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On Intuitionistic Fuzzy Metric Space and Ideal Convergence of Triple Sequence Space

Shailendra Pandit[1], Ayaz Ahmed[2] and Ayhan Esi[3*]

Abstract. The purpose of this article is to introduce the triple sequences and its convergence over instuitionistic fuzzy metric space (**IFMS**). The article also discusses ideal convergence of triple sequences, the uniqueness of ideal limits, the relationship between Pringsheim's limit and ideal limits, the ideal Cauchy sequences, and various specific spaces of triple sequences with respect to IFMS.

1. Introduction

Classical set theory, developed by George Cantor in 1800, is significant for mathematics, where we gain confirmation about an element whether it belongs to a study totally or not, but when we deal with a study in which certain components can participate partially, mathematics requires a distinct idea to cope with the problem of ambiguity or dilemma. Where we can engage with such situations. This encourages the development of new notions, fuzzy set theory [10] by Zadeh in 1965. Later in 1983, Krassimir T-Atanassov [9] developed the notion of intuitionistic fuzzy set (IFS) by establishing a non-membership function in addition to membership value function, which assign the degree of non belongingness of an element to the study. As a result, mathematics adopted IFS as a stronger tool for dealing with uncertainty and ambiguity. A human being who belongs to a study may not have the same degree of non-belongingness to the same study, as a complement of degree of belongingness. There may be a disparity between a sense of belonging and a sense of non-belonging. This disparity was regarded as

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hesitation in the research. This viewpoint indicates that verbal negation and logical negation do not always coincide.

Pringsheim's originally examined the idea of convergence of multiple sequences in 1900. Since then, numerous authors have [1, 2, 4, 12] examined the ideas in their own ways and extended them to triple sequence convergent space as well as statistical convergence of triple sequence space with respect to some generalised spaces.

In 1951 and 1985, Fast [3] and Fridy [5] individually introduced the notion of statistical convergence. Following that, numerous scholars examined the concept in their own way and generalised it to the ideal convergence [13].

We now recall that, the family I of subsets of $\mathbb{N}$, where throughout the paper $\mathbb{N}$ deals the set of positive integers, forms an Ideal on $\mathbb{N}$ if (i) $\phi \in I$, (ii) if $A_1 \in I$ and $A_2 \subseteq A_1$ then $A_2 \in I$, (iii) $A_1 \in I, A_2 \in I$ then $A_1 \cup A_2 \in I$. An ideal I is called admissible if $\{\alpha\} \in I$ for all $\alpha \in \mathbb{N}$ and the ideal I called non trivial if $\mathbb{N} \notin I$ and $I \neq \{\phi\}$. Throughout the study $\mathbb{N}^3$ denotes the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and I_3 denotes the ideal on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

The proposed work is divided into four sections: an introduction, basics, and preliminary material, ideal convergence, and key results on the sequence space. The proposed work aims to deal with the ideal convergence of triple sequences, uniqueness of ideal limit, and completeness of the space of bounded triple sequence space with respect to intuitionistic fuzzy normed spaces.

2. Basics and Preliminaries

Definition 2.1 ([7])**.** A continuous function $o : [0,1]^2 \longrightarrow [0,1]$ is triangular norm (t-norm) if o holds, (i) the law of commutative, (ii) the law of associative, (iii) $\alpha o 1 = \alpha$ for all $\alpha \in [0,1]$, (iv) o is monotonically increasing $i.e$ $\alpha\ o\ \beta \leq \gamma\ o\ \delta$ whenever $\alpha \leq \gamma$ and $\beta \leq \delta$ for all $\alpha, \beta, \gamma, \delta \in [0,1]$.

Definition 2.2 ([7])**.** A continuous function $* : [0,1]^2 \longrightarrow [0,1]$ is triangular co-norm (t-co-norm) if $*$ holds (i) Law of commutative, (ii) Law of associative, (iii) $\alpha * 0 = \alpha$ for all $\alpha \in [0,1]$, (iv) $*$ is monotonically increasing $i.e$ $\alpha * \beta \leq \gamma * \delta$ whenever $\alpha \leq \gamma$ and $\beta \leq \delta$ for all $\alpha, \beta, \gamma, \delta \in [0,1]$.

