frac{g_2'(w)}{g_2(w)}\right)\left(\frac{\frac{h_2'(w)}{h_2(w)}-e^{2i\gamma}\frac{g_2'(w)}{g_2(w)}}{\frac{h_2'(w)}{h_2(w)}+e^{2i\gamma}\frac{g_2'(w)}{g_2(w)}}\right)\\&=\vartheta'(w)\left(\frac{1-e^{2i\gamma}a_2}{1+e^{2i\gamma}a_2}\right)\\&=\vartheta'(w)p(w),\end{aligned}$$

where

$$p(w)=\left(\frac{1-e^{2i\gamma}a_2}{1+e^{2i\gamma}a_2}\right).$$

But it is obvious that $\Re(p(w))>0$. For the convexity of $\log f$ in the direction of γ we will use Lemma 1.2. Now

$$\begin{aligned}&\Re\left(\frac{we^{-i\gamma}\left(\frac{h'(w)}{h(w)}-e^{2i\gamma}\frac{g'(w)}{g(w)}\right)}{k(w)}\right)\\&=\Re\left(\frac{we^{-i\gamma}}{k(w)}\left[u\frac{h_1'(w)}{h_1(w)}+(1-u)\frac{h_2'(w)}{h_2(w)}-e^{2i\gamma}\left(u\frac{g_1'(w)}{g_1(w)}+(1-w)\frac{g_2'(w)}{g_2(w)}\right)\right]\right)\\&=u\Re\left(\frac{we^{-i\gamma}\varphi'(w)}{k(w)}\right)+(1-u)\Re\left(\frac{we^{-i\gamma}\vartheta'(w)p(w)}{k(w)}\right)\\&=u+(1-u)\Re(p(w))\\&>0.\end{aligned}$$

So $e^{-i\gamma}\left(\log h-e^{2i\gamma}\log g\right)$ is convex in the position of real axis, and hence the function $\left(\log h-e^{2i\gamma}\log g\right)$ is convex in the position γ or $\log f$ is convex in the position γ. This completes the proof. □

Theorem 2.10. *Let* $f_1(z)=h_1(z)\overline{g}_1(z)\in S_{LH}^-(k,\varphi)$ *and* $f_2(z)=h_2(z)\overline{g}_2(z)\in S_{LH}^-(k,\varphi)$ *where*

$$\varphi(z)=\frac{1}{2}\log\frac{1+z}{1-z},\quad (z\in E)$$

then $f(z)=f_1^u(z)f_2^{1-u}(z)\in S_{LH}^-(k,\varphi)$, $(0\leq u\leq 1)$ *and* $\log f$ *is convex.*

Proof. By considering Theorem 2.8, we know that

$$f(w) = f_1^u(w) f_2^{1-u}(w) \in S_{LH}^-(k,\varphi),$$

where $(0 \leq t \leq 1)$. On the other hand by Lemma 1.1 the convexity of $\log f$ is equivalent that analytic functions $(\log h - e^{2i\theta}\log g)$ are univalent and convex in the direction θ, for all $(0 \leq \theta < \pi)$.

Hence it is sufficient to show that the functions $F_\theta = ie^{-i\theta}(\log h - e^{2i\theta}\log g)$ are convex in the direction of the imaginary axis and are univalent. But

$$\begin{aligned}
&(\log h(w))' - (\log g(w))' \\
&\quad = \log\left(h_1^u(w) h_2^{1-u}(w)\right)' - \log\left(g_1^u(w) g_2^{1-u}(w)\right)' \\
&\quad = (u\log h_1(w) + (1-u)\log h_2(w) - u\log g_1(w) - (1-u)\log g_2(w))' \\
&\quad = u\frac{h_1'(w)}{h_1(w)} + (1-w)\frac{h_2'(w)}{h_2(w)} - u\frac{g_1'(w)}{g_1(w)} - (1-u)\frac{g_2'(w)}{g_2(w)} \\
&\quad = u\left(\frac{h_1'(w)}{h_1(w)} - \frac{g_1'(w)}{g_1(w)}\right) + (1-u)\left(\frac{h_2'(w)}{h_2(w)} - \frac{g_2'(w)}{g_2(w)}\right) \\
&\quad = \frac{1}{1-w^2}.
\end{aligned}$$

