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Bounds for the Operator Norm on Weighted Cesàro Fractional Difference Sequence Spaces

Kuldip Raj[1], Anu Choudhary[2] and Mohammad Mursaleen[3*]

Abstract. In this paper, we determine the upper and lower bounds for the norm of lower triangular matrix operators on Cesàro weighted $(p, v)-$fractional difference sequence spaces of modulus functions. We consider the matrix operators acting between $\ell_p(w)$ and $C_p(v, \omega, \Delta^{(\eta,\ell)}, \mathcal{F})$ and identify their bounds and vice-versa. We also investigate the same characteristics for Nörlund and weighted mean matrix operators.

1. Introduction

Let W denote the following sequence space

$$W = \{x = (x_s) : x_s \in \mathbb{R} \quad \text{or} \quad \mathbb{C}\},$$

where $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively. For $1 < p < \infty$, the sequence space ℓ_p is defined as

$$\left\{x = (x_s) \in W : \sum_{s=1}^{\infty} |x_s|^p < \infty\right\},$$

and the space ℓ_p is a Banach space with respect to the norm

$$\|x\|_p = \left(\sum_{s=1}^{\infty} |x_s|^p\right)^{\frac{1}{p}}, \quad \text{for } p > 1.$$

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For $\omega = (\omega_s) \geq 0$, the weighted sequence space $\ell_p(\omega)$ is defined as follows:

$$\ell_p(\omega) = \left\{ x = (x_s) \in W : \sum_{s=1}^{\infty} \omega_s |x_s|^p < \infty \right\},$$

and

$$\|x\|_{p,\omega} = \left(\sum_{s=1}^{\infty} \omega_s |x_s|^p \right)^{\frac{1}{p}}.$$

Recently, Shahraki and Ledari in [19] worked on $\ell_p(c_0)$ sequence space. Initially, the fractional difference operators $\Delta^{\eta}, \Delta^{(\eta)}, \Delta^{-\eta}, \Delta^{(-\eta)}$ were introduced in [7] and discussed some topological results for these spaces. In [3], Baliarsingh et al. studied approximation theorems and statistical convergence in fractional difference sequence spaces. The binomial fractional difference sequence spaces by clubbing binomial matrix and fractional difference operators were studied in [14]. In [4], the double difference fractional order sequence spaces were examined. Recently, Choudhary et al. [5] investigated some interesting results on the space of double difference sequences of fractional order (see also [21–23]).

Let $\mathbb{N}$ be a set of natural numbers and ℓ be a real number. Then $(\eta, \ell)-$fractional difference operator $\Delta^{(\eta,\ell)} : W \to W$ is defined by:

$$\Delta^{(\eta,\ell)}(x_s) = \sum_{i=0}^{s} \frac{(-\eta)_{i,\ell}}{i!} x_{s-i},$$

where η is a positive proper fraction and the Pochhammer symbol

$$(\eta)_{s,\ell} = \begin{cases} 1, & \text{when } s = 0, \\ \eta(\eta+\ell)(\eta+2\ell)(\eta+3\ell)\cdots(\eta+(s-1)\ell), & \text{when } s \in \mathbb{N}. \end{cases}$$

Nakano [15] introduced the concept of the modulus function. For definitions and results, see [1, 2, 18].

In 1970, Shiue [20] introduced the Cesàro sequence space ces_p, for $p > 1$ and is defined by

$$ces_p = \left\{ x \in W : \sum_{s=1}^{\infty} \left(\frac{1}{s} \sum_{\wp=1}^{s} |x_\wp| \right)^p < \infty \right\},$$

and this sequence space is a Banach space with respect to a norm

$$\|x\| = \left(\sum_{s=1}^{\infty} \left(\frac{1}{s} \sum_{\wp=1}^{s} |x_\wp| \right)^p \right)^{1/p}.$$

Afterward, many authors studied these sequence spaces (see [6, 9, 12, 13, 16, 17]). Recently, the norms and lower bounds for matrix operators on weighted difference sequence spaces were determined in [8].

