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Abstract. A new class of retro Banach frames called retro bi-Banach frame has been introduced and studied with illustrative examples. Relationships of a retro bi-Banach frame with various existing classes of Banach frame are presented. In the sequel, we deal with characterizations of the near-exact retro Banach frame and discuss the invariance of near-exact retro Banach frames under block perturbation. Finally, applications regarding the rank of a matrix and eigenvalue problems have been demonstrated.

1. Introduction

In 1952, the concept of frame was introduced by Duffin and Schaeffer [8], in the setting of Hilbert spaces. In fact, Duffin and Schaeffer abstracted Gabor's method [11] to define Hilbert frames. Years later, in 1986, Daubechies, Grossmann, and Meyer [9] realized the potential of frames and found new applications to wavelets, Gabor transforms, and in various areas of pure and applied mathematics, like, powerful tools from the operator theory and Banach spaces are being employed to study frames [12, 13, 17–20, 22]. More detailed towards the utility of frames can be found in the excellent books by O. Christensen [6, 7]. After this landmark work [9], the theory of frames began to be widely studied.

Recall that, a frame for the Hilbert space H is a sequence $\{f_i\}_{i=1}^{\infty} \subset H$ of vectors satisfying:

$$C\left\|g\right\|^2 \leq \sum_{i=1}^{\infty} \left|\langle g, f_i\rangle\right|^2 \leq D\left\|g\right\|^2, \quad \text{for all } g \in H,$$

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where C and D are bounds with $0 < C \leq D < \infty$.

The positive constants C and D are known as the lower and upper frame bounds, respectively. These constants are not unique. The operator $\mathcal{T} : l^2 \to H$ defined by

$$\mathcal{T}(\{\alpha_k\}_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_k f_k, \quad \text{for all } \{\alpha_k\}_{k\in\mathbb{N}} \in l^2,$$

is called the pre-frame operator or the synthesis operator. The adjoint operator $\mathcal{T}^* : H \to l^2$ defined by

$$\mathcal{T}^*(f) = \{\langle f, f_k\rangle\}_{k\in\mathbb{N}}, \quad \text{for all } f \in H,$$

is known as the analysis operator. The frame operator is obtained by composing $\mathcal{T}$ and $\mathcal{T}^*$ as $\mathcal{S} = \mathcal{T}\,\mathcal{T}^* : H \to H$ defined by

$$\mathcal{S}(f) = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, \quad \text{for all } f \in H.$$

Thus, it follows that

$$\langle \mathcal{S}f, f\rangle = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2, \quad \text{for all } f \in H.$$

So, the frame operator $\mathcal{S}$ is a self adjoint, positive and invertible operator on H. Hence the reconstruction formula, for all $f \in H$, is given by

$$\begin{aligned} f &= \mathcal{S}\mathcal{S}^{-1} f \\ &= \sum_{k=1}^{\infty} \left\langle \mathcal{S}^{-1} f, f_k \right\rangle f_k \\ &= \sum_{k=1}^{\infty} \left\langle f, \mathcal{S}^{-1} f_k \right\rangle f_k. \end{aligned}$$

Nowadays, frames play a crucial role in many areas of pure and applied mathematics and engineering sciences. An important and very useful property of frames is that they allow the representation of each element of the space as a linear combination of the elements in the frame, although, linear independence among the frame elements is not required, i.e. the corresponding coefficients are not necessarily unique. Thus, a frame might not be a basis that shows the advantage of frames over the basis.

A remarkable fact in Hilbert spaces is that the reconstruction formula can be obtained by the norm equivalence hypothesis. However, in Banach spaces, this does not hold in general. Motivated by this fact, Feichtinger and Grochenig [10], considered a decomposition of a Banach space and introduced atomic decomposition in 1988. Later, in 1991, the Banach frame was introduced by Grochenig [13] in the setting of Banach spaces. The frames were further generalized by many authors, namely, fusion frames [4], G-frames [23, 29], p-frames [2], PG-frames [1], K-atomic decomposition [25], Λ-Banach frame [12], $\tilde{X}$-frame [14], X_d-frames [5], Schauder frames [3], Frechet frame [26] and etc.

In 2004, Jain et al.[17] extended Banach frames to the conjugate Banach spaces and introduced retro Banach frames. In 2008, Raj Kumar and Sharma [24] studied the exactness of retro Banach frames and introduced near-exact retro Banach frames. In the sequel, in 2017, S. Jahan introduced the notion of a strong retro Banach frame [15] and a $\mathcal{J}$-frame [16] in Banach spaces. Recently, N. N. Jha and Shalu Sharma [21] deal with a crucial aspect of the retro Banach frame called block sequences.

1.1. **Purpose and Significance of Main Results.** In this paper, we interplayed the concept of retro Banach frames and bi-Banach frames and introduced the idea of retro bi-Banach frames. More precisely, the purpose of this paper is to extend bi-Banach frames to the conjugate Banach spaces and analyze the relationship of various existing frames in Banach spaces to the retro bi-Banach frames. These observations are given by an arrow diagram which provides a systematic approach to further studying the Banach frame theory. Our main result provides conditions for the existence of a retro bi-Banach frame and solves the problems raised in [27]. In the sequel, some characterization of near-exact retro Banach frames and their block perturbation are discussed. Finally, we conclude with applications regarding rank and eigenvalue.

The rest of the paper is organized as follows. Section 2 contains some notations, needed definitions, and key results which will be used in the subsequent part of the paper. In section 3, the retro bi-Banach frame has been defined and studied with the help of examples. In the sequel, some observations on retro bi-Banach frames have been discussed. Also, we deal with problems raised by Shalu Sharma in [27] and provide positive answers. In section 4, some characterizations of near-exact retro Banach frames are given. Finally, in section 5, we provide applications of the bi-Banach frame and near-exact retro Banach frames.

