# Pseudosymmetric Spaces as Generalization of Symmetric spaces

Bilal Bilalov, Yusuf Zeren, Betule Alizade and Feyza Elif Dal

**Sahand Communications in Mathematical Analysis**

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 1
Pages: 61-79

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.551403.1090

Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 1 (2023), 61-79
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.551403.1090

# Pseudosymmetric Spaces as Generalization of Symmetric spaces

Bilal Bilalov[1] *, Yusuf Zeren[2], Betule Alizade[3] and Feyza Elif Dal[4]

Abstract. In this paper, the concept of a pseudosymmetric space which is a natural generalization of the concept of a symmetric space is defined. All basic concepts such as the Luxemburg representation theorem, the Boyd indices, the fundamental function and its properties, Calderon's theorem, etc., is transferred over the pseudosymmetric case. Examples are given for pseudosymmetric spaces. The quasi-symmetric spaces expand the scope of the application of symmetric space results.

## 1. Introduction

Function spaces play an imperative role in various areas of mathematics, including harmonic analysis, Fourier series, the theory of approximation and basis, the theory of partial derivative equations, the spectral theory of differential equations, etc. Historically, differential equations were first studied in Banach spaces of continuous and Hölder functions. The use of Lebesgue spaces became necessary to solve problems over time (predominantly Hilbert space $L_2$). Subsequently, both theoretically and from the point of view of concrete problems of mechanics and mathematical physics, novel function spaces appeared. Along with this, various problems which belong to these spaces of mathematics began to be studied. More details on related topics can be found in the monographs [1, 13–15, 17, 18, 25]. Unlike abstract Banach spaces, the rich structure (lattice structure, different types of convergence, etc.) of Banach function spaces allows for deeper exploration of various analysis

2020 *Mathematics Subject Classification.* 46E30, 46E35.
*Key words and phrases.* Symmetric spaces, Calderon theorem, Boyd indices, Fundamental function.
Received: 05 April 2022, Accepted: 09 October 2022.
* Corresponding author.

problems (including harmonic analysis) and the theory of differential equations in these spaces. The general definition of Banach function spaces (shortly BFS) belongs to Luxemburg [22] (see also [2, 19–21]). Numerous articles by different authors have been devoted this course (e.g., see [3–12, 16, 23, 24, 26–29]) and this tendency continues to develop. It should be noted that symmetric spaces play a crucial role in this course. A separate theory has been created about these spaces and thanks to the well-known theorems of Luxemburg, Calderon, Boyd, etc., many facts of harmonic analysis have been transferred over such spaces. It turns out that an anological theory can be created when the spaces are not symmetric but are close to symmetric spaces. This article is devoted to exactly this problem.

The concept of a pseudosymmetric space which is a natural generalization of the concept of a symmetric space is defined. All basic facts such as the Luxemburg representation theorem, the Boyd indices, the fundamental function and its properties, Calderon's theorem, etc., are transferred over the pseudosymmetric case. Examples are given for pseudosymmetric spaces. The concept of pseudosymmetric spaces expands the range of application of results about symmetric spaces.

## 2. Essential Information

We will use the following standard notations and concepts. $N$ is the set of all positive integers; $Z_+ = \{0\} \bigcup N; Z = \{-N\} \bigcup Z_+$; $R_+ = (0, +\infty)$; $\chi_M(\cdot)$ is the characteristic function of the set $M$; $R$ is the set of all real numbers; $C$ is the set of all complex numbers; $\omega = \{z \in C : |z| < 1\}$ is a unit disk in $C$; $\gamma = \partial\omega$ is a unit circle; $\bar{M}$ is a closure of the set $M$ with respect to the appropriate norm; $\overline{(\cdot)}$ is the complex conjugate; $[x]$ is an integer part of the number $x \in R$. By $[X; Y]$ we denote the space of bounded linear operators with the Banach space $X$ as a domain and the Banach space $Y$ as a range. In case $[X; X]$, we write $[X]$; $R_T$ is a range of the operator $T$, and $KerT$ is its kernel. By $|M|$ we denote the Lebesgue measure of the measurable set $M$.

We will need some concepts and facts from the theory of Banach function spaces (see, e.g., [24, 28]). Let $(\Omega; \mu)$ be a measure space, and $\mathscr{M}^+$ be a cone of $\mu$-measurable functions on $\Omega$ whose values lie in $[0, +\infty]$.

**Definition 2.1.** A mapping $\rho : \mathscr{M}^+ \to [0, +\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N$, in $\mathscr{M}^+$, for all constants $a \geq 0$ and for all $\mu$-measurable subsets $E \subset \Omega$, the following properties hold:

(P1) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0$ $\mu$-a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ $\mu$-a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$;
(P3) $0 \leq f_n \uparrow f$ $\mu$-a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$;
(P4) $\mu(E) < +\infty$ $\Rightarrow$ $\rho(\chi_E) < +\infty$;
(P5) $\mu(E) < +\infty$ $\Rightarrow$ $\int_E f d\mu \leq C_E \rho(f)$, for some constant $C_E$ : $0 < C_E < +\infty$ depending on $E$ and $\rho$, but independent of $f$.

Let $\mathscr{M}$ denote the collection of all extended scalar-valued (real or complex) $\mu$-measurable functions and $\mathscr{M}_0 \subset \mathscr{M}$ denote the subclass of functions that are finite $\mu$-a.e.

**Definition 2.2.** Let $\rho$ be a function norm. The collection $X = X(\rho)$ of all functions $f$ in $\mathscr{M}$ for $\rho(|f|) < +\infty$ is called a Banach function space. Define $\|f\|_X = \rho(|f|)$ for each $f \in X$.

**Theorem 2.3.** *Let $\rho$ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as the definition above. Then, under natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$\mathscr{M}_s \subset X \subset \mathscr{M}_0,$$

*hold, where $\mathscr{M}_s$ is a set of $\mu$-simple functions. In particular, if $f_n \to f$ in $X$, then $f_n \to f$ in measure over finite-dimensional sets, and hence some subsequence converges pointwise $\mu$-a.e. to $f$.*

If $X$ is equipped with the norm $\|f\|_X = \rho(|f|)$, $X$ is called a Banach function space. Let

$$\rho'(g) = \sup\left\{ \int_\gamma f(\tau) g(\tau) |dt| : f \in \mathscr{M}^+; \rho(f) \leq 1 \right\}, \quad \forall g \in \mathscr{M}^+.$$

A space

$$X' = \left\{ g \in \mathscr{M} : \rho'(|g|) < +\infty \right\},$$

is called an associate space (Kothe dual) of $X$.