The concepts of t-norm and t-co-norm basically characterise the fuzzy intersection and unions respectively [6, 11]. These norms were used at very first by Menger [8] while studying of statistical metric spaces. Later, many authors proposed various examples on this concepts.

Remark 2.3 ([14]). (i) For any ϵ_1 and $\epsilon_2 \in (0,1)$ with $\epsilon_1 > \epsilon_2$, we can find ϵ_3 and $\epsilon_4 \in (0,1)$ such that $\epsilon_1 o \epsilon_3 \geq \epsilon_2$,

(ii) For any $\epsilon_0 \in (0,1)$, we can find ϵ_5 *and* $\epsilon_6 \in (0,1)$ such that $\epsilon_5 * \epsilon_5 \geq \epsilon_0$ and $\epsilon_6 * \epsilon_6 \leq \epsilon_0$.

Definition 2.4 ([7]). A five-tuple $(X, d, \underline{d}, o, *)$ is called an instuitionistic fuzzy metric space **(IFMS)** if d and $\underline{d}$ are fuzzy sets on $X \times X \times R^+$, where X is arbitrary set o is continuous t-norm and $*$ is a continuous t co-norm holds the following criteria. For all $u, v, w \in X$ and $\xi,\ \xi_0 \in R^+$

(i) $d(u, v, \xi) : R^+ \longrightarrow (0, 1]$ is a continuous function of ξ,
(ii) $0 < d(u, v, \xi) \leq 1$,
(iii) $d(u, v, \xi) = 1$ iff $u = v$,
(iv) $d(u, v, \xi) = d(v, u, \xi)$,
(v) $d(u, v, \xi) + \underline{d}(v, u, \xi) \leq 1$,
(vi) $d(u, v, \xi) o d(v, u, \xi_0) \leq d(u, v, \xi + \xi_0)$,
(vii) $\underline{d}(u, v, \xi) : R^+ \longrightarrow [0, 1)$ is a continuous function of ξ,
(viii) $0 \leq \underline{d}(u, v, \xi) < 1$,
(ix) $d(u, v, \xi) = 0$ iff $u = v$,
(x) $\underline{d}(u, v, \xi) = \underline{d}(v, u, \xi)$,
(xi) $\underline{d}(u, v, \xi + \xi_0) \leq \underline{d}(u, v, \xi) * \underline{d}(u, v, \xi_0)$.

Note: *Throught the paper, IFMS is abbreviation of intuitionistic fuzzy metric space and X deals the IFMS* $(X, d, \underline{d}, o, *)$.

Remark 2.5. $(d, \underline{d})$ defined as intuitionistic fuzzy metric over X and the functions $d(u, v, \xi)$ and $\underline{d}(u, v, \xi)$ standing for degree of nearness and non-nearness between u and v respectively with respect to ξ. Further $d(u, v, \xi)$ is a non-decreasing function of ξ for all u, v. i.e. if $\xi_1 < \xi_0$ we have $d(u, v, \xi_1) \leq d(u, v, \xi_0)$ for all u, v. In same way $\underline{d}(y, x, \xi)$ is a non-increasing function of ξ. i.e. if $\xi_1 < \xi_0$ we have $\underline{d}(u, v, \xi_1) \geq \underline{d}(u, v, \xi_0)$ for all u, v.

We should also be aware that every fuzzy metric space (X, d, o) can be extended to an IFMS $(X, d, \underline{d}, o, *)$ by defining $\underline{d} = 1 - d$. Moreover, $u * v = 1 - (1 - u)o(1 - v)$ for all $u, v \in X$.

