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Hamza Bouda, Chafik Allouch and Mohammed Tahrichi

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Legendre Superconvergent Degenerate Kernel and Nyström Methods for Fredholm Integral Equations

Hamza Bouda [1], Chafik Allouch [2*] and Mohammed Tahrichi[3]

Abstract. In this paper, polynomial-based superconvergent degenerate kernel and Nyström methods for solving Fredholm integral equations of the second kind with the smooth kernel are studied. By using an interpolatory projection based on Legendre polynomials of degree $\leq n$, we analyze the convergence of these methods and we establish superconvergence results for their iterated versions. Two numerical examples are given to illustrate the theoretical estimates.

1. Introduction

Consider the Fredholm integral equation defined on $\mathbb{X} = \mathcal{C}[-1, 1]$ by

$$(I - T)u = f, \tag{1.1}$$

where T is the compact linear integral operator given by

$$(Tu)(s) = \int_{-1}^{1} \kappa(s, t)u(t)dt, \quad s \in [-1, 1], \tag{1.2}$$

with kernel $\kappa(.,.) \in \mathcal{C}([-1, 1] \times [-1, 1])$, f is a real continuous function and $u \in \mathbb{X}$ is the unknown function to be determined. Assume that the homogenous integral equation $u - Tu = 0$ has only the trivial solution in $\mathbb{X}$, then the operator $I - T$ is invertible and therefore, equation (1.1) has a unique solution.

A standard technique for solving (1.1) numerically is to replace T with a finite rank operator. The approximate solution is then obtained by solving a system of linear equations. The projection, Nyström and

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* Corresponding author.

degenerate kernel methods are commonly used methods for this purpose. In the literature (see [5, 6, 9, 13] and the references cited therein), they have been extensively studied. In [16], Sloan introduced the iterated projection solution obtained by one step of iteration which improves upon the projection solution. Recently, Kulkarni proposed in [10] a new method (so-called modified projection or multi-projection method) based on projections for solving (1.1). It is shown that if κ and f are suitably smooth, the resulting solution converges faster than the projection and the Sloan solution. Moreover, the iterated modified projection solution obtained by performing one step of iteration converges the fastest of all. More recently, superconvergent degenerate kernel and Nyström methods, which were inspired by the modified projection method, were used in [2], to solve equation (1.1). These methods are based on an interpolatory projection and converge as rapidly as the multi-projection method.

This paper aims is to redefine these methods using global polynomial basis functions rather than piecewise polynomial basis functions which reduces highly the size of the linear system. In particular one can use Legendre polynomials as basis functions for the approximating subspace. These polynomials are generated recursively with ease and they are less expensive computationally. As in the case of piecewise polynomial basis, we show that these methods improve upon the degenerate kernel and Nyström methods. We show that the approximate solutions in both Legendre degenerate kernel and Nyström methods converge with the order $\mathcal{O}(n^{-r})$, whereas the Legendre superconvergent degenerate kernel and Nyström solutions converge with the order $\mathcal{O}\left(n^{\frac{1}{2}-2r}\right)$, in the infinity norm, where r denotes the smoothness of the kernel and n denotes the degree of the Legendre polynomials employed. By using Sloan's iteration, we prove that the order of convergence of the proposed methods can be improved to reach $\mathcal{O}\left(n^{-2r}\right)$.

In several recent papers, various polynomially based numerical methods for linear integral equations were studied. The discrete Galerkin method using Legendre polynomials was introduced by Golberg [8] and its iterated version was proposed by Kulkarni and Gnaneshwar [11]. The convergence of the Legendre-Galerkin solution in the case of weakly singular kernels was considered by Panigrahi and Gnaneshwar [14]. Moreover, the Legendre multi-projection as well as its iterated version were studied in [12].

The organization of this paper is as follows. In Section 2, we define notations and discuss Legendre degenerate kernel and Nyström methods for Fredholm integral equations with a smooth kernel. In Section 3, we discuss the existence of the approximate solutions and their convergence rates. In Section 4, we describe superconvergent degenerate kernel and Nyström methods based on the interpolatory projection π_n and we give details on the linear system which needs to be solved to obtain the corresponding approximate solutions and superconvergence rates. In Section 5, we illustrate our results with numerical examples.

2. Legendre Degenerate Kernel and Nyström Methods

Let $\mathbb{X}_n$ denote the space of all polynomials of degree $\leq n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the Legendre polynomials $\{L_0, L_1, \dots, L_n\}$ are defined by the following three-term recurrence relation

$$\begin{aligned} &L_0(s) = 1, \qquad L_1(s) = s, \quad s \in [-1,1] \\ &(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1,2,\dots,n-1, \end{aligned} \tag{2.1}$$

form an orthogonal basis for $\mathbb{X}_n$. For $u, v \in \mathbb{X}$, the inner product is given by $\langle u, v\rangle = \int_{-1}^{1} u(t)v(t)dt$ and $\|u\|_{L^2} = \left(\int_{-1}^{1} |u(t)|^2 dt\right)^{\frac{1}{2}}$. Let $\pi_n : \mathbb{X} \longrightarrow \mathbb{X}_n$ be the interpolatory operator which is defined as follows

$$(\pi_n u)(s) = \sum_{i=0}^{n} u(\tau_i)\ell_{n,i}(s), \tag{2.2}$$

and satisfying

$$(\pi_n u)(\tau_i) = u(\tau_i), \quad i = 0,1,\dots,n,$$

where $\{\tau_i : \ i = 0,1,\dots,n\}$ are the zeros of the Legendre polynomial L_{n+1} and $\{\ell_{n,i} : \ i = 0,1,\dots,n\}$ is the Lagrange basis of $\mathbb{X}_n$. Throughout this paper, we assume that c is a generic constant which is independent of n.