The functions $f; g \in \mathscr{M}_0$ are called equimeasurable, if

$$\mu(\{\tau \in \Omega : |f(\tau)| > \lambda\}) = \mu(\{\tau \in \Omega : |g(\tau)| > \lambda\}), \quad \forall \lambda \geq 0.$$

Banach function norm $\rho : \mathscr{M}^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in \mathscr{M}_0^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space $X$ with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz spaces are r.i.s.

We will also need some results concerning the Fourier series in r.i.s. First, let's state some concepts and notations.

**Definition 2.4.** Let $X$ be a Banach function space. The closure of the set of simple functions $\mathscr{M}_s$ in $X$ is denoted by $X_b$.

**Definition 2.5.** Suppose $f(\cdot)$ belongs to $\mathscr{M}_0$. The decreasing rearrangement of $f(\cdot)$ is the function $f^*$ defined on $[0,\infty)$ by

$$f^*(t)=\inf\{\lambda:\mu_f(\lambda)\le t\}, \quad t\ge 0,$$

where $\mu_f(\lambda)=\mu\{t:|f(t)|>\lambda\}$, $\lambda\ge 0$, is a distribution function of $f(\cdot)$.

**Theorem 2.6** (Hardy, Littlewood)**.** *If $f(\cdot)$ and $g(\cdot)$ belong to $\mathscr{M}_0$, then*

$$\int_\Omega |f\,g|\,d\mu\le\int_0^\infty f^*(s)\,g^*(s)\,ds. \tag{2.1}$$

*An immediate consequence of the Hardy-Littlewood inequality* (2.1) *is that*

$$\int_\Omega |f\,\tilde g|\,d\mu\le\int_0^\infty f^*(t)\,g^*(t)\,dt, \tag{2.2}$$

*for every function $\tilde g$ on $\Omega$ equimeasurable with $g$.*

**Definition 2.7.** If the supremum on $\tilde g$ of the integrals on the left of (2.2) coincides with the value on the right, such measure space is called resonant. If the supremum is attained, then the measure space is called strongly resonant.

Throughout this paper, we will assume that the considered functions are defined on the interval $(-\pi,\pi]$, and we will equate $(-\pi,\pi]$ with $\gamma$.

Denote by $T_s$ the translation operator $(T_s f)(t)=f\left(e^{i(s+t)}\right)$, $-\pi<s;t\le\pi$, and by $\omega_X(f,\cdot)$ the $X$-modulus of continuity of $f$:

$$\omega_X(f;\delta)=\sup_{|s|\le\delta}\|T_s f-f\|_X, \quad 0\le\delta\le\pi.$$

**Definition 2.8.** Let $X$ be a rearrangement-invariant Banach space (r.i.s.) over a resonant space $(\Omega;\mu)$. For each finite value of $t$ belonging to the range of $\mu$, let $E$ be a subset of $\Omega$ with $\mu(E)=t$ and let

$$\varphi_X(t)=\|\chi_E\|_X\,.$$

The function $\varphi_X$ is called the fundamental function of $X$.

If $f$ belongs to $L_1(\gamma)$, then for each integer $n$ the $n$-th Fourier coefficient of $f$ is defined by

$$\hat f(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)e^{-in\theta}d\theta, \quad n\in Z.$$

The so-called "multiplier" operator $m$ is defined on trigonometric polynomials

$$P\left(e^{i\theta}\right)=\sum_{n=-r}^{r}a_n e^{in\theta} \quad \text{by} \quad mP\left(e^{i\theta}\right)=\sum_{n=-r}^{r}-i\,sign\,n\,a_n e^{in\theta}.$$

It is evident that

$$\left(\overset{\wedge}{mP}\right)(n) = \begin{cases} -i\operatorname{sign} n\, a_n, & \forall n = \overline{-r,r}, \\ 0, & n \neq \overline{-r,r}, \end{cases}$$

for arbitrary trigonometric polynomial $P\left(e^{i\theta}\right) = \sum\limits_{n=-r}^{r} a_n e^{in\theta}$.

Let $S_n$ be partial sums of the Fourier series of the function $f$ :

$$S_n(f) = \sum_{|k|\le n} \hat{f}(k)\, e^{ikt}.$$

**Theorem 2.9.** *Let $X$ be a separable r.i.s. on $\gamma$. Fourier series converge in the norm in $X$ if and only if the Boyd indices of $X$ satisfy $0 < \alpha_X; \beta_X < 1$.*

Recall that the conjugate-function operator $\tilde{f}$ is defined by

$$\tilde{f}\left(e^{i\theta}\right) = \frac{1}{2\pi}\lim_{\varepsilon\to+0}\int_{\varepsilon<|s|\le\pi} f\left(e^{i(\theta-s)}\right)\cot\frac{s}{2}ds, \quad \forall\theta : -\pi < \theta \le \pi.$$

If any one of these conditions holds, then $mf = \tilde{f}$ a.e. $\forall f \in X_b$.

We will also need the following theorem in the sequel.

**Theorem 2.10.** *Suppose $X$ is an r.i.s. on $\gamma$ whose fundamental function satisfies $\varphi_X(+0) = 0$. Then the following conditions are equivalent:*

(i) *Fourier series converge in the norm in $X_b$;*
(ii) *partial sum operators $S_n$ are uniformly bounded in $X_b$;*
(iii) *the multiplier operator $m$ is bounded in $X_b$;*
(iv) *the conjugate-function operator is bounded in $X_b$;*
(v) *Calderon operator*

$$S\, f^*(t) = \int_0^1 f^*(s)\min\left(1, \frac{s}{t}\right)\frac{ds}{s},$$

*is bounded in $(X_b)^-$ (the Luxemburg representation of $X_b$ on the interval $[0,1]$).*

## 3. Pseudosymmetric Spaces

Let $(\Omega; \sigma_\Omega; \mu)$ be a resonant measure space and let $X(\Omega)$ be some b.f.s. on it.