2. Basic Definitions and Needed Results

Throughout this paper, a Banach space and its first dual will be denoted by E and E^*, respectively, over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). E_d and $(E^*)_d$ denote the Banach spaces of scalar-valued sequences associated with E and E^*, respectively. Also, $[x_n]$ denotes the closed linear subspace of E, spanned by $\{x_n\}$. The set of positive integers will be denoted by $\mathbb{N}$. We shall refer to the abbreviation RBF for the retro Banach frame.

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$ and is said to be total if $\{f \in E^* : f(x_n) = 0, \forall n \in \mathbb{N}\} = \{0\}$.

The concept of the Banach frame was defined by Grochenig [13] as follows:

Definition 2.1 ([13])**.** Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a Banach frame for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$,

(ii) there exist constants A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq A_2\|x\|_E, \quad x \in E, \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S\left(\{f_n(x)\}\right) = x, \quad x \in E.$$

The positive constants A_1 and A_2 are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : E_d \to E$ is called the reconstruction operator. The inequality (2.1) is called the Banach frame inequality.

The Banach frame $(\{f_n\}, S)$ is said to be exact if there exists no reconstruction operator S_0 such that $(\{f_n\}_{n\neq i}, S_0)$ is a Banach frame for E.

Remark 2.2 ([18])**.** Let $(\{f_n\}, S)$ be an exact Banach frame for E. Then there exists a sequence $\{x_n\} \subset E$, called an admissible sequence of vectors to $(\{f_n\}, S)$ such that

$$\begin{aligned} f_i(x_j) &= \delta_{i,j} \\ &= \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \text{for all } i, j \in \mathbb{N}. \end{aligned}$$

Next, we see the definition of a retro Banach frame which was the extension of frames in conjugate Banach spaces.

Definition 2.3 ([17])**.** Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences indexed by $\mathbb{N}$ associated with E^*. Let $\{x_n\}$ be a sequence in E and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a retro Banach frame (RBF) for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$,

(ii) there exist constants A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1 \|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq A_2 \|f\|_{E^*}, \quad \text{for all } f \in E^*, \tag{2.2}$$

(iii) T is a bounded linear operator such that

$$T(\{f(x_n)\}) = f, \quad \text{for all } f \in E^*.$$

The positive constants A_1 and A_2 are called, lower and upper frame bounds of the retro Banach frame $(\{x_n\}, T)$, respectively. The operator $T : (E^*)_d \to E^*$ is called the reconstruction operator for the retro Banach frame $(\{x_n\}, T)$. The inequality (2.2) is called the retro frame inequality.

An RBF $(\{x_n\}, T)$ is called tight if $A_1 = A_2$ and is called normalized tight if $A_1 = A_2 = 1$. If removal of one x_n renders the collection $\{x_n\} \subset E$ no longer an RBF for E^*, then $(\{x_n\}, T)$ is called an exact RBF.

In 2014, S. Sharma [27] introduced the notion of the bi-Banach frame and gave the following definition:

Definition 2.4 ([27])**.** Let E be a Banach space. A pair $(\{x_n\}, \{f_n\})$ (where $\{f_n\} \subset E^*$ and $\{x_n\} \subset E$) is called a bi-Banach frame for E if there exist associated Banach spaces E_d and $(E^*)_d$ and bounded linear operators $S : E_d \to E$, $T : (E^*)_d \to E^*$ such that $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, T)$ is a retro Banach frame for E^*.

Definition 2.5 ([3])**.** Let E be a Banach space and let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^*. Then the pair $(\{x_n\}, \{f_n\})$ is called a Schauder frame for E if

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \quad \text{for all } x \in E.$$

Definition 2.6 ([20])**.** Let $(\{f_n\}, S)$ (where $\{f_n\} \subset E^*$ and $S : E_d \to E$) be a Banach frame for E with respect to E_d. Let $\{\Phi_n\} \in E^{**}$ be a sequence such that $\Phi_i(f_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. If there exists an associated Banach space $(E^*)_d$ and a reconstruction operator $T : (E^*)_d \to E^*$ such that $(\{\Phi_n\}, T)$ is a Banach frame for E^*, then the system $((\{f_n\}, S), (\{\Phi_n\}, T))$ is called a Banach frame system for E.

We finish this section with some key results required for the subsequent part of the paper. In the sequel, the first result is very useful in the theory of frames in Banach spaces as we shall see in the next section that this result provides the existence of associated Banach spaces and frames in Banach spaces.

Lemma 2.7 ([28])**.** *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the BK-space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Lemma 2.8 ([17])**.** *Let E be a Banach space. Then E^* has a retro Banach frame if and only if E is separable.*

Lemma 2.9 ([17]). *Let* $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ *be a retro Banach frame for* E^* *with respect to* $(E^*)_d$. *Then* $(\{x_n\}, T)$ *is exact if and only if* $x_n \notin [x_i]_{i \neq n}$.

3. Retro bi-Banach Frame

In 2014, the Banach frame was generalized to the bi-Banach frame as a pair of Banach frame and retro Banach frame, by Shalu Sharma [27]. In this section, we extend this concept to the conjugate Banach space and define the retro bi-Banach frame as follows:

Definition 3.1. Let E be a Banach space. Let $\{x_n\}$ and $\{f_n\}$ be sequences in E and E^*, respectively. Then the system $(\{x_n\}, \{f_n\})$ is called a retro bi-Banach frame for E^* if there exist associated Banach spaces $(E^*)_d$ and $(E^{**})_d$ and bounded linear operators $S : (E^*)_d \to E^*$, $T : (E^{**})_d \to E^{**}$ such that $(\{x_n\}, S)$ is an RBF for E^* with respect to $(E^*)_d$ and $(\{f_n\}, T)$ is an RBF for E^{**} with respect to $(E^{**})_d$.