Example 2.6. Let $\mathbb{N}$ is set of positive integers. Now for $\alpha,\ \beta \in \mathbb{N}$ define

$$\alpha o \beta = \begin{cases} \min(\alpha, \beta), & \alpha + \beta > 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$d(u, v, \xi) = \begin{cases} \frac{u}{v}, & u \leq v, \\ \frac{v}{u}, & u \geq v. \end{cases}$$

Thus, we notice that $(\mathbb{N}, d, o)$ is a fuzzy metric space so this can be thought as a intuitionistic fuzzy metric space $(\mathbb{N}, d, \underline{d}, o, *)$ where $\underline{d} = 1 - d$

.i.e

$$\underline{d}(u,v,\xi) = \begin{cases} 1-\frac{u}{v}, & u \leq v, \\ 1-\frac{v}{u}, & u \geq v, \end{cases}$$

and

$$\alpha * \beta = \begin{cases} 1-\min(1-\alpha, 1-\beta) \text{ or } \max(\alpha,\beta), & \alpha+\beta<1, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.7. Let $(X, d, \underline{d}, o, *)$ be an IFMS then a sequence (x_n) of the elements of X is said to be convergent to a number x if for every $\epsilon > 0$ and with respect to every positive real ξ. There exist a $n_0 \in \mathbb{N}$ such that $1 - d(x, x_n, \xi) \leq \epsilon$ and $\underline{d}(x, x_n, \xi) \leq \epsilon$ for all $n \geq n_0$.

Definition 2.8. A triple sequence over IFMS $(X, d, \underline{d}, o, *)$ is defined as a function f :$\mathbb{N}\times\mathbb{N}\times\mathbb{N}\rightarrow (X, d, \underline{d}, o, *)$ given by $f(m, n, p) = x_{mnp}$.

Definition 2.9. A triple sequence (x_{mnp}) said to be convergent to a number x in Pringsheim's sense if for every $\epsilon > 0$ there exist $(m_0, n_0, p_0) \in \mathbb{N}^{\,3}$ such that such that $|x_{mnp} - x| < \epsilon$ for all $m \geq m_0, \;\; n \geq n_0, \;\; p \geq p_0$.

Example 2.10.

$$x_{mnp} = \begin{cases} m+1, & p=q=1, \\ n+1, & m=p=1, \\ p-1, & m=n=1, \\ \frac{1}{m^2+n^2+p^2}, & \text{otherwise,} \end{cases}$$

the given sequence convergent to 0 in Pringsheim's sense, although the given sequence is unbounded. Hence, we conclude that a convergent triple sequence in Pringsheim's sense may not be bounded.

Definition 2.11. Let $(X, d, \underline{d}, o, *)$ is an IFMS. Then, a triple sequence (x_{mnp}) over X is called intuitionistic fuzzy Cauchy sequence, if for every $\epsilon > 0$ and with respect to a positive real ξ there exist $(m_0, n_0, p_0) \in \mathbb{N}^3$ such that $1 - d(x_{ijk}, x_{mnp}, \xi) \leq \epsilon$ and $\underline{d}(x_{ijk}, x_{mnp}, \xi) \leq \epsilon$ for all $i, m \geq m_0, \;\; j, n \geq n_0, \;\; k, p \geq p_0$. Furthermore, an IFMS X is called complete if every Cauchy sequence over X has a limit point in X.

Remark 2.12 ([13])**.** An Ideal I_3 on $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is dense if for every infinite subset A of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ there is an infinite subset B of A and $B \in I$. An ideal I_3 is a P-ideal if, for every countable family $I_c \subseteq I_3$ there is $A \in I_3$ such that $F - A$ is finite for every $F \in I_c$. The ideal I_3 is called maximal ideal over $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ if there doesn't exist any other ideal J_3 between I_3 and $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $I_3 \subset J_3 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. The ideal $I_3 = I_f$ is defined as family of all the finite subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

3. Ideal Convergence

Definition 3.1. Let I_3 be an ideal on $\mathbb{N}^3$ and $(X, d, \underline{d}, o, *)$ be an IFMS. Then, a triple sequence (x_{mnp}) is said to be I_3 - convergent to a number x if for every $\epsilon > 0$ and with respect to every positive real ξ. The set $\{(m,n,p) \in \mathbb{N}^3 : 1 - d(x, x_{mnp}, \xi) > \epsilon \textit{ and } \underline{d}(x, x_{mnp}, \xi) > \epsilon\} \in I_3$.