We will say that the space $X(\Omega)$ is pseudosymmetric if

$$\exists\delta > 0 : f; g \in X(\Omega) : \mu_f = \mu_g\,(f \sim g),$$

then the estimate

$$\delta\,\|f\|_{X(\Omega_1)} \le \|g\|_{X(\Omega_2)} \le \delta^{-1}\,\|f\|_{X(\Omega_1)}, \tag{3.1}$$

holds. When $\delta = 1$ in (3.1), then the pseudosymmetric space $X(\Omega)$ is symmetric. Suppose

$$\|f\|_{X(\Omega)} = \rho_\Omega(f), \qquad S_\Omega = \{f \in X(\Omega) : \rho_\Omega(f) \leq 1\}.$$

Let $X'(\Omega)$ be a corresponding associative space with the norms $\rho'_\Omega(\cdot)$ and $S'_\Omega = \{g \in X'(\Omega) : \rho'_\Omega(g) \leq 1\}$.

The following proposition holds.

**Proposition 3.1.** *If b.f.s.* $X(\Omega)$ *is pseudosymmetric, then* $X'(\Omega)$ *is also pseudosymmetric. Expressions*

$$\nu'_\Omega(g) = \sup\left\{\int_0^\infty f^*(s)\, g^*(s)\, ds : f \in S_\Omega\right\},$$

$$\nu_\Omega(f) = \sup\left\{\int_0^\infty f^*(s)\, g^*(s)\, ds : g \in S'_\Omega\right\},$$

*define equivalent b.f.n. in* $X'(\Omega)$ *and* $X(\Omega)$, *respectively, moreover there are estimates*

$$\left\{\begin{array}{ll} \delta\nu'_\Omega(g) \leq \rho'_\Omega(g) \leq \nu'_\Omega(g), & \forall g \in X'(\Omega), \\ \delta\nu_\Omega(f) \leq \rho_\Omega(f) \leq \nu_\Omega(f), & \forall f \in X(\Omega), \end{array}\right. \tag{3.2}$$

*where,* $\delta \in (0,1]$ *is the constant in* (3.1).

*Proof.* First, let us prove the validity of (3.2). We have

$$\rho'_\Omega(g) = \sup_{f \in S_\Omega} \int_\Omega |fg|\, d\mu.$$

The Hardy-Littlewood inequality implies (see. e.g., [2, p. 44])

$$\rho'_\Omega(g) \leq \sup_{f \in S_\Omega} \int_0^\infty f^*(s)\, g^*(s)\, ds.$$

Let $f$ and $\tilde{f}$ be equimeasurable functions in $X(\Omega)$, so that, $\mu_f = \mu_{\tilde{f}}$.

If $f \in S_\Omega$, then (3.1) implies $\delta\, \tilde{f} \in S_\Omega$. Therefore, it is clear that

$$\rho'_\Omega(g) \geq \sup_{f \in S_\Omega} \sup_{\tilde{f} \sim f} \int_\Omega \left|\delta\, \tilde{f}\, g\right| d\mu,$$

holds. Since the space $(\Omega; \sigma_\Omega; \mu)$ is resonant, then we obtain

$$\rho'_\Omega(g) \geq \delta \sup_{f \in S_\Omega} \int_0^\infty f^*(s)\, g^*(s)\, ds.$$

The first inequality in (3.2) is proved. Now, let $\tilde{g}, g \in X'(\Omega)$ and $\mu_{\tilde{g}} = \mu_g$. This immediately implies $\tilde{g}^*(s) = g^*(s), \forall s > 0$. As a result, we have

$$\delta\nu'_\Omega(\tilde{g}) \leq \nu'_\Omega(g) \leq \delta^{-1}\nu'(\tilde{g}).$$

Applying the first inequality to the functions $\tilde{g}$ and $g$, we obtain

$$\rho'_\Omega(\tilde{g}) \le \nu'_\Omega(\tilde{g}) \le \delta^{-1}\nu'_\Omega(g) \le \delta^{-2}\rho'_\Omega(g),$$
$$\rho'_\Omega(g) \le \nu'_\Omega(g) \le \delta^{-1}\nu'(\tilde{g}) \le \delta^{-2}\rho'_\Omega(\tilde{g}).$$

It follows that $X'(\Omega)$ is pseudosymmetric. Applying the G.G. Lorentz and W.H.J. Luxemburg theorem (see [2, p. 10]), the second inequality in (3.2) is established completely similar to the previous case.

Therefore the proof is complete. □

This proposition immediately implies.

**Corollary 3.2.** *Let all conditions of Proposition 3.1 be satisfied. Then the following Hölder inequality*

$$\int_\Omega |fg|\,d\mu \le \int_0^\infty f^*(s)\,g^*(s)\,ds$$
$$\le \delta^{-1}\rho_\Omega(f)\,\rho'_\Omega(g),$$

*holds, where* $\delta \in (0,1]$ *is the constant in* (3.1).

The following theorem is also true.

**Theorem 3.3.** *Let all conditions of Proposition 3.1 be satisfied. Then* $f_1 \prec f_2$ *(i.e.,* $f_1^{**}(s) \le f_2^{**}(s), \forall s > 0$*) implies,* $\rho(f_1) \le \delta^{-1}\rho(f_2)$*, where* $\delta \in (0,1]$ *is the constant in* (3.1).

*Proof.* Let $f \prec f_2$. As determined in the monograph [2, p. 61, Theorem 4.6], we have

$$\int_0^\infty f_1^*(s)\,g^*(s)\,ds \le \int_0^\infty f_2^*(s)\,g^*(s)\,ds, \quad \forall g : \rho'_\Omega(g) \le 1.$$

Then from Proposition 3.1, we obtain

$$\delta^2\nu_\Omega(f_1) \le \delta\rho_\Omega(f_1) \le \delta\nu_\Omega(f_1) \le \delta\nu_\Omega(f_2) \le \rho_\Omega(f_2).$$

Thus the proof is complete. □

Using this theorem, the following theorem is similarly proved to Theorem 4.8 of the monograph [2, p. 61].

**Theorem 3.4.** *Let* $(\Omega;\sigma_\Omega;\mu)$ *be a resonant space and* $\{E_k\}_{k\in J} \subset \sigma_\Omega :$ $E_i \bigcap E_j = \emptyset,\ i \ne j, \mu(E_k) < +\infty, \forall k \in J$. *Let* $E = \Omega \backslash \bigcup_{k\in J} E_k$. *Suppose*

$$Af = f\chi_E + \sum_{k\in J}\left(\frac{1}{\mu(E_k)}\int_{E_k} f\,d\mu\right)\chi_{E_k}.$$

*If the conditions of Proposition 3.1 are satisfied, then*

$$\|Af\|_{X(\Omega)} \le \delta^{-1}\|f\|_{X(\Omega)}.$$

The following theorem is also valid.