A retro bi-Banach frame $(\{x_n\}, \{f_n\})$ is said to be

- exact if each of retro Banach frames corresponding to E^* and E^{**} are exact;
- near-exact if either of retro Banach frames corresponding to E^* and E^{**} is near-exact;
- tight if each of retro Banach frames corresponding to E^* and E^{**} are tight.

Towards the existence of retro bi-Banach frame, we furnish the following examples. Note that, in Example 3.2 and Example 3.3 we consider the Banach space $E = c_0$ which is a normed linear space of null sequences with norm given by $\|x\| = \sup |x_k|$, $x = \{x_k\} \in E$.

Example 3.2. Let $E = c_0$. Let $\{x_n\}$ and $\{f_n\}$ are the sequences of unit vectors in E and E^*, respectively. Then, by Lemma 2.7 there exist associated Banach spaces $(E^*)_d$, $(E^{**})_d$ and bounded linear operators $S : (E^*)_d \to E^*$, $T : (E^{**})_d \to E^{**}$ such that $(\{x_n\}, S)$ is an RBF for E^* and $(\{f_n\}, T)$ is an RBF for E^{**}. Hence, $(\{x_n\}, \{f_n\})$ is a retro bi-Banach frame for E^*.

Example 3.3. Let $E = c_0$. Define sequences $\{x_n\} \subset E$ and $\{h_n\} \subset E^*$ by

$$x_n = \frac{1}{n}(e_{n+1} - e_n), \quad \text{for all } n \in \mathbb{N},$$

and

$$h_n = f_{n+1} - f_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{e_n\}$ and $\{f_n\}$ are sequences of unit vectors in E and E^*, respectively. Lemma 2.7 ensures the existence of associated Banach spaces

$(E^*)_d$, $(E^{**})_d$ and bounded linear operators $S : (E^*)_d \to E^*$, $T : (E^{**})_d \to E^{**}$ such that $(\{x_n\}, S)$ is an RBF for E^* and $(\{h_n\}, T)$ is an RBF for E^{**}. Hence, $(\{x_n\}, \{h_n\})$ is a retro bi-Banach frame for E^*.

Example 3.4. Let $E = l^1$ and let $\{x_n\}$ be a sequence of unit vectors in E. Then, Lemma 2.7 ensures the existence of associated Banach space $(E^*)_d$ and a bounded linear operator $S : (E^*)_d \to E^*$ such that $(\{x_n\}, S)$ is an RBF for E^*. However, according to Lemma 2.8, there does not exists a reconstruction operator $T : (E^{**})_d \to E^{**}$ such that $(\{f_n\}, S)$ is an RBF for E^{**}. Hence, $(\{x_n\}, \{f_n\})$ is not a retro bi-Banach frame for E^*.

Notice that the following observations show the essence of the retro bi-Banach frame in the study of frames in conjugate Banach spaces.

(O1) Let E be a reflexive Banach space. If E^* has a retro bi-Banach frame then E has a bi-Banach frame. Indeed, let $(\{x_n\}, \{f_n\})$ be a retro bi-Banach frame for E^*. So that, $(\{x_n\}, S)$ is an RBF for E^* and $(\{f_n\}, T)$ is an RBF for E^{**}. Let π be the canonical isomorphism of E^* into E^{***}. Then $(\{\pi(f_n)\}, T)$ is a Banach frame for $E^{**} = E$ (see page 715 in [17]). Hence, $(\{x_n\}, \{\pi(f_n)\})$ is a bi-Banach frame for E.

(O2) Converse of **(O1)** need not be true. For example, let $E = l^p$ $(1 < p < \infty)$. Define $\{x_n\} \in E$ and $\{f_n\} \in E^*$ by

$$x_1 = e_1, \qquad x_n = (-1)^n e_1 + e_n, \quad n \geq 2,$$

and

$$f_n = \hat{e}_n, \quad n \in \mathbb{N},$$

where $\{e_n\}$ and $\{\hat{e_n}\}$ be sequences of unit vectors in E and E^*, respectively. Then by Lemma 2.7, there exist associated Banach spaces E_d and $(E^*)_d$ and bounded linear operators $S : E_d \to E$, $T : (E^*)_d \to E^*$ such that $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, T)$ is an RBF for E^*. Hence, $(\{x_n\}, \{f_n\})$ is a bi-Banach frame for E. However, if we define $\{\varphi_n\} \in E^*$ by

$$\varphi_1 = f_2, \qquad \varphi_n = f_n, \quad n \geq 2,$$

then there exists no reconstruction operator T_0 such that $(\{\varphi_n\}, T_0)$ is an RBF for E^{**}. Hence, E^* has no retro bi-Banach frame.

(O3) Let E be a reflexive Banach space. If E^* has a retro bi-Banach frame then E has a Banach frame system. Indeed, let $(\{x_n\}, \{f_n\})$ be a retro bi-Banach frame for E^*. So that, $(\{x_n\}, S)$ is an RBF for E^* and $(\{f_n\}, T)$ is an RBF for E^{**}. Let $\pi : E \to E^{**}$ be the canonical isomorphism defined by $\pi(x_n) = \varphi_n$, $n \in \mathbb{N}$. Then $(\{\varphi_n\}, S)$ is a Banach frame for E^*.