Example 3.2. Let I_3 be an admissible ideal on $\mathbb{N}^3$ and sequence (x_{mnp}) over IFMS $(\mathbb{R}, d, \underline{d}, o, *)$ defined by

$$x_{mnp} = \begin{cases} \frac{1}{m} + \frac{1}{n} + \frac{1}{p}, & m \neq n \neq p, \\ 1, & m = n = p, \end{cases}$$

$d(u, v, \xi) = \frac{\xi}{\xi + |u - v|}$ and $\underline{d}(x, y, \xi) = \frac{|u-v|}{\xi + |u-v|}$ here we observe that $I_3 - x_{mnp} = 0$ since

$$d(0, x_{mnp}, \xi) = \frac{\xi}{\xi + \left|\frac{1}{m} + \frac{1}{n} + \frac{1}{p}\right|}, \qquad \underline{d}(0, x_{mnp}, \xi) = \frac{\left|\frac{1}{m} + \frac{1}{n} + \frac{1}{p}\right|}{\xi + \left|\frac{1}{m} + \frac{1}{n} + \frac{1}{p}\right|},$$

for $m \neq n \neq p$ for every $\epsilon > 0$ and with respect to every positive real ξ . The set $\left\{(m,n,p) \in \mathbb{N}^3 : 1 - d(0, x_{mnp}, \xi) > \epsilon \text{ and } \underline{d}(0, x_{mnp}, \xi) > \epsilon\right\} \in I_3$.

3.1. Uniqueness of ideal limit.

Theorem 3.3. *The ideal limit of triple sequence over IFMS is not unique in general.*

Proof. We verify the result by an appropriate example. Consider the ideal I_3, a admissible maximal ideal over $\mathbb{N}^3$ and sequence x defined by

$$x_{mnp} = \begin{cases} 1, & n = p = 1, \\ \frac{1}{2}, & m = p = 1, \\ \frac{1}{3}, & m = n = 1, \\ 0, & \text{otherwise}, \end{cases}$$

over IFMS $(\mathbb{R}, d, \underline{d}, o, *)$ where d and $\underline{d}$ are defined as in Example 3.2.

Notice that the ideal limit of given sequence about considered ideal are $0, \frac{1}{2}, \frac{1}{3}$ and 1, which completes our requirement. One thing is more that the Pringsheim's limit of this sequence is 0, which states that in case of triple sequence, the Pringsheim's limit doesn't coincides with ideal limit in general.

Now, we have another example which tells that Pringsheim's doesn't guarantee the ideal limit in general.

Take the sequence described in Example 2.10 over same IFMS as that of above example now change the Ideal I_3 set new Ideal $I_3 = I_{fin}$. Thus we get Pringsheim's limit $= 0$, but ideal limit doesn't exist. □

Theorem 3.4. Let I_3 be an admissible ideal on $\mathbb{N}^3$ and if each sub-sequence of triple sequence (x_{mnp}) of elements of IFMS $(X, d, \underline{d}, o, *)$ is I_3 - convergent to x then the sequence (x_{mnp}) is also I_3-convergent to x.