**Theorem 3.5.** *Let $(\Omega;\sigma_\Omega;\mu)$ be a totally $\sigma$–finite measure space and let $\lambda$ be some pseudosymmetric norm on $(R^+;m)$ ($m$ –Lebesgue measure). Then, $\underline{\lambda}(f)=\lambda(f^*)\,,\forall f\in\mathscr{F}_0(\Omega)$ defines a pseudosymmetric norm on $(\Omega;\mu)$.*

*Proof.* $\underline{\lambda}$ is f.n. as shown in [2, p. 62]. Let us prove that $\underline{\lambda}$ is pseudo–symmetric. Let $f_1\overset{\delta}{\sim}f_2$, since $\mu_{f_k}=m_{f_k^*}, k=1,2$, this implies $f_1^*\overset{\delta}{\sim}f_2^*$. Hence, we obtain

$$\delta\lambda(f_1^*)\le\lambda(f_2^*)\le\delta^{-1}\lambda(f_1^*)\,.$$
$$\Downarrow$$
$$\delta\underline{\lambda}(f_1)\le\underline{\lambda}(f_2)\le\delta^{-1}\underline{\lambda}(f_1)\,.$$

The proof is complete. □

Now, let us prove the following analogue of the Luxemburg representation theorem.

**Theorem 3.6.** *Let $\rho_\Omega$ be a pseudosymmetric norm on a resonant space $(\Omega;\sigma_\Omega;\mu)$. Then, there is a (not unique) pseudosymmetric norm $\overline{\rho}_\Omega$ on $(R^+;m)$ such that*

$$\delta\overline{\rho}_\Omega(f^*)\le\rho_\Omega(f)\le\overline{\rho}_\Omega(f^*)\,,\quad\forall f\in\mathscr{F}_0(\Omega)\,. \tag{3.3}$$

*Moreover, if for a symmetric norm $\lambda$ on $(R^+;m)$*

$$\delta\lambda(f^*)\le\rho_\Omega(f)\le\lambda(f^*)\,,\quad\forall f\in\mathscr{F}_0(\Omega)\,, \tag{3.4}$$

*then*

$$\delta\lambda'(g^*)\le\rho_\Omega^{'}(g)\le\delta^{-1}\lambda'(g^*)\,,\quad\forall g\in\mathscr{F}_0(\Omega)\,, \tag{3.5}$$

*is also true for the associative norms $\rho_\Omega^{'}$ and $\lambda'$.*

*Proof.* Suppose

$$\overline{\rho}_\Omega(h)=\sup_{g\in S_\Omega'}\int_0^\infty g^*(s)\,h^*(s)\,ds,\quad\forall g\in F_0(\Omega)\,,$$

where $\overline{\rho}_\Omega$ is f.n. on $(\Omega;\mu)$ as shown in [2, p. 62]. It is quite obvious that $\overline{\rho}_\Omega$ is a symmetric norm on $(R^+;m)$. It follows from Proposition 3.1 that $\overline{\rho}_\Omega$ satisfies (3.3).

Let us consider the second part of the theorem. Let $\lambda$ be a symmetric norm that satisfies (3.4) on $(R^+;m)$. Since $(R^+;m)$ is a resonant space, it is clear that an associative norm $\lambda^{'}$ is defined by

$$\lambda^{'}(f)=\sup_{\lambda(h)\le1}\int_0^\infty\varphi^*(s)\,h^*(s)\,ds.$$

Paying attention to Proposition 3.1, we have

$$\begin{aligned}\rho'_\Omega(g) &\le \nu'_\Omega(g)\\ &= \sup_{f\in S_\Omega}\int_0^\infty f^*(s)\,g^*(s)\,ds\\ &= \delta^{-1}\sup_{\rho_\Omega(f)\le 1}\int_0^\infty (\delta f)^*(s)\,g^*(s)\,ds.\end{aligned}$$

According to (3.4),

$$\begin{aligned}\rho_\Omega(f)\le 1 \quad &\Rightarrow \quad \lambda(\delta f^*)\le 1\\ &\Rightarrow \quad \{f:\rho_\Omega(f)\le 1\}\subset\{f:\lambda(\delta f^*)\le 1\}.\end{aligned}$$

Hence

$$\rho'_\Omega(g)\le \delta^{-1}\sup\int_0^\infty \tilde{f}^*(s)\,g^*(s)\,ds\le \delta^{-1}\lambda'(g^*).$$

On the other hand, as determined in the monograph [2, p. 63], we have

$$\lambda'(g^*)=\sup_{f\in S_\Omega}\int_0^\infty f^*(s)\,g^*(s)\,ds.$$

Then from Proposition 3.1 (inequality (3.2)), we obtain, $\delta\lambda'(g^*)\le\rho'_\Omega(g)$, which completes the proof.

□

3.1. **Example for Pseudosymmetric Space.** Let $(\Omega;\sigma_\Omega;\mu)$ be a measure space and let $X(\Omega)$ be a symmetric space on $\Omega$. Let $\varphi:\Omega\to C$ be a $\mu$-measurable function, what is more $\left\|\varphi^{\pm 1}\right\|_{L_\infty(\Omega)}<+\infty$, (the norm in space $L_\infty(\Omega)$ is taken with respect to the measure $\mu$). Let us consider a new norm

$$\|f\|_{X_\varphi(\Omega)}=\|\varphi f\|_{X(\Omega)},\quad \forall f\in X(\Omega).$$

It is clear that the norms $\|\cdot\|_{X_\varphi(\Omega)}$ and $\|\cdot\|_{X(\Omega)}$ are equivalent. Generally (depending on the function $\varphi$), the space $X_\varphi(\Omega)$ is not symmetric. Let us show that $X_\varphi(\Omega)$ is pseudosymmetric. Suppose $f$ and $g$ be equimeasurable (according to measure $\mu$) functions, so that, $f\sim g$. Hence, $\|f\|_{X(\Omega)}=\|g\|_{X(\Omega)}$. We have

$$\begin{aligned}\|f\|_{X_\varphi(\Omega)} &\le \|\varphi\|_{L_\infty(\Omega)}\|f\|_{X(\Omega)}\\ &= \|\varphi\|_{L_\infty(\Omega)}\|g\|_{X(\Omega)}\\ &\le \|\varphi\|_{L_\infty(\Omega)}\left\|\varphi^{-1}\right\|_{L_\infty(\Omega)}\|g\|_{X_\varphi(\Omega)}.\end{aligned}$$

Similarly, it is established

$$\|g\|_{X_\varphi(\Omega)}\le\delta^{-1}\|f\|_{X_\varphi(\Omega)},$$

where $\delta^{-1} = \|\varphi\|_{L_\infty(\Omega)} \left\|\varphi^{-1}\right\|_{L_\infty(\Omega)} \geq 1$. This immediately implies

$$\delta \|g\|_{X_\varphi(\Omega)} \leq \|f\|_{X_\varphi(\Omega)} \leq \delta^{-1} \|g\|_{X_\varphi(\Omega)},$$

and means $X_\varphi(\Omega)$ is pseudosymmetric.