According to the analysis of (Canuto et al. [7], p. 289), the crucial properties of π_n are given in the following proposition.

Proposition 2.1. *There exists a constant $p > 0$ independent of n such that for any $n \in \mathbb{N}$ and any $u \in L^2[-1,1]$,*

$$\|\pi_n u\|_{L^2} \leq p \|u\|_{L^2}, \tag{2.3}$$

$$\|u - \pi_n u\|_{L^2} \leq c \inf_{v \in \mathbb{X}_n} \|u - v\|_{L^2} \to 0, \quad n \to \infty. \tag{2.4}$$

Moreover, for any $u \in \mathcal{C}^r[-1,1]$, there exists a constant c_1 independent of n such that

$$\|u - \pi_n u\|_{L^2} \leq c_1 n^{-r} \left\|u^{(r)}\right\|_{L^2}, \tag{2.5}$$

$$\|u - \pi_n u\|_{\infty} \leq c_1 n^{\frac{1}{2}-r} \left\|u^{(r)}\right\|_{\infty}. \tag{2.6}$$

The estimate (2.5) shows that $\|u-\pi_n x\|_{L_2} \to 0$, as $n \to \infty$ for any $u \in \mathcal{C}^r[-1,1]$, whereas the estimate (2.6) implies that $\|u-\pi_n u\|_\infty \nrightarrow 0$, as $n \to \infty$ for any $u \in \mathcal{C}^r[-1,1]$.

2.1. **Approximate solutions.** Let us consider the following degenerate kernel

$$\begin{aligned} \kappa_n(s,t) &= \pi_n \kappa(s,t) \\ &= \sum_{i=0}^{n} \kappa(s,\tau_i)\ell_{n,i}(t), \end{aligned} \tag{2.7}$$

which is obtained by interpolating π_n and the kernel $\kappa(s,t)$ considered as a function of t. The associated degenerate kernel operator is given by

$$\begin{aligned} (T_{n,1}u)(s) &= \int_{-1}^{1} \kappa_n(s,t)u(t)dt \\ &= \sum_{i=0}^{n} \kappa(s,\tau_i) \int_{-1}^{1} \ell_{n,i}(t)u(t)dt, \quad s \in [-1,1]. \end{aligned} \tag{2.8}$$

On the other hand, the Nyström operator based on π_n is defined by

$$\begin{aligned} (T_{n,2}u)(s) &= \int_{-1}^{1} \pi_n[\kappa(s,.)u(.)](t)dt \\ &= \sum_{i=0}^{n} \omega_{n,i}\kappa(s,\tau_i)u(\tau_i), \quad s \in [-1,1], \end{aligned} \tag{2.9}$$

where $\omega_{n,i} = \int_{-1}^{1} \ell_{n,i}(t)dt,\ \ i = 0,1,\dots,n$.

We denote by $u_{n,1}$ and $u_{n,2}$ the approximate solutions obtained by using the degenerate kernel and Nyström methods respectively, then for $i = 1,2$, we have

$$(I - T_{n,i})u_{n,i} = f. \tag{2.10}$$

Using the formula (2.8), the approximate equation (2.10) becomes

$$u_{n,1}(s) - \sum_{j=0}^{n} \kappa(s,\tau_j) \int_{-1}^{1} \ell_{n,j}(t)u_{n,1}(t)dt = f(s). \tag{2.11}$$

Hence, the degenerate kernel solution $u_{n,1}$ takes the form

$$u_{n,1}(s) = f(s) + \sum_{j=0}^{n} a_j \kappa(s,\tau_j). \tag{2.12}$$

To determine $\{a_j\}$, multiply (2.11) by $\ell_{n,i}(s)$ and integrate over $[-1,1]$. This yields the system

$$a_i - \sum_{j=0}^{n} a_j \int_{-1}^{1} \ell_{n,i}(s)\kappa(s,\tau_j)ds = \int_{-1}^{1} \ell_{n,i}(s)f(s)ds, \quad 0 \le i \le n. \tag{2.13}$$

Similarly, using the Nyström operator $T_{n,2}$ defined by (2.9), it follows from (2.10) that

$$u_{n,2}(s) - \sum_{j=0}^{n} \omega_{n,j}\kappa(s,\tau_j)u_{n,2}(\tau_j) = f(s), \quad s \in [-1,1]. \tag{2.14}$$