Now, take $\psi_n \in E^{***} = E^*$ such that $\varphi_i(\psi_j) = \delta_{ij}$ and define, again, a canonical isomorphism $\pi' : E^* \to E^{***}$ by $\pi'(f_n) = \psi_n$, $n \in \mathbb{N}$. Then $(\{\psi_n\}, T)$ is a Banach frame for $E^{**} = E$. Hence, $((\{\psi_n\}, T), (\{\varphi_n\}, S))$ is a Banach frame system for E.

(O4) Converse of **(O3)** need not be true. For example, let $E = l^1$ and let $\{f_n\}$ be the sequence of unit vectors in E^*. Define $\{\varphi_n\} \subset E^{**}$ by

$$\varphi_n(x) = \xi_n, \qquad \text{where } x = \{\xi_n\} \in l^\infty, \quad n \in \mathbb{N}.$$

Then, there exists associated Banach spaces $E_d = \{\{f_n(x)\} : x \in E\}$ and $(E^*)_d = \{\{\varphi_n(f)\} : f \in E^*\}$ with norm given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$ and $\|\{\varphi_n(f)\}\|_{(E^*)_d} = \|f\|_{E^*}$, $f \in E^*$, respectively. Define $S : E_d \to E$ by $S(\{f_n(x)\}) = x$, $x \in E$ and $T : (E^*)_d \to E^*$ by $T(\{\varphi_n(f)\}) = f$, $f \in E^*$. Then S and T are bounded linear operators such that $((\{f_n\}, S), (\{\varphi_n\}, T))$ is a Banach frame system for E. However, E^* is not separable, it follows by Lemma 2.8 that, E^{**} has no RBF. Hence, E^* has no retro bi-Banach frame.

(O5) *If* $(\{x_n\}, \{f_n\})$ *is a Schauder frame for* E*, then* $(\{x_n\}, \{f_n\})$ *is a tight retro bi-Banach frame for* E^**.* Indeed, since $\{x_n\}$, $\{f_n\}$ are total sequences, it follows that, there are associated Banach spaces $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ and $(E^{**})_d = \{\{\phi(f_n)\} : \phi \in E^{**}\}$ and bounded linear operators $S : (E^*)_d \to E^*$, $T : (E^{**})_d \to E^{**}$ such that $(\{x_n\}, S)$ is a tight RBF for E^* and $(\{f_n\}, T)$ is a tight RBF for E^{**}. Hence, $(\{x_n\}, \{f_n\})$ is a tight retro bi-Banach frame for E^*.

(O6) Converse of **(O5)** need not be true. For example, let $E = c_0$. Define sequences $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ by

$$x_1 = e_1, \qquad x_n = \frac{1}{n} e_{n-1}, \quad n \geq 2,$$

where $\{e_n\}$ is a sequence of unit vectors in E and

$$f_1(x) = \xi_1, \qquad f_2(x) = \xi_2, \qquad f_n(x) = \xi_{n-1}, \quad n \geq 3,$$

for every $x = \{\xi_n\} \in E$. Then, there exists associated Banach spaces $(E^*)_d$ and $(E^{**})_d$ with norms $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$, $f \in E^*$ and $\|\{g(f_n)\}\|_{(E^{**})_d} = \|g\|_{E^{**}}$, $g \in E^{**}$, respectively. Define operators $T : (E^*)_d \to E^*$ and $U : (E^{**})_d \to E^{**}$ by $T(\{f(x_n)\}) = f$, $f \in E^*$ and $U(\{g(f_n)\}) = g$, $g \in E^{**}$, respectively . Then $(\{x_n\}, T)$ is an RBF for E^* and $(\{f_n\}, U)$ is an RBF for E^{**}. Hence, $(\{x_n\}, \{f_n\})$ is a retro bi-Banach frame for E^*. However,

$$\sum_{n=1}^{\infty} f_n(e_2)x_n = \frac{1}{2}e_1 + \frac{1}{3}e_2$$
$$\neq e_2,$$

which shows that $(\{x_n\}, \{f_n\})$ is not a Schauder frame.

(O7) *If E^* has an exact retro bi-Banach frame then E^* also has an exact RBF but converse need not be true.* For example, let $E = c_0$ and $\{x_n\}$ be a sequence of unit vectors in E. Then by Lemma 2.7, there exists an associated Banach space $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ and a bounded linear operator $T : (E^*)_d \to E^*$ such that $(\{x_n\}, T)$ is an RBF for E^*. Since, $x_n \notin [x_i]_{i \neq n}$. Hence $(\{x_n\}, T)$ is an exact RBF for E^*. Next, define $\{f_n\} \subset E^*$ by,

$$f_1 = x_1 \qquad \text{and} \qquad f_n = x_{n-1}, \quad \text{for } n \geq 2.$$

Lemma 2.7 ensures the existence of an associated Banach space $(E^{**})_d = \{\{\Phi(f_n)\} : \Phi \in E^{**}\}$ with norm given by

$$\|\{\Phi(f_n)\}\|_{(E^{**})_d} = \|\Phi\|_{E^{**}}, \quad \Phi \in E^{**},$$

together with a reconstruction operator $S : (E^{**})_d \to E^{**}$ such that $(\{f_n\}, S)$ is an RBF for E^{**} with respect to $(E^{**})_d$. Further, since $f_1 \in [f_n]_{n \neq 1}$, it follows that $(\{f_n\}, S)$ is a non-exact RBF. Hence, $(\{x_n\}, \{f_n\})$ is not an exact retro bi-Banach frame.

In the following result, we will obtain a necessary and sufficient condition for the existence of a retro bi-Banach frame.