**Remark 3.7.** It should be noted that the same space can be considered on different measure spaces (without changing the norm). For instance, let $w : (a, b) \to R_+$ be a weight function, $\mathscr{L}_{(a,b)}$ is $\sigma$-algebra of Lebesgue measurable subsets of $(a, b)$. Consider the space $L_{p;w}(a, b)$ on $\left((a; b); \mathscr{L}_{(a,b)}; dx\right)$ which has norm

$$\|f\|_{p;w} = \left(\int_a^b |f|^p \, w dx\right)^{\frac{1}{p}}, \quad p \geq 1.$$

Consider the same space on a measure space $\left((a; b); \mathscr{L}_{(a,b)}; d\mu\right)$, where $d\mu = w dx$. We denote it by $\mathscr{L}_{p;d\mu}(a, b)$. Hence, $\|f\|_{p;d\mu} = \left(\int_a^b |f|^p \, d\mu\right)^{\frac{1}{p}}$, it is quite clear that $\mathscr{L}_{p;d\mu}(a, b)$ is symmetric.

## 4. Fundamental Function of Pseudosymmetric Spaces

Let $X(\Omega)$ be a $\delta$-symmetric space on a measure space $(\Omega; \sigma_\Omega; \mu)$. A set function $\varphi_X(E) = \|\chi_E(\cdot)\|_{X(\Omega)}$, $\forall E \in \sigma_\Omega$, is called the fundamental function of $X$. In particular, we have

$$\varphi_{L_p}(E) = \mu^{\frac{1}{p}}(E), \quad 1 \leq p < \infty,$$
$$\varphi_{L_\infty}(E) = \begin{cases} 0, & \mu(E) = 0, \\ 1, & \mu(E) > 0, \end{cases}$$

for $L_p(\Omega; \mu)$ with the norm

$$\|f\|_{L_p(\Omega;\mu)} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}.$$

The following theorem is true.

**Theorem 4.1.** *Let $X$ be a pseudosymmetric space over a resonant measure space $(\Omega; \sigma_\Omega; \mu)$ and let $X'$ be its associate space. Then*

$$\mu(E) \leq \varphi_X(E) \varphi_{X'}(E) \leq \delta^{-2} \mu(E), \quad \forall E \in \sigma_\Omega. \tag{4.1}$$

*Proof.* Let $\mu(E) = 0$. Since functions from $X$ are identified $\mu$-a.e., it is clear that $\|\chi_E\|_X = \|\chi_\emptyset\|_X = 0$.

Let us consider the case: $0 < \mu(E) < +\infty$. Hölder's inequality implies

$$\mu(E) = \int_E d\mu \leq \|\chi_E\|_X \|\chi_E\|_{X'} = \varphi_X(E) \varphi_{X'}(E).$$

On the other hand, we have

$$\varphi_X(E) = \|\chi_E\|_X = \sup_{g\in S'_\Omega} \int_E |g|\, d\mu.$$

Suppose

$$h = \left(\frac{1}{\mu(E)} \int_E |g|\, d\mu\right) \chi_E.$$

It follows from Proposition 3.1 that $X'$ is a pseudosymmetric space. Let $E^c = \Omega \backslash E$. Hence, $h \le h + \chi_{E^c} = Ag$. Then applying Theorem 3.4 with respect to the space $X'$, we obtain

$$\delta^2 \|h\|_{X'} \le \delta^2 \|Ag\|_{X'} \le \delta^{-2} \|g\|_{X'} \le 1.$$

Hence

$$\frac{\delta^2}{\mu(E)} \int_E |g|\, d\mu \, \|\chi_E\|_{X'} \le 1, \quad \forall g \in S'_\Omega.$$

Taking sup with respect to $g \in S'_\Omega$, we obtain

$$\delta^2 \|\chi_E\|_X \|\chi_E\|_{X'} \le \mu(E),$$

$$\Downarrow$$

$$\varphi_X(E)\, \varphi_{X'}(E) \le \delta^{-2} \mu(E).$$

Inequality (4.1) for $\mu(E) < +\infty$ is established.

Let $E \subset \sigma_\Omega : \mu(E) = +\infty$. It is quite clear that

$$\exists \{E_n\} \subset \sigma_\Omega : \mu(E_n) < +\infty, \forall n \in N; E_1 \subset E_2, \ldots, E = \bigcup_n E_n.$$

Inequality (4.1) holds for $E_n$. Levy's theorem immediately implies

$$\varphi_X(E_n) \to \varphi_X(E), \qquad \varphi_{X'}(E_n) \to \varphi_X(E), \quad n \to \infty.$$

Therefore

$$\mu(E_n) \le \varphi_X(E_n)\, \varphi_{X'}(E_n) \le \delta^{-2} \mu(E_n), \quad \forall n \in N.$$

By taking the limit as $n \to \infty$, we obtain

$$\varphi_X(E)\, \varphi_{X'}(E) = +\infty.$$

Thus the proof is complete. □

Let's take the notation $\varphi \sim \psi$ which means that $\exists A > 0 : c\varphi(t) \le \psi(t) \le c^{-1}\varphi(t)$, for all possible $t$. We say that $\varphi(\cdot)$ is almost decreasing (almost increasing) if $\varphi \sim \psi$, where $\psi$ is decreasing (increasing). It is quite obvious that if $E_1 \subset E_2$ then $\varphi_X(E_1) \le \varphi_X(E_2)$ (including $\varphi_{X'}(E_1) \le \varphi_{X'}(E_2)$). Therefore, $\varphi_X(\cdot)$ $(\varphi_{X'}(\cdot))$ is a non-decreasing function. Then, it follows from (4.1) that $\frac{\varphi_X(\cdot)}{\mu(\cdot)}$ is almost non-increasing. Moreover, it is clear that $\varphi_X(E) = 0 \Leftrightarrow \mu(E) = 0$. It is quite obvious that if $E_n \uparrow E$then $\varphi_X(E_n) \uparrow \varphi_X(E)$. Therefore, the following is true:

**Corollary 4.2.** *Assume that all conditions of Theorem 4.1 are satisfied. Then the function $\varphi_X(\cdot)$ satisfies the following conditions:*

(i) $\varphi_X(\cdot)$ *is non-decreasing;* $\varphi_X(E) = 0 \quad \Leftrightarrow \quad \mu(E) = 0$;
(ii) $\frac{\varphi_X(\cdot)}{\mu(\cdot)}$ *is almost non-increasing.*

In exactly the same way as in monograph [2, p. 67] (Theorem 5.5) the following theorem is proved.