Collocating the above equation at the nodes τ_i, yields

$$u_{n,2}(\tau_i) - \sum_{j=0}^{n} \omega_{n,j}\kappa(\tau_i,\tau_j)u_{n,2}(\tau_j) = f(\tau_i), \quad 0 \le i \le n, \tag{2.15}$$

which is a linear system of size $n+1$ having as unknown the vector

$$u_{n,2} = [u_{n,2}(\tau_0), u_{n,2}(\tau_1), \ldots, u_{n,2}(\tau_n)]^T.$$

Once this system is solved, the Legendre Nyström solution $u_{n,2}$ follows immediately from (2.14) and it is

$$u_{n,2}(s) = f(s) + \sum_{j=0}^{n} \omega_{n,j}\kappa(s,\tau_j)u_{n,2}(\tau_j), \quad s \in [-1,1].$$

In practice, the integrals involving (2.13) and (2.15) are evaluated by an appropriate quadrature formula.

2.2. **Convergence rates.** To discuss the existence and uniqueness of the approximate solutions we need first to recall the following definition of ν-convergence and a lemma from [1].

Definition 2.2 (ν-convergence)**.** Let $\mathcal{X}$ be Banach space and $BL(\mathcal{X})$ be space of bounded linear operators from $\mathcal{X}$ into $\mathcal{X}$. Let $A, A_n \in BL(\mathcal{X})$. We say that A_n is ν–convergent to A if

- (H_1) $\|A_n\| \le c < \infty$,
- (H_2) $\|(A_n - A)A\| \to 0$ as $n \to \infty$,
- (H_3) $\|(A_n - A)A_n\| \to 0$ as $n \to \infty$.

Lemma 2.3 (Ahues et al. [1])**.** *Let $\mathcal{X}$ be a Banach space and A, A_n be bounded linear operators on $\mathcal{X}$. If $\|A_n - A\| \to 0$, as $n \to \infty$ or A_n is ν-convergent to A and $(I - A)^{-1}$ exists, then for n large enough $(I - A_n)^{-1}$ exists and is uniformly bounded on $\mathcal{X}$.*

For $r \geq 1$, we suppose that $\kappa(.,.) \in \mathcal{C}^r([-1,1]\times[-1,1])$. Then the range $R(T)$ of T is contained in $\mathcal{C}^r[-1,1]$ which implies if $f \in \mathcal{C}^r[-1,1]$, the exact solution u of (1.1) belongs to $\mathcal{C}^r[-1,1]$. We set

$$\begin{aligned} D^{i,j}\kappa(s,t) &= \frac{\partial^{i+j}\kappa}{\partial s^i \partial t^j}(s,t), \quad s,t \in [-1,1], \\ \|\kappa\|_{r,\infty} &= \max\left\{\left\|D^{i,j}\kappa\right\|_\infty : \ i,j = 0,1,\ldots,r\right\}, \\ \|u\|_{r,\infty} &= \max\left\{\left\|u^{(i)}\right\|_\infty : \ i = 0,1,\ldots,r\right\}. \end{aligned}$$

Lemma 2.4. *Let $T_{n,i} : \mathbb{X} \longrightarrow \mathbb{X}$, $i = 1,2$ be the linear operators defined by (2.8) and (2.9) respectively. Assume that the inverse of $(I-T)$ exists and is uniformly bounded. Then, for a sufficiently large n, the operators $(I-T_{n,i})^{-1}$, $i=1,2$ exist. Moreover,*

$$\left\|(I-T_{n,i})^{-1}\right\|_\infty \leq C_1, \tag{2.16}$$

for a suitable constant C_1 independent of n.

Proof. For a fixed $s \in [-1,1]$, let κ_s be the s section of κ and let $u \in \mathcal{C}[-1,1]$. By using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} |(T-T_{n,1})u(s)| &= \left|\int_{-1}^{1} u(t)(I-\pi_n)\kappa_s(t)dt\right| \\ &\leq \|(I-\pi_n)\kappa_s\|_{L^2}\|u\|_{L^2} \\ &\leq \sqrt{2}\,\|(I-\pi_n)\kappa_s\|_{L^2}\|u\|_\infty . \end{aligned} \tag{2.17}$$

Hence from (2.4)

$$\|(T-T_{n,1})\|_\infty \leq \sqrt{2}\max_{s\in[-1,1]}\|(I-\pi_n)\kappa_s\|_{L^2} \to 0, \quad n\to\infty.$$

The desired result is now immediate from Lemma 2.3. For the Nyström operator we need to show that $T_{n,2}$ is ν-convergent to T. Let $u \in \mathcal{C}[-1,1]$ and let $s \in [-1,1]$. Using Holder's inequality, we have

$$\begin{aligned} |(T_{n,2}u)(s)| &= \left|\int_{-1}^{1}[\pi_n\kappa(s,.)u(.)](t)dt\right| \\ &\leq \left(\int_{-1}^{1}|\pi_n\kappa(s,.)u(.)(t)|^2\,dt\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, by (2.3), we have

$$\begin{aligned} |(T_{n,2}u)(s)| &\leq p\,\|\kappa_s u\|_{L^2} \\ &\leq 2p\,\|\kappa\|_{0,\infty}\|u\|_\infty . \end{aligned} \tag{2.18}$$