Theorem 3.5. *Let E be a Banach space. The conjugate space E^* has a retro bi-Banach frame if and only if E^{**} has an RBF.*

Proof. Direct implication is obvious. For converse, let $(\{f_n\}, U)$ (where $\{f_n\} \subset E^*$, $U : (E^{**})_d \to E^{**}$) be an RBF for E^{**} with respect to $(E^{**})_d$. It follows by Lemma 2.8 that, E^* is separable and hence E is separable. Consequently, there exists a sequence $\{x_n\}$ in E such that $[x_n] = E$. Again by Lemma 2.7, there exists a bounded linear operator $T : \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is an RBF for E^*. Hence, $(\{x_n\}, \{f_n\})$ is a retro bi-Banach frame for E^*. $\square$

Moreover, Shalu Sharma [27], on the discussion of bi-Banach frames, raised the following two problems:

(P1) Let E has a Banach frame system. Does E has a bi-Banach frame?

(P2) Let E has a Banach frame system. Does E^* has an RBF?

In order to solve these problems, we need to prove the following result, which gives positive answers to these questions.

Theorem 3.6. *Let E be a reflexive Banach space. If E has a Banach frame system then E is separable.*

Proof. Let $((\{f_n\}, S), (\{\varphi_n\}, T))$ be a Banach frame system for E, where $\{f_n\}$ and $\{\varphi_n\}$ are sequences in E^* and E^{**}, respectively, and S, T are corresponding reconstruction operators. On contrary, assume that E is not separable. Then $[\varphi_n] \neq E$. Therefore there exists a nonzero functional $f \in E^*$ such that $\varphi_n(f) = 0$, for each $n \in \mathbb{N}$. Thus the frame inequality for $(\{\varphi_n\}, T)$ yields, $f = 0$. This contradicts the fact that f is nonzero. Hence, E is separable. $\square$

Remark 3.7. Converse of Theorem 3.6 need not be true. It follows in view of Example 3.8.

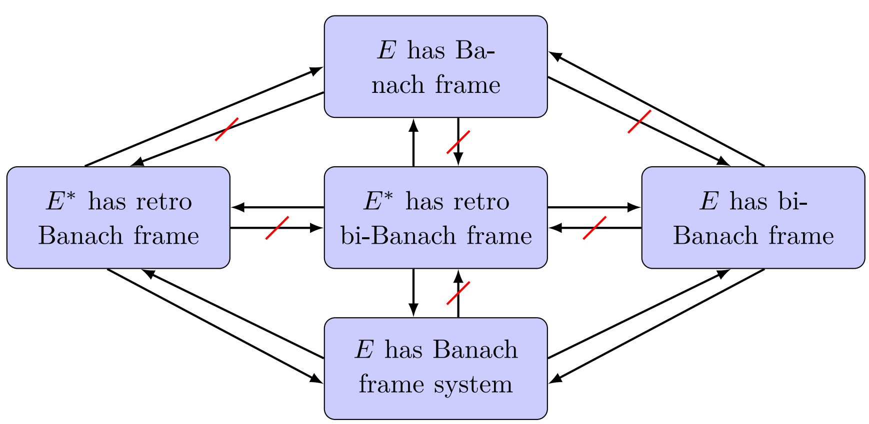
Example 3.8. Let $E = l^1$ and let $\{f_n\}$ be the sequence of unit vectors in E^*. Then, Lemma 2.7 ensures the existence of an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ with norm $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$. Define $S : E_d \to E$ such that $S(\{f_n(x)\}) = x$, $x \in E$. Then, S is a bounded linear operator such that $(\{f_n\}, S)$ is a Banach frame for E. Further, define $\{\varphi_n\} \subset E^{**}$ by

$$\varphi_1(f) = \xi_2, \qquad \varphi_n(f) = \xi_n, \quad n \geq 2, \quad (f = \{\xi_n\} \subset E^*, n \in \mathbb{N}).$$

Then, there exists no reconstruction operator T such that $(\{\varphi_n\}, T)$ is a Banach frame for E^{**}. Hence $((\{f_n\}, S), (\{\varphi_n\}, T))$ is not a Banach frame system for E.

Remark 3.9. Positive answers for both the problems (P1) and (P2), follows, in view of Theorem 3.6 and Lemma 2.8.

So far from our discussion and observations are given in [27], the final picture regarding the relationship among various concepts with the newly introduced notion of retro bi-Banach frame is given in the following diagram. (Here, E is a reflexive Banach space)



4. Near-Exact Retro Banach Frame

Near-exact retro Banach frame was introduced by Kumar and Sharma in [24]. They gave the following definition.

Definition 4.1 ([24])**.** A retro Banach frame $(\{x_n\}, T)$ (where $\{x_n\} \subset E$, $T : (E^*)_d \to E^*$) for E^* is said to be near-exact retro Banach frame for E^*, if it can be transformed into an exact retro Banach frame by omitting a finite number of its elements.

The sufficient condition for the existence of a near-exact Banach frame is given in Theorem 3.4 in [19]. Using a similar argument, we can obtain a sufficient condition for the existence of near-exact RBF as an extension of Theorem 3.4 in [19].

Theorem 4.2. *Let $(\{x_n\}, T)$ be an RBF for a separable Banach space E^*, where $\{x_n\} \subset E$ and T be a reconstruction operator. Then the system $(\{x_n\}, T)$ is near-exact if for every infinite sequence $\{\alpha(k)\}_{k=1}^{\infty}$ of positive integers,*

$$[x_i]_{i \neq \alpha(1), \alpha(2), \dots} \neq E.$$

Proof. A slight variation in the similar arguments given in the proof of Theorem 3.4 in [19], follows the conclusion. □

Let E and F be two Banach spaces. Let τ be a continuous linear operator from F onto E and τ^* is its adjoint. If E^* has an RBF, then by the following result we may obtain an RBF for F^*. This result generalizes Theorem 3.2 of [22].