**Theorem 4.3.** *Let $(\Omega; \sigma_\Omega; \mu)$ be a nonatomic totally $\sigma$–finite (that is, not containing atoms) measure space and assume $X$ be a pseudosymmetric space on it.*

a) *Then the following conditions are equivalent:*
  (i) $\lim\limits_{\mu(E)\to 0} \varphi_X(E) = 0$;
  (ii) $X_a = X_b$;
  (iii) $(X_b)^* = X'$;
  *Moreover, if the measure $\mu$ is separable, then conditions (i)-(iii) are equivalent to condition:*
  (iv) $X_b$ *is separable;*

b) *Moreover, if* $\lim\limits_{\mu(E)\to 0} \varphi_X(E) > 0$, *then* $X_a = \{0\}$.

We need the following definition for the next explanation.

**Definition 4.4.** A function $\varphi : R^+ = [0, \infty) \to R^+$ is called pseudo-quasiconvex if the following holds:

(i) $\varphi(\cdot)$ is non-decreasing & $\varphi(t) = 0 \quad \Leftrightarrow \quad t = 0$;
(ii) $\frac{\varphi(t)}{t}$ almost decreasing on $(0, \infty)$.

It is quite obvious that a quasiconvex function (with respect to a quasiconvex function, see, e.g.,[2, p. 69]) is also pseudo-quasiconvex, but the opposite is not true. For instance, function $\varphi(t) = \left(1 + \sin^2 t\right) \max(1; t)$, $t > 0$ is pseudo-quasiconvex, but not quasiconvex (since $\frac{\varphi(t)}{t}$ is non-decreasing).

## 5. The Calderon Theorem. Boyd's Indices

It follows from Proposition 3.1 that pseudosymmetric spaces can be transformed into symmetric spaces by replacing the norm with an equivalent norm. Then it is absolutely clear that all the results of the monograph [2] concerning symmetric spaces are also valid in pseudosymmetric spaces (with appropriate corrections). For example, an analogue of Theorem 6.6 [2, p. 77] has the following form.

**Theorem 5.1.** *Let $X$ be a pseudosymmetric space over a resonant space. Then, the following continuous embeddings are valid*

$$L_1 \bigcap L_\infty \subset X \subset L_1 + L_\infty.$$

Theorem 2.2 [2, p. 106] has the following analogue.

**Theorem 5.2.** *Let $X$ be a pseudosymmetric space over a resonant measure space. Then $X$ is an interpolation space between $L_1$ and $L_\infty$.*

In fact, replacing the norm of $X$, we transform it into a symmetric space and denote it by $\tilde{X}$. Then $\tilde{X}$ by Theorem 2.2 [2, p. 106] is an exact interpolation space between $L_1$ and $L_\infty$. It is quite obvious that $X$ is an interpolation between $L_1$ and $L_\infty$.

Theorem 2.12 (A.P.Calderon) has the following generalization.

**Theorem 5.3.** *Let $X$ be a b.f.s over a resonant measure space. Then $X$ which is an interpolation between $L_1$ and $L_\infty$ is pseudosymmetric.*

*Proof.* If $X$ is pseudosymmetric, then from Theorem 3.4 it is an interpolation between $L_1$ and $L_\infty$. Conversely, let $X$ be an interpolation space between $L_1$ and $L_\infty$. Let us show that $X$ is pseudosymmetric. Suppose $g \sim f$, i.e. $\mu_g = \mu_f \Rightarrow g \prec f$ (also $f \prec g$). Then by Corollary 2.11 [2, p. 115] there is an admissible operator $T : \|T\|_{\mathscr{A}} \leq 1 \& Tf = g$. Since $X$ is an interpolation space between $L_1$ and $L_\infty$, then

$$T \in [X] \quad \Rightarrow \quad \|g\|_X \leq \|T\|_{[X]} \|f\|_X .$$

Considering Proposition 1.11 [2, p. 103], we have

$$\|g\|_X \leq C \|T\|_{\mathscr{A}} \|f\|_X = C_1 \|f\|_X ,$$

where $C_1 > 0$ is the constant which is independent of $f$ and $g$. In a similar way, we establish $\exists C_2 > 0$ ($C_2$ is independent of $f$ and $g$):

$$\|f\|_X \leq C_2 \|g\|_X .$$

It follows that $X$ is pseudosymmetric.

Therefore, the proof is complete. □

The following analogue of Theorem 5.7 (A.P. Calderon) [2, p. 144] is also valid.

**Theorem 5.4.** *Let $1 \leq p_0 < p_1 \leq +\infty, 1 \leq q_0; q_1 < +\infty$ and $q_0 \neq q_1$. Let $X$ and $Y$ be pseudosymmetric spaces over resonant spaces $(\Omega_0; \sigma_{\Omega_0}; \mu_0)$ and $(\Omega_1; \sigma_{\Omega_1}; \mu_1)$, respectively. Let $S_\sigma (f^*) (1) < \infty, \forall f \in X$, where $S_\sigma (f^*) (1) = \int_0^1 s^{\frac{1}{p_0}} f^* (s) \frac{ds}{s} + \int_1^\infty s^{\frac{1}{p_1}} f^* (s) \frac{ds}{s}$, ($S_\sigma$-the Calderon operator).*

*Then, the following assertions are equivalent:*

(i) *every linear operator of joint weak type* $(p_0, q_0; p_1, q_1)$ *belongs to* $[X; Y]$*;*
(ii) *every quasilinear operator of joint weak type* $(p_0, q_0; p_1, q_1)$ *belongs to* $[X; Y]$*;*
(iii) *if* $f \in X$ *and* $g \in \mathscr{F}_0(\Omega_1; \mu_1)$*, then the inequality* $g^* \leq S_\sigma(f^*)$ *implies* $g$ *belongs to* $Y$ *and* $\|g\|_Y \leq c\|f\|_X$*, where* $c$ *is the constant independent of* $f$ *and* $g$*.*

In fact, if we denote symmetric spaces with equivalent norms by $\tilde{X}$ and $\tilde{Y}$, it is easy to see that all conditions of Theorem 5.7 (A.P. Calderon) with respect to the spaces $\tilde{X}$ and $\tilde{Y}$ are satisfied. The rest is obvious.