For $p>1$, $\Delta^{(\eta,\ell)}$ a $(\eta,\ell)-$fractional difference operator, $\mathcal{F}=(f_{\wp})$ a sequence of modulus functions, $v=(v_{\wp})$ a sequence of positive real numbers and $V_s=v_1+v_2+\cdots+v_s$, the Cesàro weighted $(p,v)-$fractional difference sequence space is defined as follows:

$$C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$$
$$=\left\{x=(x_s)\in W:\sum_{s=1}^{\infty}\omega_s\left(\frac{1}{V_s}\sum_{\wp=1}^{s}v_{\wp}\left(f_{\wp}\left|\Delta^{(\eta,\ell)}x_{\wp}\right|\right)\right)^p<\infty\right\},$$

and

$$\|x\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}=\left(\sum_{s=1}^{\infty}\omega_s\left(\frac{1}{V_s}\sum_{\wp=1}^{s}v_{\wp}\left(f_{\wp}\left|\Delta^{(\eta,\ell)}x_{\wp}\right|\right)\right)^p\right)^{\frac{1}{p}}.$$

For $\omega_s=1$, sequence spaces $\ell_p(\omega)$ and $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ reduce to ℓ_p and $C_p\left(v,\Delta^{(\eta,\ell)},\mathcal{F}\right)$, respectively. Suppose $D=(d_{m,j})$ is a lower triangular matrix such that for all m,j, $d_{m,j}\geq 0$, $p^*=\frac{p}{(p-1)}$, the set $\delta^+=\max(\delta,0)$ and $\delta^-=\min(\delta,0)$.

We can denote this by

(i) $\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}$, the norm of a matrix D as an operator from $\ell_p(\omega)$ into $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$,

(ii) $\|D\|_{p,C_p\left(v,\Delta^{(\eta,\ell)},\mathcal{F}\right)}$, the norm of a matrix D as an operator from ℓ_p to $C_p\left(v,\Delta^{(\eta,\ell)},\mathcal{F}\right)$,

(iii) $\|D\|_{C_p(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}),p,\omega}$, the norm of a matrix D as an operator from $C_p(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F})$ into $\ell_p(\omega)$,

(iv) $\|D\|_{C_p(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}),p}$, the norm of a matrix D as an operator from $C_p(v,\Delta^{(\eta,\ell)},\mathcal{F})$ to ℓ_p,

(v) $\|D\|_{p,\omega}$, the norm of a matrix D as an operator from $\ell_p(\omega)$ into itself and

(vi) $\|D\|_p$, the norm of a matrix D as an operator from ℓ_p into itself.

Let us define N_D and n_D as follows:

$$N_D=\sup_{s\geq 1}\left\{\sum_{\wp=1}^{s}\frac{s-\wp+1}{V_s}\left(\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}\right)^+\right\}.$$

$$n_D=\sup_{S\geq 1}\inf_{s\geq S}\left\{\frac{s}{V_s}\sum_{q=S}^{s}d_{q,S}+\frac{s}{V_s(s-S+1)}\sum_{\wp=S+1}^{s}(s-\wp+1)\right.$$

$$\left(\sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} d_{q,\wp-1}\right)^{-}\Bigg\},$$

Let Y and Z be two normed vector spaces. A linear map $D : Y \to Z$ is continuous if and only if there exists a real number R such that

$$\|Dy\| \leq R\|y\|, \quad \forall y \in Y.$$

The continuous linear operators are also known as bounded operators.

Let (d_k) be a non-negative sequence with $d_1 > 0$ and $D_s = \sum_{k=1}^{s} d_k$. The Nörlund matrix $\tilde{N}_d = (d_{s,\wp})$ is defined by

$$d_{s,\wp} = \begin{cases} d_{s-\wp+1}/D_s, & 1 \leq \wp \leq s, \\ 0, & \text{otherwise}, \end{cases}$$

and the weighted mean matrix $\tilde{W}_d = (d_{s,\wp})$ is defined by

$$d_{s,\wp} = \begin{cases} d_{\wp}/D_s, & 1 \leq \wp \leq s, \\ 0, & \text{otherwise}. \end{cases}$$

2. Main Results

Lemma 2.1. *Let $v = (v_\wp)$ be a sequence of positive real numbers, $\Delta^{(\eta,\ell)}$ be a $(\eta, \ell)-$fractional difference operator, and d, x be two non-negative sequences. Then*

$$\sum_{\wp=1}^{s} \frac{1}{V_s} d_\wp v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right|\right) \leq \left\{\max_{1\leq\wp\leq s} \frac{1}{s-\wp+1} \sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right\} \sum_{\wp=1}^{s} \frac{(s-\wp+1)}{V_s} \left(d_\wp - d_{\wp-1}\right)^{+},$$

for all $s \geq 0$.