Theorem 4.3. *Let E and F be two reflexive Banach spaces and let E^* having an RBF $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ with respect to $(E^*)_d$. Let $\{y_n\} \subset F$. If there exists a continuous linear mapping τ from E^* onto F^* such that $\tau^*(y_n) = x_n$, $n \in \mathbb{N}$, then there exists a system $(\{y_n\}, S)$ which is normalized tight RBF for F^* with respect to the Banach space $(F^*)_d$ associated with F^*, where $S : (F^*)_d \to F^*$ is a reconstruction operator. Moreover, if $(\{x_n\}, T)$ is exact, then $(\{y_n\}, S)$ is also exact.*

Proof. Since the linear mapping $\tau : E^* \to F^*$ is surjectjive, for every $g \in F^*$, there is a functional $f \in E^*$ such that $\tau(f) = g$. Let $g(y_n) = 0$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned} f(x_n) &= f(\tau^*(y_n)) \\ &= \tau^{**}(f)(y_n) \\ &= (\tau f)(y_n) \\ &= g(y_n) \end{aligned}$$

$$= 0, \quad \forall\, n \in \mathbb{N}.$$

Therefore, it follows, by retro frame inequality of the RBF $(\{x_n\}, T)$ that, $f = 0$ and so $g = 0$. Consequently, $\{y_n\}$ is total over F. Lemma 2.7, ensures the existence of an associated Banach space $(F^*)_d = \{\{g(y_n)\} : g \in F^*\}$ with the norm $\|\{g(y_n)\}\|_{(F^*)_d} = \|g\|_{F^*}$, $g \in F^*$. Define $S \in B((F^*)_d, F^*)$ by $S(\{g(y_n)\}) = g$, $g \in F^*$. Then $(\{y_n\}, S)$ is a normalized tight RBF for F^* with respect to $(F^*)_d$.

Further, we have $x_n \notin [x_i]_{i \neq n}$, for all $n \in \mathbb{N}$, since the RBF $(\{x_n\}, T)$ is exact. Therefore, there exists a sequence $\{f_n\} \subset E^*$ satisfying $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Now, take $\tau(f_n) = g_n$, for all $n \in \mathbb{N}$. Then, the sequence $\{g_n\} \in F^*$ is such that

$$\begin{aligned} g_i(y_j) &= \tau(f_i)(y_j) \\ &= f_i(\tau^*(y_j)) \\ &= f_i(x_j) \\ &= \delta_{i,j}, \quad i, j \in \mathbb{N}. \end{aligned}$$

This shows that $y_n \notin [y_i]_{i \neq n}$, for all $n \in \mathbb{N}$. Consequently, it follows by Lemma 2.9 that, the system $(\{y_n\}, S)$ is exact. □

Perturbation theory plays an important role in many areas of applied mathematics. In frame theory, the fundamental perturbation result was given by Young [30] in 1980. Since then, various generalizations of perturbations to Hilbert and Banach frame theory have been studied. One of them is block perturbation. In the conjugate Banach space, block perturbation of a retro Banach frame was defined by Jain et al.[19] as follows:

Definition 4.4 ([19])**.** Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E)$ be a retro Banach frame for E^* and let $\{m_n\}$, $\{p_n\}$ be increasing sequences of positive integers, where $m_0 = 0$ and $m_{n-1} + 1 \leq p_n \leq m_n$, $n \in \mathbb{N}$. Define a sequence $\{y_n\} \subset E$, $n \in \mathbb{N}$, by

$$y_k = \begin{cases} x_k, & \text{if } k \neq p_n, \\ x_{p_n} + z_n, & \text{if } k = p_n, \end{cases}$$

where $z_n = \sum\limits_{i=m_{n-1}+1}^{p_n - 1} \alpha_i x_i + \sum\limits_{i=p_n+1}^{m_n} \alpha_i x_i$, for all $n \in \mathbb{N}$. Then $\{y_n\}$ is called the block perturbation of $\{x_n\}$.

It is natural to ask whether block perturbation of a near-exact RBF is invariant. In order to show the invariance of a near-exact RBF under block perturbation, we derive the following result.

Theorem 4.5. *Let $(\{x_n\}, T)$ be a near-exact RBF for E^* with respect to $(E^*)_d$. Then its block perturbation is also a near-exact RBF.*

Proof. Since $(\{x_n\}, T)$ is a near-exact RBF, by omitting finite number of elements $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}\}$, say, from $\{x_n\}_{n=1}^{\infty}$, there is an associated Banach space $(E^*)_{d_0}$ and a bounded linear operator $T_1 : (E^*)_{d_0} \to E^*$ such that $\big(\{x_n\}_{n \neq \sigma(1), \sigma(2), \ldots, \sigma(m)}, T_1\big)$ is an exact RBF for E^* with respect to $(E^*)_{d_0}$. Consider a block perturbation $\{y_n\}$ of $\{x_n\}$. Then, by Theorem 4.1 in [19], there is an associated Banach space $(E^*)_{d_1}$ and a bounded linear operator $S : (E^*)_{d_1} \to E^*$ such that $(\{y_n\}, S)$ is an RBF for E^*.