We denote by $X \sim Y$ if the b.f.s. $X$ and $Y$ are given on the same measure space and their norms are equivalent. Let $\alpha_X$ and $\beta_X$ denote the lower and upper Boyd indices of the space $X$, respectively.

**Lemma 5.5.** *Let* $X$ *and* $Y$ *be symmetric spaces over an infinite, nonatomic, totally* $\sigma$*-finite measure space. If* $X \sim Y$*, then* $\alpha_X = \alpha_Y$ *and* $\beta_X = \beta_Y$*.*

*Proof.* Let

$$c_1 \|x\|_X \leq \|x\|_Y \leq c_2 \|x\|_X, \quad \forall x \in X, \tag{5.1}$$

where the constants $c_1; c_2 > 0$ are independent of $x$. Let $E_t$ denote the dilation operator

$$(E_t f)(s) = f(st), \quad 0 < s < +\infty.$$

By definition, we have

$$h_X(t) = \left\|E_{\frac{1}{t}}\right\|_{[\overline{X}]},$$
$$h_Y(t) = \left\|E_{\frac{1}{t}}\right\|_{[\overline{Y}]},$$

where $\overline{X}; \overline{Y}$ are the Luxemburg representations of the spaces $X$ and $Y$, respectively. Estimate (5.1) and Theorem 4.10 [2, p. 62] (Luxemburg representation theorem) immediately imply that

$$A_1 \left\|E_{\frac{1}{t}}\right\|_{[\overline{X}]} \leq \left\|E_{\frac{1}{t}}\right\|_{[\overline{Y}]} \leq c_2 \left\|E_{\frac{1}{t}}\right\|_{[\overline{X}]},$$

where the constants $c_1; c_2 > 0$ are independent of $t$. Hence

$$c_1 h_X(t) \leq h_Y(t) \leq c_2 h_X(t), \quad \forall t > 0.$$

This implies

$$\log c_1 + \log h_X(t) \leq \log h_Y(t) \leq \log c_2 + \log h_X(t),$$
$$\Downarrow$$
$$\alpha_X = \lim_{t \to +0} \frac{\log h_X(t)}{\log t} \leq \lim_{t \to +0} \frac{\log h_Y(t)}{\log t} \leq \lim_{t \to +0} \frac{\log h_X(t)}{\log t} = \alpha_X,$$

$$\Downarrow$$

$\Rightarrow \alpha_X = \alpha_Y$. Similarly, $\beta_X = \beta_Y$ is established.
Thus, the lemma is proved. □

Based on this lemma, we accept the following

**Definition 5.6.** Let $X$ be a pseudosymmetic space over an infinite, nonatomic, totally $\sigma$–finite measure space. Let's take $\alpha_X = \alpha_Y$ and $\beta_X = \beta_Y$, where $Y \sim X$is an arbitrary symmetric space.

It follows from Proposition 3.1 that $\exists Y : Y \sim X$ and $Y$ are symmetric and it follows from Lemma 5.5 that the definitions of the quantities $\alpha_X; \beta_X$ are correct. $\alpha_X$ and $\beta_X$ will be called the lower and upper Boyd indices of the space $X$. It is quite obvious, if $X \sim Y$, then $X' \sim Y'$. Then it immediately follows from Proposition 5.13 [2, p. 149] the following proposition.

**Proposition 5.7.** *Let $X$ be a pseudosymmetric space over an infinite, nonatomic, totally $\sigma$–finite measure space. Then*

$$0 \le \alpha_X \le \beta_X \le 1;$$
$$\alpha_{X^{-1}} = 1 - \beta_X; \beta_{X^{-1}} = 1 - \alpha_X.$$

Using this fact, the following analogue of Boyd's theorem is easily determined.

**Theorem 5.8.** *Let $1 \le p < q \le \infty$ and let $X$ be a pseudosymmetric space over an infinite, nonatomic, totally $\sigma$–finite measure space. Then every linear(quasilinear) operator $T$ of joint type $(p, p; q, q)$ belongs to $[X]$ if and only if the Boyd indices $\alpha_X; \beta_X$ of the space $X$ satisfy*

$$\frac{1}{q} < \alpha_X \le \beta_X < \frac{1}{p}.$$

The following analogs of the theorems of G.G.Lorentz & T.Shimogaki and D.W.Boyd are also valid.

**Theorem 5.9.** *Let a pseudosymmetric space $X$ on $R^n$ satisfy the conditions of Theorem 5.8. Then the Hardy-Littlewood maximal operator $M \in [X] \Leftrightarrow \alpha_X < 1$.*

The following theorem also holds.

**Theorem 5.10.** *Let a pseudosymmetric space $X$ on $R^n$ satisfy the conditions of Theorem 5.8. Then the Hilbert transform $H \in [X] \Leftrightarrow 0 < \alpha_X \le \beta_X < 1$.*

In a similar way, we determine the validity of the following main theorem.