Thus, (H_1) is satisfied with $c = 2p\,\|\kappa\|_{0,\infty}$. Similarly to (2.17), we have

$$\|(T-T_{n,2})Tu\|_\infty \leq \max_{s\in[-1,1]}\|(I-\pi_n)\kappa_s Tu\|_{L^2}$$

$$\leq \sqrt{2} \max_{s\in[-1,1]} \left\|(I-\pi_n)\kappa_s T\right\|_{L^2} \left\|u\right\|_\infty .$$

Since T is a compact linear integral operator in $L^2[-1,1]$ and π_n, converges to the identity operator pointwise, then it follows that $\left\|(I-\pi_n)\kappa_s T\right\|_{L^2} \to 0$ as $n\to\infty$. Hence, we deduce that

$$\left\|(T-T_{n,2})T\right\|_\infty \leq \sqrt{2} \max_{s\in[-1,1]} \left\|(I-\pi_n)\kappa_s T\right\|_{L^2} \to 0 \quad \text{as } n\to\infty.$$

This proves (H_2). The condition (H_3) can be checked in the same way as (H_2). □

Proposition 2.5. *Let $\kappa \in \mathcal{C}^r[-1,1]^2$. In the case of the degenerate kernel method, we assume that $u \in \mathcal{C}[-1,1]$, while in the case of the Nyström method we assume that $u \in \mathcal{C}^r[-1,1]$. Then, for $i=1,2$, there holds*

$$\left\|(T-T_{n,i})u\right\|_\infty = \mathcal{O}(n^{-r}). \tag{2.19}$$

Proof. By (2.5) and (2.17), we have for any $s\in[-1,1]$

$$\begin{aligned} |(T-T_{n,1})u(s)| &\leq \|(I-\pi_n)\kappa_s\|_{L_2}\|u\|_{L_2} \\ &\leq c_1\sqrt{2}n^{-r}\|\kappa_s^{(r)}\|_{L_2}\|u\|_\infty \\ &\leq 2c_1\|\kappa\|_{r,\infty}\|u\|_\infty n^{-r}. \end{aligned}$$

Taking norms, we obtain (2.19). The proof for $T_{n,2}$ is essentially the same. □

Theorem 2.6. *In the case of the degenerate kernel method, we assume that $\kappa(s,.) \in \mathcal{C}^r[-1,1]$ for all $s\in[-1,1]$ and $u\in\mathcal{C}[-1,1]$, while in the case of the Nyström method we assume that $\kappa(s,.)u(.) \in \mathcal{C}^r[-1,1]$ for all $s\in[-1,1]$ and $u\in\mathcal{C}^r[-1,1]$. Then, for $i=1,2$, there holds*

$$\|u-u_{n,i}\|_\infty = \mathcal{O}(n^{-r}). \tag{2.20}$$

Proof. According to [5, theorem 2.1.1],

$$\|u-u_{n,i}\|_\infty \leq \left\|(I-T_{n,i})^{-1}\right\|_\infty \left\|(T-T_{n,i})u\right\|_\infty. \tag{2.21}$$

Then (2.20) is deduced by combining the above estimate with (2.19) and (2.16). □

3. Superconvergent Nyström and Degenerate Kernel Methods

3.1. **Description of the Methods.** We propose to approximate T by the following two finite rank operators

$$T_n = \pi_n T + T_{n,i} - \pi_n T_{n,i}, \quad i=1,2.$$

The corresponding approximate of (1.1) becomes

$$u_{n,i}^S - (\pi_n T + T_{n,i} - \pi_n T_{n,i}) u_{n,i}^S = f, \tag{3.1}$$

where $u_{n,1}^S$ and $u_{n,2}^S$ are the superconvergent degenerate kernel and Nyström solutions, respectively. The iterated solutions are defined by

$$\widetilde{u}_{n,i}^S = T u_{n,i}^S + f. \tag{3.2}$$

Now, we consider the reduction of (3.1) to a system of linear equations. Set

$$\bar{\kappa}_j = \kappa(., \tau_j), \qquad \kappa_j^* = T\bar{\kappa}_j, \qquad \tilde{\ell}_{n,j} = T\ell_{n,j}, \quad j = 0, \dots, n.$$

We successively obtain

$$\pi_n T u = \sum_{i=0}^{n} \left(\int_{-1}^{1} \kappa(\tau_i, t) u(t) dt \right) \ell_{n,i} \tag{3.3}$$

$$= \sum_{i=0}^{n} a_i' \ell_{n,i},$$

$$T_{n,1} u = \sum_{j=0}^{n} \kappa(., \tau_j) \int_{-1}^{1} \ell_{n,j}(t) u(t) dt \tag{3.4}$$

$$= \sum_{j=0}^{n} b_j \kappa(., \tau_j).$$

Then

$$\pi_n T_{n,1} u = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} b_j \kappa(\tau_i, \tau_j) \right) \ell_{n,i}.$$