Proof. By applying summation by parts, we have

$$\begin{aligned}\sum_{\wp=1}^{s} \frac{1}{V_s} d_\wp v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right|\right) &= \sum_{\wp=1}^{s} \frac{1}{V_s} \sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right) \left(d_\wp - d_{\wp-1}\right) \\ &\leq \left\{\max_{1\leq\wp\leq s} \frac{1}{s-\wp+1} \sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right\} \\ &\quad \sum_{\wp=1}^{s} \frac{(s-\wp+1)}{V_s} \left(d_\wp - d_{\wp-1}\right)^{+}.\end{aligned}$$

□

Lemma 2.2. *Let $v = (v_\wp)$ be a sequence of positive real numbers, $V_s = v_1 + v_2 + \cdots + v_s$ and $S \geq 1$. If $x_S \geq x_{S+1} \geq \cdots \geq 0$ and $x_s = 0$, for $s < S$. Then*

$$\frac{1}{V_s}\sum_{\wp=1}^{s} d_\wp v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right|\right) \geq \left(\frac{1}{s}\sum_{k=1}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right) \times \left\{\frac{s}{V_s} d_S + \frac{s}{V_s\,(s-S+1)} \sum_{\wp=S+1}^{s} (s-\wp+1)\,(d_\wp - d_{\wp-1})^{-}\right\},$$

for all $s \geq 0$.

Proof. We know that the result holds for $s < S$. Now, let $s \geq S$. Then for $(S \leq \wp \leq s)$ we have

$$\begin{aligned}\frac{1}{s-S+1}\sum_{k=1}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right) &= \frac{1}{s-S+1}\sum_{k=S}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\\ &\geq \frac{1}{s-\wp+1}\sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right),\end{aligned}$$

since $x_S \geq x_{S+1} \geq \cdots \geq 0$ and $x_s = 0$ for $s < S$. Now, by using summation by parts, we have

$$\begin{aligned}&\frac{1}{V_s}\sum_{\wp=1}^{s} d_\wp v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right|\right)\\ &\quad = d_S \left(\frac{1}{V_s}\sum_{k=S}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right)\\ &\qquad + \frac{1}{V_s}\sum_{\wp=S+1}^{s} (d_\wp - d_{\wp-1}) \left(\sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right)\\ &\quad \geq \left(\frac{1}{s}\sum_{k=1}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right|\right)\right)\\ &\qquad \left\{\frac{s d_S}{V_s} + \frac{s}{V_s(s-S+1)} \sum_{\wp=S+1}^{s} (s-\wp+1)\,(d_\wp - d_{\wp-1})^{-}\right\}. \qquad \square\end{aligned}$$

Lemma 2.3 ([11])**.** *Let $p > 1, S \geq 1$ and $\omega = (\omega_s) \geq 0$ be a decreasing sequence and $\sum_{s=1}^{\infty} \frac{\omega_s}{s}$ be divergent and $C_S = \left(c_{s,\wp}^S\right)$ be a matrix with*

$$c_{s,\wp}^S = \begin{cases} \frac{1}{s+S-1}, & s \geq \wp, \\ 0, & otherwise. \end{cases}$$

Then the norm $\|C_S\|_{p,\omega} = p^*$.

Lemma 2.4 ([11])**.** *Let $v = (v_\wp)\,(p > 1)$ be a sequence of positive real numbers, $y = (y_\wp) = v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right|\right) \geq 0$ and $\omega \geq 0$ be decreasing sequence. Then*

$$\sum_{k=1}^{\infty} \omega_k \max_{1 \leq q \leq k} \left(\frac{1}{k-q+1} \sum_{\wp=q}^{k} y_\wp \right)^p \leq (p^*)^p \sum_{\wp=1}^{\infty} \omega_\wp \left(y_\wp\right)^p .$$

Theorem 2.5. *Let $\mathcal{F} = (f_\wp)$ be a sequence of modulus functions, $\omega = (\omega_s)$ be a non-decreasing sequence, $v = (v_s)$ be a sequence of positive real numbers and $V_s = v_1 + v_2 + \cdots + v_s$. If $D = (d_{s,\wp}) \geq 0$ be a lower triangular matrix, then*

(i) $\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \leq p^* N_D$. *In addition if $N_D < \infty$, then D is a bounded matrix operator from $\ell_p(\omega)$ into $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$.*

(ii) $\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \geq p^* n_D$, *if $\sum_{s=1}^{\infty} \frac{\omega_s}{s}$ is divergent and $\left(\frac{\omega_s}{\omega_{s+1}}\right)$ is decreasing.*

Accordingly, if $\omega = (\omega_s)$ be a decreasing sequence with non-negative entries, $\left(\frac{\omega_s}{\omega_{s+1}}\right)$ is decreasing and $\sum_{s=1}^{\infty} \frac{\omega_s}{s} = \infty$, then

$$p^* n_D \leq \|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \leq p^* N_D.$$

In particular, if $\omega_s = 1$, for all s and if $N_D < \infty$, then $D : \ell_p \to C_p\left(v,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ is a bounded matrix operator.