**Theorem 5.11.** *Let $X$ be a pseudosymmetric space on the unit disc $\omega$ with linear Lebesgue measure and let $\varphi_X(E) \to 0, |E| \mapsto 0$. Then the following conditions are equivalent:*

(i) *the Fourier series converge in the norm in $X_b$, where $X_b$–the closure of the simple functions;*

(ii) *the partial-sum operators $S_N$:*

$$(S_N f)\left(e^{i\theta}\right) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}$$

*are uniformly bounded on $X_b$, where*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta} d\theta;$$

(iii) *$X_b$ is the closure in $X$ of the trigonometric polynomials;*

(iv) *translation is continuous in $X_b$, that is*

$$\lim_{\theta\to 0} \|T_\theta f - f\|_X = 0, \quad \forall f \in X_b,$$

*where $(T_\theta f)\left(e^{it}\right) = f\left(e^{i(\theta+t)}\right), -\pi < \theta; t \le \pi$;*

(v) $\lim\limits_{t\to+0} \omega_X(f;t) = 0, \forall f \in X_b$, *where*

$$\omega_X(f;t) = \sup_{|\theta|\le t} \|T_\theta f - f\|_X, \quad 0 \le t \le \pi;$$

(vi) *the multiplier operator $\mathscr{M}$ is bounded on $X_b$, where on trigonometric polynomials $P\left(e^{i\theta}\right) = \sum\limits_{n=-N}^{N} a_n e^{in\theta}$ the operator $\mathscr{M}$ is defined by*

$$(\mathscr{M} P)\left(e^{i\theta}\right) = \sum_{n=-N}^{N} -i\, sign\, n a_n e^{in\theta};$$

(vii) *the conjugate-function operator $\tilde{f}$ is bounded on $X_b$, where*

$$\tilde{f}\left(e^{i\theta}\right) = \frac{1}{2\pi}\lim_{\varepsilon\to+0}\int_{\varepsilon<|\theta|\le\pi} f\left(e^{i(\theta-t)}\right) ctg\frac{t}{2} dt;$$

(viii) *the Calderon operator*

$$Sf^*(t) = \int_0^1 f^*(s)\min\left(1; \frac{s}{t}\right)\frac{ds}{s},$$

*is bounded on $\overline{(X_b)}$– the Luxemburg representation of $X_b$ on the interval $[0,1]$. If any one of these conditions holds, then $\mathscr{M} f = \tilde{f}$ a.e. for all $f \in X_b$.*

**Corollary 5.12.** *Let $X$ be a pseudosymmetric separabel space on $\omega$. The Fourier series converge in $X$ if and only if $0 < \alpha_x; \beta_x < 1$.*

**Acknowledgement.** This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) with Azerbaijan National Academy of Sciences (ANAS), Project Number: 19042020 and by TUBITAK 2211-E Domestic Direct Doctorate Scholarship Program.

## References

1. D.R. Adams, *Morrey spaces*, Switzherland, Springer, 2016.
2. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.
3. B.T. Bilalov, *The basis property of a perturbed system of exponentials in Morrey-type spaces*, Sib. Math. J., 60(2) (2019), pp. 323-350.
4. B.T. Bilalov, T.B. Gasymov and A.A. Guliyeva, *On solvability of Riemann boundary value problem in Morrey-Hardy classes*, Turkish J. Math., 40(50) (2016), pp. 1085-1101.
5. B.T. Bilalov and A.A. Guliyeva, *On basicity of the perturbed systems of exponents in Morrey-Lebesgue space*, Internat. J. Math., 25:1450054 (2014), pp. 1-10.
6. B.T. Bilalov and Z.G. Guseynov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math, 9(3) (2012), pp. 487-498.
7. B.T. Bilalov, A.A. Huseynli and S.R. El-Shabrawy, *Basis Properties of Trigonometric Systems in Weighted Morrey Spaces*, Azerb. J. Math., 9(2) (2019), pp. 200-226.
8. B.T. Bilalov and S.R. Sadigova, *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Var. Elliptic Equ., 66(12) (2021), pp. 2117-2130.
9. B.T. Bilalov and S.R. Sadigova, *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Commun. Math. Anal., 18(2) (2021), pp. 129-148.
10. B.Bilalov and S. Sadigova, *On the fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces*, Ric. Mat. (2021).
11. B.T. Bilalov and F.Sh. Seyidova, *Basicity of a system of exponents with a piecewise linear phase in Morrey-type spaces*, Turkish J. Math., 43 (2019), pp. 1850–1866.
12. L. Caso, R. D'Ambrosio and L. Softova, *Generalized Morrey Spaces over Unbounded Domains*, Azerb. J. Math., 10(1) (2020), pp. 193-208.

13. R.E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces*, Springer, 2016.
14. D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Birkhauser, Springer, 2013.
15. P. Harjulehto and P. Hasto, *Orlicz spaces generalized Orlicz spaces*, Springer, 2019.
16. D.M. Israfilov and N.P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerb. J. Math., 1(1) (2011), pp. 99-113.
17. V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Variable Exponent Lebesgue and Amalgam Spaces, Springer, (1) 2016.
18. V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Variable Exponent Hölder, Morrey–Campanato and Grand Spaces, Springer, (2) 2016.
19. S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978.
20. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, I, Springer-Verlag, 1977.
21. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, II, Springer-Verlag, 1979.
22. W.A.J. Luxemberg, *Banach function spaces*, Ph.D. Thesis, Delft Ins. of Technology, Assen (Netherlands), 1955.
23. D.K. Palagachev, M.A. Ragusa and L.G. Softova, *Regular oblique derivative problem in Morrey spaces*, Electron. J. Differential Equations, 2000(39) (2000), pp. 1-17.
24. D.K. Palagachev and L.G. Softova, *Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's*, Potential Anal., 20 (2004), pp. 237-263.
25. M.M. Reo and Z.D. Ren, *Applications of Orlichz Spaces*, New-York-Basel, 2002.
26. I.I. Sharapudinov, *On Direct And Inverse Theorems of Approximation Theory in Variable Lebesgue and Sobolev Spaces*, Azerb. J. Math., 4(1) (2014), pp. 55-72.
27. L.G. Softova, *The Dirichlet problem for elliptic equations with VMO coefficients in generalized Morrey spaces*, Operator Theory, 229 (2013), pp. 365-380.
28. Y. Zeren, M.I. Ismailov and C. Karacam, *Korovkin-type theorems and their statistical versions in grand Lebesgue spaces*, Turkish J. Math., 44 (2020), pp. 1027-1041.
29. Y. Zeren, M. Ismailov and Fatih Sirin, *On basicity of the system of eigenfunctions of one discontinuous spectral problem for second order differential equation for grand Lebesgue space*, Turkish J.

Math., 44(5) (2020), pp. 1995-1612.

[1] Department of Non-harmonic Analysis, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan.
*Email address*: b_bilalov@mail.ru

[2] Department of Mathematics, Yildiz Technical University, Istanbul, Turkey.
*Email address*: yusufzeren@hotmail.com

[3] Department of Non-harmonic Analysis, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan.
*Email address*: b.alizade99@gmail.com

[4] Department of Mathematics, Yildiz Technical University, Istanbul, Turkey.
*Email address*: feyzadal@hotmail.com