By using (3.1), the approximate solution can be written

$$u_{n,1}^S = f + \sum_{i=0}^{n} a_i \ell_{n,i} + \sum_{j=0}^{n} b_j \bar{\kappa}_j, \tag{3.5}$$

with $a_i = a_i' - \sum\limits_{j=0}^{n} b_j \kappa(\tau_i, \tau_j)$, and the coefficients $\{a_i, b_i \;\; i = 0, 1, \dots, n\}$ are obtained by substituting $u_{n,1}^S$ from equation (3.5) into equation (3.1) then, we successively have

$$\begin{aligned} \pi_n T u_{n,1}^S &= \sum_{i=0}^{n} T u_{n,1}^S(\tau_i) \ell_{n,i} \\ &= \sum_{i=0}^{n} \left[Tf(\tau_i) + \sum_{k=0}^{n} a_k \tilde{\ell}_{n,k}(\tau_i) + \sum_{l=0}^{n} b_l \kappa_l^*(\tau_i) \right] \ell_{n,i}, \end{aligned}$$

$$
\begin{aligned}
T_{n,1}u_{n,1}^S &= \sum_{j=0}^{n} \bar{\kappa}_j \int_{-1}^{1} \ell_{n,j}(t) u_{n,1}^S(t) dt \\
&= \sum_{j=0}^{n} \bar{\kappa}_j \left\langle u_{n,1}^S, \ell_{n,j} \right\rangle \\
&= \sum_{j=0}^{n} \left[\langle f, \ell_{n,j} \rangle + \sum_{k=0}^{n} a_k \langle \ell_{n,k}, \ell_{n,j} \rangle + \sum_{l=0}^{n} b_l \langle \bar{\kappa}_l, \ell_{n,j} \rangle \right] \bar{\kappa}_j, \\
\pi_n T_{n,1} u_{n,1}^S &= \sum_{i=0}^{n} T_{n,1} u_{n,1}^S(\tau_i) \ell_{n,i} \\
&= \sum_{i=0}^{n} \left\{ \sum_{j=0}^{n} \left[\langle f, \ell_{n,j} \rangle + \sum_{k=0}^{n} a_k \langle \ell_{n,k}, \ell_{n,j} \rangle + \sum_{l=0}^{n} b_l \langle \bar{\kappa}_l, \ell_{n,j} \rangle \right] \bar{\kappa}_j(\tau_i) \right\} \ell_{n,i}.
\end{aligned}
$$

By identifying the coefficients of $\ell_{n,i}$ and $\bar{\kappa}_i$, respectively, we obtain the linear system of size $2n+2$

$$
\begin{cases}
a_i = Tf(\tau_i) + \sum\limits_{k=0}^{n} a_k \tilde{\ell}_{n,i}(\tau_i) + \sum\limits_{l=0}^{n} b_l \kappa_l^*(\tau_i) - \sum\limits_{j=0}^{n} b_j \bar{\kappa}_j(\tau_i), \\
b_j = \langle f, \ell_{n,j} \rangle + \sum\limits_{k=0}^{n} a_k \langle \ell_{n,k}, \ell_{n,j} \rangle + \sum\limits_{l=0}^{n} b_l \langle \bar{\kappa}_l, \ell_{n,j} \rangle .
\end{cases}
$$

Now, to get the solution $u_{n,2}^S$, applying π_n and $(I - \pi_n)$ to equation (3.1), we obtain

$$\pi_n u_{n,2}^S - \pi_n T u_{n,2}^S = \pi_n f, \tag{3.6}$$

$$(I - \pi_n) u_{n,2}^S - (I - \pi_n) T_{n,2} u_{n,2}^S = (I - \pi_n) f. \tag{3.7}$$

By writing

$$T u_{n,2}^S = T(I - \pi_n) u_{n,2}^S + T \pi_n u_{n,2}^S, \tag{3.8}$$

and replacing $(I - \pi_n) u_{n,2}^S$ from (3.7), equation (3.8) becomes

$$T u_{n,2}^S = \left[T(I - \pi_n) T_{n,2} + T \pi_n \right] u_{n,2}^S + T(I - \pi_n) f. \tag{3.9}$$

Now, by replacing $T u_{n,2}^S$ in equation (3.6), we obtain

$$\pi_n u_{n,2}^S - \left[\pi_n T(I - \pi_n) T_{n,2} + \pi_n T \pi_n \right] u_{n,2}^S = \pi_n f + \pi_n T(I - \pi_n) f.$$

The approximate solution is given by

$$u_{n,2}^S = \pi_n u_{n,2}^S + (I - \pi_n) T_{n,2} u_{n,2}^S + (I - \pi_n) f. \tag{3.10}$$