Proof. By employing Lemma 2.1, we have

$$\sum_{\wp=1}^{s} \left(\frac{1}{V_s} \sum_{q=\wp}^{s} d_{q,\wp} \right) v_\wp \left(f_\wp \left|\Delta^{(\eta,\ell)} x_\wp\right| \right)$$
$$\leq \left\{ \max_{1 \leq \wp \leq s} \frac{1}{s-\wp+1} \sum_{k=\wp}^{s} v_k \left(f_k \left|\Delta^{(\eta,\ell)} x_k\right| \right) \right\}$$

$$\sum_{\wp=1}^{s} \frac{s-\wp+1}{V_s} \left(\sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} d_{q,\wp-1} \right)$$
$$\leq N_D \max_{1\leq\wp\leq s} \frac{1}{s-\wp+1} \sum_{k=\wp}^{s} v_k \left(f_k \left| \Delta^{(\eta,\ell)} x_k \right| \right).$$

Thus, by using Lemma 2.4, we have

$$\sum_{s=1}^{\infty} \omega_s \left(\sum_{\wp=1}^{s} \left(\frac{1}{V_s} \sum_{q=\wp}^{s} d_{q,\wp} \right) v_\wp \left(f_\wp \left| \Delta^{(\eta,\ell)} x_\wp \right| \right) \right)^p$$
$$\leq (N_D)^p \sum_{s=1}^{\infty} \omega_s \max_{1\leq\wp\leq s} \left(\frac{1}{s-\wp+1} \sum_{k=\wp}^{s} v_k \left(f_k \left| \Delta^{(\eta,\ell)} x_k \right| \right) \right)^p$$
$$\leq (p^* N_D)^p \sum_{\wp=1}^{\infty} \omega_\wp \left(v_\wp \left(f_\wp \left| \Delta^{(\eta,\ell)} x_\wp \right| \right) \right)^p.$$

(ii) Let $z = (z_s)$ be a decreasing sequence with non-negative entries, $\|z\|_{u,\omega} = 1$, for all $s \geq 1$, $x_1 = x_2 = \cdots = x_{S-1} = 0$ and

$$x_{s+S-1} = \left(\frac{\omega}{\omega_{s+S-1}} \right)^{(1/p)} z_s.$$

Hence, $\|x\|_{p,\omega} = \|z\|_{p,\omega} = 1$. Now, by using Lemma 2.2, we have

$$\|D\|^p_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}$$
$$\geq \sum_{s=1}^{\infty} \omega_s \left(\sum_{\wp=1}^{s} \left(\frac{1}{V_s} \sum_{q=\wp}^{s} d_{q,\wp} \right) v_\wp \left(f_\wp |\Delta^{(\eta,\ell)} x_\wp| \right) \right)^p$$
$$\geq (\gamma_S)^p \sum_{s=1}^{\infty} \omega_s \left(\frac{1}{V_s} \left(\frac{V_s}{s} \right) \sum_{k=1}^{s} v_k \left(f_k \left| \Delta^{(\eta,\ell)} x_k \right| \right) \right)^p$$
$$= (\gamma_S)^p \sum_{s=1}^{\infty} \omega_s \left(\frac{1}{s} \sum_{k=1}^{s} v_k \left(f_k \left| \Delta^{(\eta,\ell)} x_k \right| \right) \right)^p$$
$$= (\gamma_S)^p \sum_{s=1}^{\infty} \omega_{s+S-1} \left(\frac{1}{s+S-1} \sum_{k=1}^{s} v_{k+S-1} \left(f_{k+S-1} \left| \Delta^{(\eta,\ell)} x_{k+S-1} \right| \right) \right)^p$$
$$= (\gamma_S)^p \sum_{s=1}^{\infty} \omega_{s+S-1} \left(\frac{1}{s+S-1} \sum_{k=1}^{s} v_{k+S-1} \left(f_{k+S-1} \left| \Delta^{(\eta,\ell)} \left(\frac{\omega_k}{\omega_{k+S-1}} \right)^{(1/p)} z_k \right| \right) \right)^p$$
$$\geq (\gamma_S)^p \|C_S z\|^p_{p,\omega},$$