Consider an increasing sequence $\{\gamma_k\}_{k=1}^{\infty}$ of positive numbers such that $\sigma(i) \notin \{\gamma_k\}_{k=1}^{\infty}$, $(i = 1, 2, \ldots, m)$ and define $\{y_{\gamma_k}\} \subset E$ by

$$y_{\gamma_i} = \begin{cases} x_{\gamma_i}, & \text{if } \gamma_i \neq p_{\gamma_k}, \\ x_{p_{\gamma_k}} + z_{\gamma_k}, & \text{if } \gamma_i = p_{\gamma_k}, \gamma_k \in \mathbb{N}, \end{cases}$$

where $z_{\gamma_k} = \sum\limits_{i=m_{\gamma_k - 1}+1}^{p_{\gamma_k}-1} \alpha_i x_i + \sum\limits_{i=p_{\gamma_k}+1}^{m_{\gamma_k}} \alpha_i x_i$, for all $\gamma_k \in \mathbb{N}$.

Again, by Theorem 4.1 of [19], there is an associated Banach space $(E^*)_{d_2} = \{\{f(y_{\gamma_k})\} : f \in E^*\}$ with norm

$$\|\{f(y_{\gamma_k})\}\|_{(E^*)_{d_2}} = \|f\|_{E^*}, \quad f \in E^*,$$

and a bounded linear operator $T_2 : (E^*)_{d_2} \to E^*$ given by $T_2(\{f(y_{\gamma_k})\}) = f$, $f \in E^*$ such that $(\{y_{\gamma_k}\}, T_2)$ is an exact RBF for E^*. Hence, $(\{y_n\}, S)$ is a near-exact RBF for E^* with respect to $(E^*)_{d_2}$. $\square$

5. Applications

Recall that, the bi-Banach frame $(\{x_n\}, \{f_n\})$ (where $\{x_n\} \subset E$, $\{f_n\} \subset E^*$) is said to be an exact bi-Banach frame if both the Banach frame $(\{f_n\}, S)$ and the retro Banach frame $(\{x_n\}, T)$ are exact. We begin this section with an application of an exact bi-Banach frame.

Theorem 5.1. *Let $(\{x_n\}, \{f_n\})$ (where $\{x_n\} \subset E$, $\{f_n\} \subset E^*$) be an exact bi-Banach frame for E. If*

$$\left\| z_j - \sum_{i=1}^{n} \zeta_{i,j} x_i \right\| \to 0, \text{ as } n \to \infty, \ 1 \leq j \leq m, \ \zeta_{i,j} \in \mathbb{R},$$

and $z_1, z_2, \ldots, z_m$ are linearly independent vectors in E, then

$$\operatorname{rank}(\zeta_{i,j}) = m, \quad (i \in \mathbb{N}; \ 1 \leq j < m).$$

Proof. Let A and B be the frame bounds for the Banach frame $(\{f_n\}, S)$ for E with respect to the associated Banach space E_d, then the frame inequality is given by

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E. \tag{5.1}$$

By assumption, for $1 \leq j < m$, we have

$$z_j = \sum_{i=1}^{\infty} \zeta_{i,j} x_i.$$

Thus, using remark 2.2, we obtain

$$f_i(z_j) = \zeta_{i,j}, \quad (i \in \mathbb{N};\ 1 \leq j < m). \tag{5.2}$$

Further, if $f_i(z_1) = 0$, for all $i \in \mathbb{N}$, then the frame inequality (5.1) yields $z_1 = 0$, which contradict the hypothesis that $z_1, z_2, z_3, \ldots, z_m$ are linearly independent. Hence, there exists an index i_1 (say) such that $f_{i_1}(z_1) \neq 0$,

Now, assume that for $k-1 < m$ there exist indices $i_1, ..., i_{k-1}$ such that

$$\begin{vmatrix} f_{i_1}(z_1) & f_{i_1}(z_2) & \cdots & f_{i_1}(z_{k-1}) \\ f_{i_2}(z_1) & f_{i_2}(z_2) & \cdots & f_{i_2}(z_{k-1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_{i_{k-1}}(z_1) & f_{i_{k-1}}(z_2) & \cdots & f_{i_{k-1}}(z_{k-1}) \end{vmatrix} \neq 0.$$

Thus it suffices to show that

$$\begin{vmatrix} f_{i_1}(z_1) & f_{i_1}(z_2) & \cdots & f_{i_1}(z_{k-1}) & f_{i_1}(z_k) \\ f_{i_2}(z_1) & f_{i_2}(z_2) & \cdots & f_{i_2}(z_{k-1}) & f_{i_2}(z_k) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{i_{k-1}}(z_1) & f_{i_{k-1}}(z_2) & \cdots & f_{i_{k-1}}(z_{k-1}) & f_{i_{k-1}}(z_k) \\ f_i(z_1) & f_i(z_2) & \cdots & f_i(z_{k-1}) & f_i(z_k) \end{vmatrix} \neq 0. \tag{5.3}$$

On contrary, suppose not, then we compute

$$f_i(z_1)\triangle_1 + f_i(z_2)\triangle_2 + \cdots + f_i(z_k)\triangle_k = 0,$$

where $\triangle_j$ $(j = 1, 2, \ldots, k)$ denote the cofactor of $f_i(z_j)$. This yields $f_i\left(\sum_{j=1}^{k} \triangle_j z_j\right) = 0$ and hence, $\sum_{j=1}^{k} \triangle_j z_j = 0$. This contradict the assumption that $z_1, z_2, \ldots, z_m$ are linearly independent, since $\triangle_k \neq 0$. Thus there exists an index i_k such that (5.3) holds. Hence, by (5.2) and (5.3) we conclude that

$$\operatorname{rank}(\zeta_{i,j}) = m, \quad (i \in \mathbb{N};\ 1 \leq j < m). \qquad \square$$

We conclude this section with the application of the near-exact retro Banach frame in eigenvalue problems. The following result generalizes the Theorem 5.2 of [22].