Since

$$\pi_n u = \pi_n v \quad \Leftrightarrow \quad u(\tau_i) = v(\tau_i), \quad i = 0, \dots, n,$$

we deduce that (3.1) is equivalent to the following system of size $n+1$

$$u_{n,2}^S(\tau_i) - (T(I - \pi_n) T_{n,2} u_{n,2}^S)(\tau_i) - (T \pi_n u_{n,2}^S)(\tau_i) \tag{3.11}$$

$$= f(\tau_i) + (T(I - \pi_n)f)(\tau_i), \quad i = 1, \dots, n.$$

Remark 3.1. The iterated solution $\widetilde{u}^S_{n,i}$ is often an improvement on $u^S_{n,i}$ and is obtained by substituting (3.5) and (3.10) into the definition (3.2). Now, applying π_n to both sides of equations (3.1) and (3.2), we have

$$\begin{aligned} \pi_n u^S_{n,i} &= \pi_n T u^S_{n,i} + \pi_n f \\ &= \pi_n \widetilde{u}^S_{n,i}, \quad i = 1, 2, \end{aligned}$$

and this yields

$$u^S_{n,i}(\tau_j) = \widetilde{u}^S_{n,i}(\tau_j), \quad j = 0, 1, \dots, n. \tag{3.12}$$

The above formula proves that at the collocation node points the convergence of $u^S_{n,i}$ to u is as rapid as that of $\widetilde{u}^S_{n,i}$ to u.

We now analyze the superconvergence of the proposed methods.

3.2. **Convergence Rates.**

Lemma 3.2. *Let T be an integral operator with a kernel $\kappa \in \mathcal{C}^r[-1,1]^2$, $r \geq 1$. Then the operators $(I - T_n)^{-1}$ exist and are uniformly bounded for a sufficiently large n, i.e. there exists a constant $C_2 > 0$ such that*

$$\left\|(I - T_n)^{-1}\right\|_\infty \leq C_2 < \infty. \tag{3.13}$$

Proof. As

$$T - T_n = (I - \pi_n)(T - T_{n,i}),$$

then

$$\|T - T_n\|_\infty \leq \|(I - \pi_n)T\|_\infty + \|(I - \pi_n)T_{n,i}\|_\infty .$$

Since π_n converges to the identity operator on $\mathcal{C}^r[-1,1]$ and $T, T_{n,i}$ are compacts, therefore, from [6, Lemma 12.1.4], we have $\|(I - \pi_n)T\|_\infty \to 0$ and $\|(I - \pi_n)T_{n,i}\|_\infty \to 0$ as $n \to \infty$. Hence

$$\|T - T_n\|_\infty \to 0 \quad \text{as } n \to \infty,$$

therefore (3.13) is a consequence of Lemma 2.3. □

The error estimates for $u^S_{n,i}$ and $\widetilde{u}^S_{n,i}$, $i = 1, 2$, can be summarized as follows.

Theorem 3.3. *For all integers n large enough and $i = 1, 2$, we have*

$$\left\|u - u^S_{n,i}\right\|_\infty \leq \left\|(I - T_n)^{-1}\right\|_\infty \|(I - \pi_n)(T - T_{n,i})u\|_\infty, \tag{3.14}$$

and

$$\left\|u - \widetilde{u}^S_{n,i}\right\|_\infty \leq \left\{1 + \sqrt{2}p \left\|T(I - T_n)^{-1}\right\|_\infty\right\} \|T(I - \pi_n)(T - T_{n,i})u\|_\infty. \tag{3.15}$$

Proof. The estimate (3.14) is proved in [10], while the estimate (3.15), can be obtained exactly as in the proof of Theorem 2.3 in [12]. □

Proposition 3.4. *Let T be an integral operator with a kernel $\kappa(.,.) \in \mathcal{C}^r[-1,1]^2$. Then for any $u \in \mathcal{C}^r[-1,1]$, the following error bound holds*

$$\|T(I-\pi_n)u\|_\infty \leq 2c_1 \|\kappa\|_{0,\infty} \left\|u^{(r)}\right\|_\infty n^{-r}. \tag{3.16}$$

Proof. Using the estimate (2.5) and Cauchy-Schwarz's inequality, for any $s \in [-1,1]$ we have

$$\begin{aligned} |[T(I-\pi_n)u]\,(s)| &= |\langle \kappa_s, (I-\pi_n)u\rangle| \\ &\leq \|\kappa_s\|_{L^2} \|(I-\pi_n)u\|_{L^2} \\ &\leq \sqrt{2}c_1 \|\kappa\|_{0,\infty} \left\|u^{(r)}\right\|_{L^2} n^{-r} \\ &\leq 2c_1 \|\kappa\|_{0,\infty} \left\|u^{(r)}\right\|_\infty n^{-r}. \end{aligned}$$

Hence taking supremum over $s \in [-1,1]$, we obtain the required estimate. ☐

Proposition 3.5. *Let T be an integral operator with a kernel $\kappa(.,.) \in \mathcal{C}^{2r}[-1,1]^2$. In the case of the degenerate kernel operator, we assume that $u \in \mathcal{C}[-1,1]$, while in the case of the Nyström operator, we assume that $u \in \mathcal{C}^r[-1,1]$. Then, for $i = 1,2$, the following estimates hold*

$$\left\|[(T-T_{n,i})u]^{(r)}\right\|_\infty = \mathcal{O}\left(n^{-r}\right), \tag{3.17}$$

$$\|(I-\pi_n)(T-T_{n,i})u\|_\infty = \mathcal{O}\left(n^{\frac{1}{2}-2r}\right), \tag{3.18}$$

$$\|T(I-\pi_n)(T-T_{n,i})u\|_\infty = \mathcal{O}\left(n^{-2r}\right). \tag{3.19}$$