where

$$\gamma_S =$$

$$\inf_{s\geq S}\left\{\frac{s}{V_s}\sum_{q=S}^{s}d_{q,S}+\frac{s}{V_s(s-S+1)}\sum_{\wp=S+1}^{s}(s-\wp+1)\left(\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}\right)^{-}\right\}.$$

Now, by Lemma 2.3, we have

$$\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\geq p^*\gamma_S.$$

Hence, $\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\geq p^* n_D$, where $n_D=\sup_{S\geq 1}\gamma_S$. □

Corollary 2.6. *Let* $u>1, D=(d_{s,\wp})$ *be a non-negative lower triangular matrix such that*

$$\sum_{q=\wp-1}^{s}d_{q,\wp-1}\leq\sum_{q=\wp}^{s}d_{q,\wp},$$

for $1<\wp\leq s$. *Then*

$$\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}p^*\sup_{s\geq 1}\frac{s}{V_s}d_{s,s}.$$

Proof. We know that the finite sequence $\left(\sum_{q=\wp}^{s}d_{q,\wp}\right)_{\wp=1}^{s}$ is increasing for each s. Hence, we have

$$\left(\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}\right)^{-}=0,$$

and

$$\begin{aligned}n_D&=\sup_{S\geq 1}\inf_{s\geq S}\frac{s}{V_s}\sum_{q=S}^{s}d_{q,S}\\&=\sup_{s\geq 1}\frac{s}{V_s}d_{s,s}.\end{aligned}$$

Also, for $(1\leq\wp\leq s)$,

$$\left(\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}\right)^{+}=\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}.$$

Thus,

$$N_D=\sup_{s\geq 1}\left\{\sum_{\wp=1}^{s}\frac{s-\wp+1}{V_s}\left(\sum_{q=\wp}^{s}d_{q,\wp}-\sum_{q=\wp-1}^{s}d_{q,\wp-1}\right)\right\}$$

$$= \sup_{s\geq 1} \frac{1}{V_s} \sum_{\wp=1}^{s} \sum_{q=\wp}^{s} d_{q,\wp}$$

$$= \sup_{s\geq 1} \frac{s}{V_s} \times \frac{1}{s} \sum_{\wp=1}^{s} \sum_{q=\wp}^{s} d_{q,\wp}$$

$$\leq \sup_{s\geq 1} \frac{s}{V_s} d_{s,s}.$$

As a consequence of Theorem 2.5, we have

$$\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} = p^* \sup_{s\geq 1} \frac{s}{V_s} d_{s,s}. \qquad \square$$

Example 2.7. Define $D = (d_{s,\wp})$ by

$$\begin{cases} 1/s^3, & \wp < s, \\ (3s-1)/s, & \wp = s, \\ 0, & \wp > s. \end{cases}$$

and a sequence $(v_k) = 2k$.

Here, the sequence $\left(\sum_{q=\wp}^{s} d_{q,\wp}\right)_{\wp=1}^{s}$ is increasing sequence for each s and $\sup_{s\geq 1} \frac{s}{V_s} d_{s,s} = 3$. Thus, by Corollary 2.6, we have

$$\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} 3p^*.$$

Corollary 2.8. *Let $u > 1$ and $D = (d_{s,\wp}) \geq 0$ be a lower triangular matrix with $\sum_{q=\wp-1}^{s} d_{q,\wp-1} \geq \sum_{q=\wp}^{s} d_{q,\wp}$, for $1 < \wp \leq s$. Then*

(2.1)
$$p^* \left(\inf_{s\geq 1} \frac{1}{V_s} \sum_{\wp=1}^{s} \sum_{q=\wp}^{s} d_{q,\wp} \right) \leq \|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \leq p^* \left(\sup_{s\geq 1} \sum_{q=1}^{s} \frac{s}{V_s} d_{q,1} \right).$$

Also, if the right hand side of (2.1) be finite, then

$$D : \ell_p(\omega) \to C_p\left(v, \omega, \Delta^{(\eta,\ell)}, \mathcal{F}\right),$$

is a bounded matrix operator.