Proof. Suppose the triple sequence (x_{mnp}) doesn't I_3-convergent to x then there exist a ϵ_0 such that for a positive real ξ, we have $A = \{(m, n, p) \in \mathbb{N}^3 : 1 - d(x, x_{mnp}, \xi) > \epsilon_0 \ and \ \underline{d}(x, x_{mnp}, \xi) > \epsilon_0\} \notin I_3$. Now A is an infinite set Since I_3 is an admissible ideal. now we form a new sequence (y_{ijk}) where triple (i, j, k) comes from the set A. thus we get (y_{ijk}) as a sub-sequence of (x_{mnp}) which is not I_3-convergent which contradicts our hypothesis. This completes the proof. □

Remark 3.5. If degree of nearness d and degree of non-nearnes $\underline{d}$ over IFMS $(X, d, \underline{d}, o, *)$ are defined by $d(u, v, \xi) = \frac{\xi}{\xi+|u-v|}$ and $\underline{d}(u, v, \xi) = \frac{|u-v|}{\xi+|u-v|}$ respectively. Then for all non zero-constant k, we have $d(ku, kv, \xi) = d\left(u, v, \frac{\xi}{|k|}\right)$ and $\underline{d}(ku, kv, \xi) = \underline{d}\left(u, v, \frac{\xi}{|k|}\right)$.

Theorem 3.6. If $\mathbf{lim} - x_{mnp} = l$ then $\mathbf{lim} - kx_{mnp} = kl$ for all non zero constant k.

Proof. Case I: if $k = 0$ result is obvious.
Case II: If $k \neq 0$, for positive real ξ, we have $1 - d(l, x_{mnp}, \xi) < \epsilon$ and $\underline{d}(l, x_{mnp}, \xi) < \epsilon$ for every ϵ. Now,

$$1 - d\left(kl, kx_{mnp}, \xi\right) = 1 - d\left(l, x_{mnp}, \frac{\xi}{|k|}\right) < \epsilon,$$

and $\underline{d}\left(kl, kx_{mnp}, \xi\right) = \underline{d}\left(l, x_{mnp}, \frac{\xi}{|k|}\right) < \epsilon$ for every $\epsilon \in (0, 1)$ by Remark 3.5, we have $\mathbf{lim} - kx_{mnp} = kl$. Hence the proof. □

Theorem 3.7. Let I_3 be an admissible ideal on $\mathbb{N}^3$ and if (x_{mnp}) be triple sequence of elements of IFMS $(X, d, \underline{d}, o, *)$ where d and $\underline{d}$ are defined by (Remark 3.5) if $I_3 - \mathbf{lim} x_{mnp} = l$ then $I_3 - \mathbf{lim} kx_{mnp} = kl$ for all non-zero constant k.

Proof. Case I: If $k = 0$ result is obvious.
Case II: if $k \neq 0$, we have
$A(\xi) = \big\{(m, n, p) \in \mathbb{N}^3 : 1 - d(l, x_{mnp}, \xi) > \epsilon \text{ and } \underline{d}(l, x_{mnp}, \xi) > \epsilon\big\} \in I_3$ for every $\epsilon \in (0, 1)$. Now, we setting new $\xi_0 = \frac{\xi}{|k|}$, and we think over the set

$$A(\xi_0) = \left\{(m, n, p) \in \mathbb{N}^3 : 1 - d\left(kl, kx_{mnp}, \xi\right) = 1 - d\left(l, x_{mnp}, \frac{\xi}{|k|}\right) > \epsilon \right.$$
$$\left. \text{and } \underline{d}\left(kl, kx_{mnp}, \xi\right) = \underline{d}\left(l, x_{mnp}, \frac{\xi}{|k|}\right) > \epsilon \right\}.$$

By Remark 3.5, since ξ was arbitrary hence ξ_0 is also be arbitrary, we get the set $A(\epsilon_0)$ must belongs the ideal I_3. Hence $I_3 - \mathbf{lim} k x_{mnp} = kl$. $\square$

4. Ideal-Cauchy Sequence

Definition 4.1. Let I_3 be an admissible ideal on $\mathbb{N}^3$ then a triple sequence (x_{mnp}) of elements of IFMS $(X, d, \underline{d}, o, *)$ is called I_3-Cauchy sequence is for positive ξ and for all $\epsilon \in (0,1)$ so, there exist $(m_0, n_0, p_0) \in \mathbb{N}^3$ such that the set

$$A = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(x_{mnp}, x_{m_0 n_0 p_0}, \xi) > \epsilon \\ \text{and } \underline{d}(x_{mnp}, x_{m_0 n_0 p_0}, \xi) > \epsilon\big\} \in I_3.$$

Theorem 4.2. The space Ω^I, of ideal Cauchy triple sequences over IFMS $(X, d, \underline{d}, o, *)$ where d and $\underline{d}$ are defined by (Remark 3.5), is a linear space.