From Lemma 1.3, by taking $\mu = \nu = \frac{\pi}{2}$ we have

$$\begin{aligned}
&\Re\left(-ie^{i\frac{\pi}{2}}\left(1 - 2we^{i\frac{\pi}{2}}cos\frac{\pi}{2} + w^2 e^{-2i\frac{\pi}{2}}\right)F_\theta'(w)\right) \\
&\quad = \Re\left((1-z^2)F_\theta'(w)\right) \\
&\quad = -\Im\left(\left(1-w^2\right)e^{-i\theta}\left[\frac{h'(w)}{h(w)} - e^{2i\theta}\frac{g'(w)}{g(w)}\right]\right) \\
&\quad = -\Im\left(\left(1-w^2\right)\left[e^{-i\theta}\frac{h'(w)}{h(w)} - e^{i\theta}\frac{g'(w)}{g(w)}\right]\right) \\
&\quad = -\Im\left(1-w^2\right)\left((\cos\theta - i\sin\theta)\frac{h'(w)}{h(w)} - (\cos\theta + i\sin\theta)\frac{g'(w)}{g(w)}\right) \\
&\quad = -\Im\left(1-w^2\right)\left(\cos\theta\left(\frac{h'(w)}{h(w)} - \frac{g'(w)}{g(w)}\right) - i\sin\theta\left(\frac{h'(w)}{h(w)} + \frac{g'(w)}{g(w)}\right)\right) \\
&\quad = -\Im\left(\frac{1}{\frac{h'(w)}{h(w)} - \frac{g'(w)}{g(w)}}\right) \\
&\qquad \times\left(\cos\theta\left(\frac{h'(w)}{h(w)} - \frac{g'(w)}{g(w)}\right) - i\sin\theta\left(\frac{h'(w)}{h(w)} + \frac{g'(w)}{g(w)}\right)\right)
\end{aligned}$$

$$= -\Im\left(\cos\theta - i\sin\theta \frac{\frac{h'(w)}{h(w)} + \frac{g'(w)}{g(w)}}{\frac{h'(w)}{h(w)} - \frac{g'(w)}{g(w)}}\right)$$
$$= \Re \sin\theta p(w),$$

where

$$p(w) = \frac{\frac{h'(w)}{h(w)} + \frac{g'(w)}{g(w)}}{\frac{h'(w)}{h(w)} - \frac{g'(w)}{g(w)}},$$

It is obvious that $\Re(p(w)) > 0$ and we conclude the F_θ is convex in the position of the imaginary axis and is univalent. □

Corollary 2.11. *Let* $f_\rho(w) = h_\rho(w)\overline{g}_\rho(w) \in S^-_{LH}(k,\chi)$, $(\rho = 1, 2, \ldots, n)$ *where*

$$\chi(w) = \frac{1}{2}\log\frac{1+w}{1-w}, \quad (w \in E)$$

then $F = f_1^{c_1} f_2^{c_2} \cdots f_n^{c_n} \in S^-_{LH}(k,\chi)$, $(0 \leq c_i, \sum_1^n c_i = 1)$ *and* $\log f$ *is convex.*

By proceeding the same as the proof of Theorem 2.10 we obtain the following result.

Theorem 2.12. *Let* $k_\rho(w) = h_\rho(w)\overline{g}_\rho(w) \in S^+_{LH}(k,\upsilon)$, $(\rho = 1, 2)$ *where*

$$\upsilon(w) = \frac{1}{2}\log\frac{1+w}{1-w}, \quad (w \in E)$$

then $f(w) = k_1^u(w)k_2^{1-u}(w) \in S^+_{LH}(k,\upsilon)$, $(0 \leq u \leq 1)$ *and* $\log f$ *is convex.*

Corollary 2.13. *Let* $k_\rho(w) = h_\rho(w)\overline{g}_\rho(w) \in S^+_{LH}(k,\kappa)$, $(\rho = 1, 2, \ldots, n)$ *where*

$$\kappa(w) = \frac{w}{1-w}, \quad (w \in E)$$

then $F(w) = k_1^{c_1}(w)k_2^{c_2}(w)\cdots k_n^{c_n} \in S^+_{LH}(k,\kappa)$, $(0 \leq c_i, \sum_1^n c_i = 1)$ *and* $\log F$ *is convex.*

3. Exponent Product

Theorem 3.1. *Let* $(\beta > -\frac{1}{2})$ *and* ϱ *be analytic convex function in the position of real axis. If* $k(w) \in S^-_{LH}(1,\varrho)$ *then* $K(w) = k(w)|k(w)|^{2\beta} \in S^-_{LH}(1,\varrho)$ *and* $\log K$ *is convex in position of real axis.*