Proof. Since, $\left(\sum_{q=\wp}^{s} d_{q,\wp}\right)_{\wp=1}^{s}$ is decreasing for each s, then for $(1 < \wp \leq s)$, we have

$$\left(\sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} d_{q,\wp-1}\right)^{-} = \sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} d_{q,\wp-1}.$$

Hence,

$$\begin{aligned} n_D &= \sup_{S\geq 1} \inf_{s\geq S} \left\{ \frac{s}{V_s} \sum_{q=S}^{s} d_{q,S} + \frac{s}{V_s\,(s-S+1)} \sum_{\wp=S+1}^{s-\wp+1} (s-\wp+1) \left(\sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} \right) \right\} \\ &= \sup_{S\geq 1} \inf_{s\geq S} \frac{s}{V_s\,(s-S+1)} \sum_{\wp=S}^{s} \sum_{q=\wp}^{s} d_{q,\wp} \\ &\geq \inf_{s\geq 1} \frac{1}{V_s} \sum_{\wp=1}^{s} \sum_{q=\wp}^{s} d_{q,\wp}. \end{aligned}$$

Also,

$$\left(\sum_{q=\wp}^{s} d_{q,\wp} - \sum_{q=\wp-1}^{s} d_{q,\wp-1}\right)^{+} = 0,$$

for $1 < k \leq s$ and

$$\left(\sum_{q=1}^{s} d_{q,1} - \sum_{q=0}^{s} d_{q,0}\right)^{+} = \sum_{q=1}^{s} d_{q,1}, \quad \text{for } k = 1.$$

Therefore,

$$N_D \sup_{s\geq 1} \sum_{q=1}^{s} \frac{s}{V_s} d_{q,1}.$$

Now, by using Theorem 2.5, we can conclude that

$$p^* \left(\inf_{s\geq 1} \frac{1}{V_s} \sum_{\wp=1}^{s} \sum_{q=\wp}^{s} d_{q,\wp} \right) \leq \|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \leq p^* \left(\sup_{s\geq 1} \sum_{q=1}^{s} \frac{s}{V_s} d_{q,1} \right).$$

□

Example 2.9. Define $D = (d_{s,\wp})$ by

$$\begin{cases} \frac{2}{s^2}, & s \geq \wp, \\ 0, & \text{otherwise.} \end{cases}$$

and a sequence $v = (v_k) = 2k$.

Here, the sequence $\left(\sum_{q=\wp}^{s} d_{q,\wp}\right)_{\wp=1}^{s}$ is decreasing sequence for each s and

$\sup_{s\geq 1}\sum_{q=1}^{s}\frac{s}{V_s}d_{q,1}=\sum_{s=1}^{\infty}\frac{2}{s(V_s)}$. Thus, by Corollary 2.8, we have

$$0\leq \|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\leq \left(\sum_{s=1}^{\infty}\frac{2}{s\left(V_s\right)}\right)p^*.$$

Corollary 2.10. *Let $p>1$, $\tilde{N}_d=(d_{s,\wp})$ be a Nörlund matrix and (d_s) be a decreasing sequence. Then*

$$p^*\leq \left\|\tilde{N}_d\right\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\leq p^*\left(\sup_{s\geq 1}\sum_{q=1}^{s}\frac{s}{V_s}\frac{d_q}{D_q}\right).$$

Corollary 2.11. *Let $p>1$, $\tilde{W}_d=(d_{s,\wp})$ be a Weighted mean matrix and (d_s) be a decreasing sequence. Then*

$$p^*\leq \left\|\tilde{W}_d\right\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\leq p^*d_1\left(\sup_{s\geq 1}\sum_{q=1}^{s}\frac{s}{V_s}\frac{1}{D_q}\right).$$

Theorem 2.12. *Let $p>1$ and $D=(d_{s,\wp})\geq 0$ be a lower triangular matrix. If $D:\ell_p(\omega)\to\ell_p(\omega)$ be a bounded matrix operator, then $D:\ell_p(\omega)\to C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ is also a bounded matrix operator and*

$$\|Dx\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\leq\left(\frac{s}{V_s}\right)p^*\|Dy\|_{p,\omega.}$$