Proof. We put $q(s,t) = \frac{\partial^r \kappa}{\partial s^r}(s,t)$. Note that for $s \in [-1,1]$, we have

$$[(T-T_{n,1})u]^{(r)}\,(s) = \int_{-1}^{1} u(t)\,[(I-\pi_n)q(s,.)]\,(t)dt,$$

where the kernel $q(.,.) \in \mathcal{C}^r[-1,1]^2$. Then (3.17) is deduced from the bound (2.19) and a similar estimate can be obtained for $i=2$.

Using the estimates (2.6) and (3.17), we get

$$\begin{aligned} \|(I-\pi_n)(T-T_{n,i})u\|_\infty &\leq c_1 \left\|[(T-T_{n,i})u]^{(r)}\right\|_\infty n^{\frac{1}{2}-r} \\ &\leq c_1 n^{\frac{1}{2}-2r}, \end{aligned}$$

and this gives (3.18).

Next, using (3.16), we obtain

$$\|T(I-\pi_n)(T-T_{n,i})u\|_\infty \leq 2c_1\|\kappa\|_{0,\infty} \left\|[(T-T_{n,i})u]^{(r)}\right\|_\infty n^{-r},$$

and combining with (3.17) the estimate (3.19) is obtained. This completes the proof. ☐

The following main result state that the proposed methods have the same rate of convergence.

Theorem 3.6. *Let $u_{n,i}^S$ and $\widetilde{u}_{n,i}^S$, $i = 1, 2$ be the approximate solutions of equation* (1.1) *defined by* (3.1) *and* (3.2)*, respectively. We assume that $\kappa(.,.) \in \mathcal{C}^{2r}([-1,1] \times [-1,1])$. In the case of the degenerate kernel operator, we assume that $u \in \mathcal{C}[-1,1]$, while in the case of the Nyström operator we assume that $u \in \mathcal{C}^r[-1,1]$. Then, for $i = 1, 2$,*

$$\left\|u - u_{n,i}^S\right\|_\infty = \mathcal{O}\left(n^{\frac{1}{2}-2r}\right), \tag{3.20}$$

$$\left\|u - \widetilde{u}_{n,i}^S\right\|_\infty = \mathcal{O}\left(n^{-2r}\right). \tag{3.21}$$

Moreover, we have the following superconvergence result for $u_{n,i}^S$ at the collocation points

$$\max_{0 \le j \le n} \left|u(\tau_j) - u_{n,i}^S(\tau_j)\right| = \mathcal{O}\left(n^{-2r}\right). \tag{3.22}$$

Proof. It suffices to apply the estimates of Theorem 3.3 and that of propositions 3.5, to obtain the estimates of (3.20) and (3.21). We recall from (3.12), that the solutions $u_{n,i}^S$ and $\widetilde{u}_{n,i}^S$ agree with the collocation node points, therefore (3.22) comes from (3.21) and this concludes the proof. □

Remark 3.7. Comparing the aspect of the two methods, we observe that the superconvergent degenerate kernel method is less demanding in terms of regularity required on the exact solution, however, it is more expansive in terms of the computational cost since to obtain an approximate solution, it is necessary to solve the system with a size twice the other method.

4. Numerical Results

In this section, numerical examples are given to illustrate the results obtained in the previous sections. Note that, all required integrals were calculated by a high accurate Gauss quadrature rule. The numerical algorithms are compiled by using Wolfram Mathematica. We give the errors obtained by the degenerate kernel and Nyström methods and the superconvergent methods as well as their iterated versions. Moreover, we give the maximum error of the solution $u_{n,i}^S$ at the collocation points defined as

$$\max_{0 \le j \le n} \left|u(\tau_j) - u_{n,i}^S(\tau_j)\right| = \max_j \left|u^j - u_{n,i}^S\right|, \quad i = 1, 2.$$

In Tables 1 and 3, we present the errors obtained by Legendre degenerate kernel and Nyström methods and their superconvergent versions. The errors $u^S_{n,i}$ at the collocation points and the iterated versions are given in Tables 2 and 4.

Example 4.1. Consider the following Fredholm integral equation

$$u(s) - \int_{-1}^{1} \cos(s)e^{s-t}u(t)dt = e^{-s} + \frac{1}{2}\cos(s)[e^{s-2} - e^{s+2}], \quad s \in [-1,1],$$

where the exact solution is $u(s) = e^{-s}$. The results are given in Tables 1 and 2.