Proof. Let $x = (x_{mnp})$ and $y = (y_{mnp})$ are in Ω we wish to show that $x + y \in \Omega$ and $\alpha x \in \Omega$ for all α (scalar). Now, there exist $(m_0, n_0, p_0) \in \mathbb{N}^3$ such that for all $m \geq m_0$, $n \geq n_0$, $p \geq p_0$ the set

$$A = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(x_{mnp}, x_{m_0 n_0 p_0}, \xi) > \epsilon \\ \text{and } \underline{d}(x_{mnp}, x_{m_0 n_0 p_0}, \xi) > \epsilon\big\} \in I_3,$$

$$B = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(y_{mnp}, y_{m_0 n_0 p_0}, \xi) > \epsilon \\ \text{and } \underline{d}(y_{mnp}, y_{m_0 n_0 p_0}, \xi) > \epsilon\big\} \in I_3,$$

$$A^c = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(x_{mnp}, x_{m_0 n_0 p_0}, \xi) \leq \epsilon \\ \text{and } \underline{d}(x_{mnp}, x_{m_0 n_0 p_0}, \xi) \leq \epsilon\big\} \in I_3^*,$$

$$B^c = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(y_{mnp}, y_{m_0 n_0 p_0}, \xi) \leq \epsilon \\ \text{and } \underline{d}(y_{mnp}, y_{m_0 n_0 p_0}, \xi) \leq \epsilon\big\} \in I_3^*.$$

Where, I_3^* denotes the filter dual to ideal I_3, then $A^c \cap B^c \in I_3^*$. Hence, there exist $(i_0, j_0, k_0) \in \mathbb{N}^3$ such that $i \geq i_0$, $j \geq j_0$, $k \geq k_0$ the set

$$A^c \cap B^c = \big\{(i,j,k) \in N^3 : 1 - d(x_{ijk} + y_{ijk}, x_{i_0 j_0 k_0} + y_{i_0 j_0 k_0}, \xi) < \epsilon \\ \text{and } \underline{d}(x_{ijk} + y_{ijk}, x_{i_0 j_0 k_0} + y_{i_0 j_0 k_0}, \xi) < \epsilon\big\} \in I_3^*.$$

Then

$$A \cup B = \big\{(i,j,k) \in \mathbb{N}^3 : 1 - d(x_{ijk} + y_{ijk}, x_{i_0 j_0 k_0} + y_{i_0 j_0 k_0}, \xi) > \epsilon \\ \text{and } \underline{d}(x_{ijk} + y_{ijk}, x_{i_0 j_0 k_0} + y_{i_0 j_0 k_0}, \xi) > \epsilon\big\} \in I_3$$

so

$$(x_{mnp}) + (y_{mnp}) \in \mathbf{\Omega^I}.$$

Also, we have

$$A = \big\{(m,n,p) \in \mathbb{N}^3 : 1 - d(x_{mnp}, x_{m_0 n_0 p_0}, \xi) > \epsilon$$

$$\text{and } \underline{d}(x_{mnp}, x_{m_0n_0p_0}, \xi) > \epsilon\} \in I_3.$$

Replacing ξ by $\frac{\xi}{|\alpha|}$ for non zero α in the set A we get the set

$$\left\{(m,n,p) \in \mathbb{N}^3 : 1 - d\left(x_{mnp}, x_{m_0n_0p_0}, \frac{\xi}{|\alpha|}\right) \geq \epsilon \right.$$
$$\left. \text{and } \underline{d}\left(x_{mnp}, x_{m_0n_0p_0}, \frac{\xi}{|\alpha|}\right) \geq \epsilon\right\} \in I_3.$$