Proof. Let $k(w) = h(w)\overline{g(w)}$, then

$$\begin{aligned} K(w) &= k(w)|k(w)|^{2\beta} \\ &= k(w)f^\beta(w)\overline{k}^\beta(w) \\ &= (h(w)\overline{g}(w))^{1+\beta}(\overline{h}(w)g(w))^\beta \end{aligned}$$

$$= M(w)\overline{N}(w),$$

where

$$M(w) = h^{1+\beta}(w)g^{\beta}(w), \qquad N(w) = h^{\beta}(w)g^{1+\beta}(w). \tag{3.1}$$

Also let $k(w) = h(w)\overline{g(w)}$ and $\hat{a}, a$ denote the second dilations of the functions $K, k($ respectively); that is

$$\frac{\overline{K}_{\overline{z}}(w)}{\overline{K}(w)} = \hat{a}\frac{K_z(w)}{K(w)}.$$

Now

$$\hat{a}(z) = \frac{\frac{\overline{K}_{\overline{z}}(w)}{\overline{K}(w)}}{\frac{K_z(w)}{K(w)}},$$

or

$$\begin{aligned}\hat{a}(w) &= \frac{(1+\beta)\frac{\overline{k}_{\overline{z}}(w)}{\overline{k}(w)} + \beta\frac{k_z(w)}{k(w)}}{(1+\beta)\frac{k_z(w)}{k(w)} + \beta\frac{\overline{k}_{\overline{z}}(w)}{\overline{k}(w)}} \\ &= \frac{(1+\beta)a(w)\frac{k_z(w)}{k(w)} + \beta\frac{k_z(w)}{k(w)}}{(1+\beta)\frac{k_z(w)}{k(w)} + \beta a(w)\frac{k_z(w)}{k(w)}} \\ &= \frac{a(w) + \frac{\beta}{1+\beta}}{1 + a(w)\frac{\beta}{1+\beta}}.\end{aligned}$$

It is clear that

$$\begin{aligned}|\hat{a}(w)| &= \left|\frac{a(w) + \frac{\overline{\beta}}{(1+\overline{\beta})}}{1 + a(w)\frac{\beta}{1+\beta}}\right| \\ &< 1,\end{aligned}$$

provided that $|\beta|^2 < |1+\beta|^2$, which evidently holds since $(\beta > -\frac{1}{2})$. On the other hand by hypothesis of Theorem and using Lemma 1.3, there are numbers α, γ with $0 \leq \alpha < 2\pi$ and $0 \leq \gamma < 2\pi$ such that

$$\Re\left(e^{i\gamma}\left(1 - 2we^{i\gamma}\cos\alpha + w^2e^{-2i\gamma}\right)\varphi'(w)\right) \geq 0, \quad (w \in E).$$

Let

$$\psi(w) = \log\frac{M(w)}{N(w)}.$$

Then

$$\begin{aligned}&\Re\left(e^{i\gamma}\left(1 - 2we^{i\gamma}\cos\alpha + w^2e^{-2i\gamma}\right)\psi'(w)\right) \\ &\quad = \Re\left(e^{i\gamma}\left(1 - 2we^{i\gamma}\cos\alpha + w^2e^{-2i\gamma}\right)\varphi'(w)\right) \\ &\quad \geq 0,\end{aligned}$$

which means that $\log K$ is convex function in the direction of real axis and the proof is complete. □

Theorem 3.2. *Let* $k_1, k_2 \in S^-_{LH}(1;\varphi)$, $\alpha_1 > -\frac{1}{2}, \alpha_2 > -\frac{1}{2}$ *and*

$$K_1(w) = k_1(w)|k_1(w)|^{2\alpha_1}, K_2(w) = k_2(w)|k_2(w)|^{2\alpha_2},$$

then

$$K(w) = K_1^{\lambda}(w)K_2^{1-\lambda}(w) \in S^-_{LH}(1,\varphi)$$

Proof. According to the definitions of K_1 and K_2 we have $K_1 \in S^-_{LH}(1,\varphi)$, $K_2 \in S^-_{LH}(1,\varphi)$ and so by Theorem 3.1, $K \in S^-_{LH}(1,\varphi)$. □

Acknowledgment. The author would like to express his gratitude to the editor and the anonymous referees for their valuable comments and suggestions which have improved the original paper.

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