TABLE 1. Degenerate kernel and Nyström methods and their superconvergent versions

n	$\\|u - u_{n,1}\\|_\infty$	$\\|u - u_{n,2}\\|_\infty$	$\\|u - u^S_{n,1}\\|_\infty$	$\\|u - u^S_{n,2}\\|_\infty$
1	1.3695×10^{-1}	2.0718×10^{-1}	2.0936×10^{-3}	2.9117×10^{-3}
2	4.5396×10^{-3}	6.8536×10^{-3}	1.2365×10^{-5}	1.7213×10^{-5}
3	7.3256×10^{-5}	1.2178×10^{-4}	7.3882×10^{-7}	1.1572×10^{-6}
4	8.5370×10^{-7}	1.3679×10^{-6}	1.6738×10^{-10}	2.6822×10^{-10}
5	6.3893×10^{-9}	1.0415×10^{-8}	5.3215×10^{-14}	8.6746×10^{-14}
6	3.4836×10^{-11}	5.7562×10^{-11}	1.7877×10^{-16}	2.9539×10^{-16}

TABLE 2. The errors $u^S_{n,i}$ at the collocation points and the iterated versions.

n	$\max_j \|u^j - u^S_{n,1}\|$	$\max_j \|u^j - u^S_{n,2}\|$	$\\|u - \widetilde{u}^S_{n,1}\\|_\infty$	$\\|u - \widetilde{u}^S_{n,2}\\|_\infty$
1	5.8070×10^{-3}	8.0771×10^{-3}	3.9294×10^{-3}	5.4656×10^{-3}
2	2.6521×10^{-6}	4.0314×10^{-6}	1.7445×10^{-6}	2.6517×10^{-6}
3	3.7550×10^{-9}	5.8819×10^{-9}	2.5089×10^{-9}	3.9299×10^{-9}
4	6.1977×10^{-13}	9.9312×10^{-13}	4.2227×10^{-13}	6.7665×10^{-13}
5	2.1312×10^{-17}	3.4755×10^{-17}	1.4636×10^{-17}	2.3868×10^{-17}

Example 4.2. Consider the following Fredholm integral equation

$$u(s) - \int_{-1}^{1} \kappa(s,t)u(t)dt = f(s), \quad s \in [-1,1],$$

with the kernel function $\kappa(s,t) = \sqrt{s + \frac{13}{10}}\sinh(s-t)$ and the function $f(s)$ is selected so that $u(s) = \sqrt{s+2}$. The results are given in Tables 3 and 4.

TABLE 3. Degenerate kernel and Nyström methods and their superconvergent versions

n	$\|u - u_{n,1}\|_\infty$	$\|u - u_{n,2}\|_\infty$	$\|u - u^S_{n,1}\|_\infty$	$\|u - u^S_{n,2}\|_\infty$
1	4.1519×10^{-3}	4.5710×10^{-3}	9.3066×10^{-4}	1.6907×10^{-3}
2	4.7490×10^{-5}	2.6668×10^{-5}	9.5062×10^{-7}	1.5621×10^{-6}
3	3.2943×10^{-7}	7.0673×10^{-7}	2.1658×10^{-8}	6.4476×10^{-8}
4	5.5513×10^{-9}	3.9286×10^{-8}	2.0842×10^{-11}	8.1424×10^{-11}
5	1.6493×10^{-10}	1.9955×10^{-9}	2.9953×10^{-13}	1.0138×10^{-12}
6	1.7805×10^{-12}	1.0799×10^{-10}	4.4408×10^{-16}	7.3274×10^{-15}

TABLE 4. The errors $u^S_{n,i}$ at the collocation points and the iterated versions.

n	$\max_j \|u^j - u^S_{n,1}\|$	$\max_j \|u^j - u^S_{n,2}\|$	$\|u - \widetilde{u}^S_{n,1}\|_\infty$	$\|u - \widetilde{u}^S_{n,2}\|_\infty$
1	6.4076×10^{-4}	8.6851×10^{-4}	9.7252×10^{-5}	1.9296×10^{-4}
2	2.0599×10^{-6}	2.4159×10^{-6}	1.1734×10^{-7}	3.0607×10^{-7}
3	9.9167×10^{-11}	1.6817×10^{-9}	8.2643×10^{-11}	2.1461×10^{-10}
4	3.2134×10^{-12}	3.4940×10^{-12}	2.2626×10^{-13}	1.0946×10^{-13}
5	5.3290×10^{-15}	2.2204×10^{-14}	1.1102×10^{-15}	8.8817×10^{-16}

From the numerical results obtained, we note that the approximation errors are in agreement with those theoretically expected. In addition, we see that the superconvergent methods (degenerate kernel and Nyström) are more precise than the classical ones.

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[1] University Mohammed I, Team MSC, FPN, LAMAO Laboratory, Nador, 62000, Morocco.

Email address: hamzabouda19@gmail.com

[2] University Mohammed I, Team MSC, FPN, LAMAO Laboratory, Nador, 62000, Morocco.

Email address: c.allouch@ump.ma

[3] University Mohammed I, Team ANAA, EST, LANO Laboratory, Oujda, 60000, Morocco.

Email address: mtahrichi@hotmail.com