Proof. For $p>1$, we have

$$\begin{aligned}
\|Dx\|^p_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} &= \sum_{s=1}^{\infty}\omega_s\left|\frac{1}{V_s}\sum_{\wp=1}^{s}\sum_{k=1}^{\wp}d_{q,k}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right|^p\\
&= \sum_{s=1}^{\infty}\omega_s\left|\frac{s}{V_s}\frac{1}{s}\sum_{\wp=1}^{s}\sum_{k=1}^{\wp}d_{q,\wp}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right|^p\\
&= \sum_{s=1}^{\infty}\omega_s\left|\left(\frac{s}{V_s}\right)\sum_{\wp=1}^{s}\left(C_1D\right)_{s,k}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right|^p\\
&= \sum_{s=1}^{\infty}\omega_s\left(\frac{s}{V_s}\right)^u\left|\sum_{\wp=1}^{s}\left(C_1D\right)_{s,k}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right|^p\\
&= \left\|\frac{s}{V_s}\left(C_1D\right)_{s,k}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right\|^p_{p,\omega}\\
&= \left(\frac{s}{V_s}\right)^p\left\|\left(C_1D\right)_{s,k}v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)\right\|^p_{p,\omega}.
\end{aligned}$$

Now, let $y = (y_k) = v_k\left(f_k\left|\Delta^{(\eta,\ell)}x_k\right|\right)$. Hence, we have

$$\|Dx\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} = \left(\frac{s}{V_s}\right)\|C_1\|_{p,\omega}\|Dy\|_{p,\omega}.$$

By using Lemma 2.3 we get

$$\left(\frac{s}{V_s}\right)\|C_1\|_{p,\omega} \le \left(\frac{s}{V_s}\right)p^*.$$

Thus,

$$\|Dx\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}\left(\frac{s}{V_s}\right)p^*\|Dy\|_{p,\omega}.$$

Thus the theorem is proved. □

Proposition 2.13 ([10]). *Let $\omega = (\omega_s) \ge 0$ be a decreasing sequence, C_1 be a Cesàro matrix and $p > 1$. Then $\|C_1\|_{p,\omega} \le p^*$.*

Theorem 2.14. *Let $p > 1$ and $D = (d_{s,\wp})$ be a non-negative lower triangular matrix. Then*

(2.2)
$$\frac{V_s}{s.p^*}\left(\frac{\|D\|_{p,\omega}}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \le \|D\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \le \sup_{s\ge1}\left(V_s\sup_{1\le\wp\le s}d_{s,\wp}\right).$$

Also, if the right hand side of (2.2) is finite, then D is a bounded matrix operator from $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ into $\ell_p(\omega)$.

Proof. Let $x \in C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$. Then

$$\begin{aligned}
\|Dx\|^p_{p,\omega} &= \sum_{s=1}^{\infty}\omega_s\left|\sum_{\wp=1}^{s}d_{s,\wp}v_\wp\left(f_\wp\left|\Delta^{(\eta,\ell)}x_\wp\right|\right)\right|^p \\
&\le \sum_{s=1}^{\infty}\omega_s\left(\sup_{1\le\wp\le s}d_{s,\wp}\sum_{\wp=1}^{s}v_\wp\left(f_\wp\left|\Delta^{(\eta,\ell)}x_\wp\right|\right)\right)^p \\
&\le \sup_{s\ge1}\left(V_s\sup_{1\le\wp\le s}d_{s,\wp}\right)^p\sum_{s=1}^{\infty}\omega_s\left(\frac{1}{V_s}\sum_{\wp=1}^{s}v_\wp\left(f_\wp\left|\Delta^{(\eta,\ell)}x_\wp\right|\right)\right)^p \\
&= \sup_{s\ge1}\left(V_s\sup_{1\le\wp\le s}d_{s,\wp}\right)^p\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}.
\end{aligned}$$

Therefore,

$$\frac{\|D\|_{p,\omega}}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}} \le \sup_{s\ge1}\left(V_s\sup_{1\le\wp\le s}d_{s,\wp}\right)$$

and

$$\|D\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \le \sup_{s\ge 1}\left(V_s \sup_{1\le\wp\le s} d_{s,\wp}\right).$$

Now, by using Proposition 2.13 we conclude that

$$\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \left\|\frac{s}{V_s} C_1 v_k\left(f_k\left|\Delta^{(\eta,\ell)} x_k\right|\right)\right\|_{p,\omega} \le \frac{s}{V_s} p^* \|y\|_{p,\omega},$$

where $y = (y_k) = v_k\left(f_k\left|\Delta^{(\eta,\ell)} x_k\right|\right)$.
This implies

$$\frac{\|Dx\|_{p,\omega}}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}} \ge \frac{V_s}{s.p^*}\frac{\|Dx\|_{p,\omega}}{\|y\|_{p,\omega}}.$$