Theorem 5.2. *Let $(\{x_n\}, U)$ be a near-exact RBF for E^*. Let $\{z_k\}_{k=1}^m \subset E$ and assume that, for every k $(1 \leq k \leq m)$ there exists $\varphi_k \in E^*$ such that $\varphi_k(x_n) = c_k^{(n)}$, for all $n \in \mathbb{N}$. If for a sequence $\{n_k\} \subset \mathbb{N} \setminus \{\sigma(i)\}_{i=1}^m$,*

$\left(\left\{x_{n_k} + \frac{1}{\lambda}\sum_{i=1}^{m} c_i^{(n_k)} z_i\right\}, V\right)$ *is an RBF for* E^* *with respect to* $(E^*)_d$, *where* $V : (E^*)_d \to E^*$ *is a reconstruction operator and* λ *be any nonzero real number, then* $-\lambda$ *is not an eigenvalue of the matrix*

$$\begin{pmatrix} \varphi_1(z_1) & \varphi_2(z_1) & \cdots & \varphi_m(z_1) \\ \varphi_1(z_2) & \varphi_2(z_2) & \cdots & \varphi_m(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1(z_m) & \varphi_2(z_m) & \cdots & \varphi_m(z_m) \end{pmatrix}.$$

Proof. We prove the result for the case m = 3. Suppose $-\lambda$ is an eigenvalue of the matrix

$$\begin{pmatrix} \varphi_1(z_1) & \varphi_2(z_1) & \varphi_3(z_1) \\ \varphi_1(z_2) & \varphi_2(z_2) & \varphi_3(z_2) \\ \varphi_1(z_3) & \varphi_2(z_3) & \varphi_3(z_3) \end{pmatrix}.$$

Then

$$\begin{vmatrix} \varphi_1(z_1) + \lambda & \varphi_2(z_1) & \varphi_3(z_1) \\ \varphi_1(z_2) & \varphi_2(z_2) + \lambda & \varphi_3(z_2) \\ \varphi_1(z_3) & \varphi_2(z_3) & \varphi_3(z_3) + \lambda \end{vmatrix} = 0.$$

So, there exists scalars α_1, α_2, α_3 not all zero such that

$$\begin{aligned} \alpha_1\varphi_1(z_1) + \alpha_2\varphi_2(z_1) + \alpha_3\varphi_3(z_1) &= -\lambda\alpha_1, \\ \alpha_1\varphi_1(z_2) + \alpha_2\varphi_2(z_2) + \alpha_3\varphi_3(z_2) &= -\lambda\alpha_2, \\ \alpha_1\varphi_1(z_3) + \alpha_2\varphi_2(z_3) + \alpha_3\varphi_3(z_3) &= -\lambda\alpha_3. \end{aligned}$$

Further, since $(\{x_n\}, U)$ is a near-exact RBF, therefore, by omitting finite number of terms $\{x_{\sigma(i)}\}_{i=1}^{m}$, say, from $\{x_n\}_{n=1}^{\infty}$, the RBF $(\{x_{n_k}\}, V)$ (where $\{n_k\} \subset \mathbb{N} \setminus \{\sigma(i)\}_{i=1}^{m}$) becomes exact. Thus we have, $[x_{n_i}]_{i\neq k} \neq E$, and so by the Hahn-Banach Theorem, $\exists\ 0 \neq \Phi \in E^*$ such that

$$\Phi(x_{n_k}) = 0, \quad \text{for all } n_k \in \mathbb{N} \setminus \{\sigma(i)\}_{i=1}^{m}.$$

To see this, let us put $\Phi = -\lambda\alpha_1\varphi_1 - \lambda\alpha_2\varphi_2 - \lambda\alpha_3\varphi_3$, then

$$\begin{aligned} \Phi(x_{n_k}) &= -\lambda\alpha_1\varphi_1(x_{n_k}) - \lambda\alpha_2\varphi_2(x_{n_k}) - \lambda\alpha_3\varphi_3(x_{n_k}) \\ &= -\lambda\alpha_1 c_1^{(n_k)} - \lambda\alpha_2 c_2^{(n_k)} - \lambda\alpha_3 c_3^{(n_k)}, \end{aligned}$$

where $c_i^{(n_k)} = \varphi_i(x_{n_k})$, $i = 1, 2, 3$. Also,

$$\Phi(z_k) = -\lambda\alpha_1\varphi_1(z_k) - \lambda\alpha_2\varphi_2(z_k) - \lambda\alpha_3\varphi_3(z_k), \quad \text{for } k = 1, 2, 3, \ldots.$$

Therefore, $\Phi(z_1) = \alpha_1\lambda^2$, $\Phi(z_2) = \alpha_2\lambda^2$, $\Phi(z_3) = \alpha_3\lambda^2$. Hence,

$$\Phi\left(x_{n_k} + \frac{1}{\lambda}\left(c_1^{(n_k)} z_1 + c_2^{(n_k)} z_2 + c_3^{(n_k)} z_3\right)\right) = 0, \text{ for all } \{n_k\} \subset \mathbb{N}\setminus\{\sigma(i)\}_{i=1}^{m}.$$

Since $\left(\left\{x_{n_k} + \frac{1}{\lambda}\sum_{i=1}^{3} c_i^{(n_k)} z_i\right\}, V\right)$ is an RBF for E^*, by the retro frame inequality $\Phi = 0$, which gives a contradiction. $\square$

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