By using the result $d(kx, ky, \xi) = d\left(x, y, \frac{\xi}{|k|}\right)$ and $\underline{d}(kx, ky, \xi) = \underline{d}\left(x, y, \frac{\xi}{|k|}\right)$ for all non zero constant k, then we get the set

$$\{(m,n,p) \in \mathbb{N}^3 : 1 - d(\alpha x_{mnp}, \alpha x_{m_0n_0p_0}, \xi) \geq \epsilon$$
$$\text{and } \underline{d}(\alpha x_{mnp}, \alpha x_{m_0n_0p_0}, \xi) \geq \epsilon\} \in I_3.$$

Then, $\alpha(x_{mnp}) \in \mathbf{\Omega^I}$. This completes the proof. □

Lemma 4.3. *If degree of nearness d and degree of non-nearness $\underline{d}$ over IFMS $(X, d, \underline{d}, o, *)$ are defined by $d(u,v,\xi) = \frac{\xi}{\xi + |u-v|}$ and $\underline{d}(u,v,\xi) = \frac{|u-v|}{\xi+|u-v|}$ respectively. Then $d(u,v,\xi) \to 1$ and $\underline{d}(u,v,\xi) \to 0$ iff $|u-v| \to 0$.*

Proof. Proof is easy, therefore we omit it. □

Theorem 4.4. *IFMS $(l^3_\infty, d, \underline{d}, o, *)$,where d and $\underline{d}$ are given by Lemma 4.3, is complete.*

Proof. Let $(x^n) = \left(x^n_{ijk}\right)$ be a Caucy sequence of elements of **IFMS** $\left(l^3_\infty, d, \underline{d}, o, *\right)$ then there exist $n_0 \in \mathbb{N}$ such that $d(x^n, x^m, \xi) \to 1$ and $\underline{d}(x^n, x^m, \xi) \to 0$ for all $n, m > n_0$, i.e.

$$\|x^n - x^m\| = \max_{i,j,k} \left|x^n_{ijk} - x^m_{ijk}\right| < \epsilon,$$

for every $\epsilon > 0$ (by previous lemma).

Now consider the sequence (x^n_{ijk}) with fixed i, j, k now the elements of this sequence coming from $\mathbb{R}$ so there exist a natural number $n_\epsilon \in \mathbb{N}$ such that $\forall n, m \geq n_\epsilon$ we have, $\max\left|x^n_{ijk} - x^m_{ijk}\right| < \epsilon$ for every $\epsilon > 0$. Since

$$\max\left|x^n_{ijk} - x^m_{ijk}\right| \leq \left\|x^n_{ijk} - x^m_{ijk}\right\| < \epsilon.$$

The sequence $\left(x^n_{ijk}\right)$ for a fixed i, j, k is a Caucy sequence in $\mathbb{R}$ thus, there exist $(x_{ijk}) \in \mathbb{R}$ such that $d(x_{ijk}, x^n_{ijk}, \xi) \to 1$ and $\underline{d}(x_{ijk}, x^n_{ijk}, \xi) \to 0$ for all $n \geq n_\epsilon$ hence $x^n_{ijk} \to x_{ijk}$ for all i, j, k.

Now, $|x_{ijk}| \leq \left|x_{ijk} - x^n_{ijk}\right| + \left|x^n_{ijk}\right|$ so

$$\max_{i,j,k} |x_{ijk}| \leq \max_{i,j,k} \left|x_{ijk} - x^n_{ijk}\right| + \max_{i,j,k} \left|x^n_{ijk}\right| < \epsilon + M.$$

Since $\max_{i,j,k} \left|x_{ijk} - x^n_{ijk}\right| < \epsilon$ and $\max_{i,j,k} \left|x^n_{ijk}\right| \leq M$ then $(x_{ijk}) \in l^3_\infty$, this completes the proof. □

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