Hence,

$$\|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \ge \frac{V_s}{s.p^*}\frac{\|Dx\|_{p,\omega}}{\|y\|_{p,\omega}}.$$

Thus,

$$\frac{V_s}{s.p^*}\left(\frac{\|Dx\|_{p,\omega}}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \le \|D\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)} \le \sup_{s\ge 1}\left(V_s \sup_{1\le\wp\le s} d_{s,\wp}\right).$$

□

Corollary 2.15 ([11]). *Let $\tilde{N}_d = (d_{s,\wp})$ be a Nörlund matrix and (d_s) be a decreasing sequence with $d_s \downarrow \beta$ and $\beta > 0$. Then*

$$\left\|\tilde{N}_d\right\|_{p,\omega} = p^*.$$

Corollary 2.16 ([11]). *Let $\tilde{W}_d = (d_{s,\wp})$ be a weighted mean matrix and (d_s) be an increasing sequence with $d_s \uparrow \beta$ and $\beta < \infty$. Then*

$$\left\|\tilde{W}_d\right\|_{p,\omega} = p^*.$$

Corollary 2.17. *Let $\Delta^{(\eta,\ell)}$ be a (η,ℓ)-fractional difference operator, $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p > 1$. Then generalized Cesàro matrix C_S is bounded from $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ into $\ell_p(\omega)$ and*

$$\frac{V_s}{s}\left(\frac{1}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \le \|C_S\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \le \frac{V_s}{s}.$$

Proof. For C_S a generalized Cesàro matrix,

$$\sup_{s\ge 1}\left(V_s \sup_{1\le\wp\le s} d_{s,\wp}\right) = \sup_{s\ge 1}\frac{V_s}{s+S-1}$$

$$= \frac{V_s}{s}.$$

Now, from Theorem 2.14, we have

$$\frac{V_s}{s.p^*}\left(\frac{\|C_S\|_{p,\omega}}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \leq \|C_S\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \leq \frac{V_s}{s}.$$

Also, $\|C_S\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} = p^*$, from Lemma 2.3, we have

$$\frac{V_s}{s}\left(\frac{1}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right)\|C_S\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \leq \frac{V_s}{s}.$$ □

Corollary 2.18. *Let $\tilde{N}_d = (d_{s,\wp})$ be a Nörlund matrix and (d_s) be a decreasing sequence with $d_s \downarrow \beta$ and $\beta > 0$. Then*

$$\frac{V_s}{s}\left(\frac{1}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \leq \left\|\tilde{N}_d\right\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \leq d_1 \sup_{s\geq 1} \frac{V_s}{D_s}.$$

Proof. As a consequence of Corollary 2.15 and Theorem 2.14 we get

$$\frac{V_s}{s}\left(\frac{1}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \leq \left\|\tilde{N}_d\right\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \leq d_1 \sup_{s\geq 1} \frac{V_s}{D_s}.$$ □

Corollary 2.19. *Let $\tilde{W}_d = (d_{s,\wp})$ be a weighted mean matrix and (d_s) be an increasing sequence with $d_s \uparrow \beta$, $\beta < \infty$ and $p > 1$. Then*

$$\frac{V_s}{s}\left(\frac{1}{\|x\|_{p,\omega,C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)}}\right) \leq \left\|\tilde{W}_d\right\|_{C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right),p,\omega} \leq \sup_{s\geq 1} \frac{V_s d_s}{D_s}.$$

Proof. As a consequence of Corollary 2.16 and Theorem 2.14 we get the desired result. □

3. Conclusion

The determination of bounds is a powerful tool for balancing vectors in any norm and is extremely helpful in hereditary discrepancy problems. The theory of difference sequence spaces plays an important role in enveloping the classical theory of fractional calculus and numerical analysis. The study of fractional calculus has a direct impact on the theory involving the solution of diverse problems in mathematics, science and engineering. Also, Cesàro sequence spaces serve many applications in physical chemistry and crystallography. In this paper, we have determined the bounds for the norm of matrix operators from $\ell_p(w)$ into $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ and from $C_p\left(v,\omega,\Delta^{(\eta,\ell)},\mathcal{F}\right)$ into $\ell_p(w)$. We have also obtained the norm of matrix operators on some lower triangular matrices such as Nörlund and weighted mean matrices.

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