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Mohammad Jahanshahi and Reza Danaei

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Analytical-Numerical Solution for a Third Order Space-time Conformable Fractional PDE with Mixed Derivative by Spectral and Asymptotic Methods

Mohammad Jahanshahi[1]* and Reza Danaei[2]

Abstract. Initial-boundary value problems including space-time fractional PDEs have been used to model a wide range of problems in physics and engineering fields. In this paper, a non-self adjoint initial boundary value problem containing a third order fractional differential equation is considered. First, a spectral problem for this problem is presented. Then the eigenvalues and eigenfunctions of the main spectral problem are calculated. In order to calculate the roots of their characteristic equation, the asymptotic expansion of the roots is used. Finally, by suitable choice of these asymptotic expansions, related eigenfunctions and Mittag-Lefler functions, the analytic and numerical solutions to the main initial-boundary value problem are given.

1. Introduction

Depending on the type of problem and its differential equation, mathematicians, physicists and engineers use different methods to solve boundary value problems and initial value problems. In cases where the differential equation is of higher order, we use the asymptotic expansion of the roots to calculate the roots of its characteristic equation. On the other hand, fractional partial differential equations involving mixed derivatives have not received much attention, but this class of differential equations has many applications in science and engineering to describe several physical occurrences or real world problems.

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* Corresponding author.

In the following, we refer to some of the methods presented in recent articles, using contour, asymptotic and spectral methods [13]. In [14] three effective numerical methods for solving the Riesz fractional diffusion equation (RFDE) and Riesz fractional advection-dispersion equation (RFADE) on a finite domain with homogeneous Dirichlet boundary conditions have been described. In [6] The solution was found in the form of a series of expansions in the eigenfunctions of the Laplace operator with nonlocal boundary conditions for a mixed problem by a partial differential equation of high order with fractional Riemann-Liouville derivatives with respect to time. In [3] authors used the Residual Power Series Method (RPSM), is one of the most efficient methods, to find the solutions of fractional-order space-time partial differential equations. In [4] the solvability of nonlinear fractional partial differential equations (FPDEs) with mixed partial derivatives has been considered. The invariant subspace method is used to obtain an exact solution for a wide class of mixed fractional partial differential equations considered in [12]. In [2], an approximate method for solving fractional partial differential equations is studied. In [8], [7] and [11] authors have applied the contour integral and asymptotic methods for solving some initial-boundary value problems involving fractional PDEs. In [10] authors obtained analytical solutions for the time-fractional Fisher's nonlinear differential equation.

The equation that we will consider in this paper is the third order fractional partial differential equation with mixed derivative with some initial and boundary conditions. In this paper, we study and find an analytical-numerical to a non-self adjoint initial boundary value problem containing the third order fractional differential equation with a the mixed term by using of its spectral problem and asymptotic analysis. We first present the asymptotic expansion of the roots of the characteristic equation and then we offer the form of the final solution. In the end, we offer the desired numerical examples along with diagrams of the answers.

2. Preliminary Knowledge

First, we introduce some definitions of the conformable fractional derivative and Mittag-Leffler functions used in this paper.

Definition 2.1 ([9])**.** Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0,1)$. If f is $\alpha-$differentiable in an open interval I and $\lim_{t\to 0^+} f^\alpha(t)$ exists, then define $f^\alpha(0) = \lim_{t\to 0^+} f^\alpha(t)$.

Theorem 2.2 ([9])**.** *Let* $\alpha \in (0,1]$ *and* f *be* $\alpha-$*differentiable, then*

$$T_\alpha f(t) = t^{1-\alpha} f'(t).$$

Now, we define the Mittag-Leffeler function that naturally occurs in the solution of fractional equations.

Definition 2.3. The one parameter Mittag-Leffeler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

and the two-parameter Mittag-Leffeler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

We use the continuous integro-differential operator with defined as

$${}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^\alpha, & \alpha < 0. \end{cases}$$

The general modified Mittag-Leffeler function is defined by the following series

$$T(t) = h_\alpha(t,r) = \sum_{k=1}^{\infty} \frac{r^k t^{k\alpha - 1}}{(k\alpha - 1)!}.$$

By using Mittag-Leffeler function, we can solve the fractional equations

$$a_m T^{\left(\frac{m}{n}\right)}(t) + a_{m-1} T^{\left(\frac{m-1}{n}\right)}(t) + \cdots + a_1 T^{\left(\frac{1}{n}\right)}(t) + a_0 T(t) = 0, \tag{2.1}$$

therefore the related characteristics equation is

$$a_m r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0 = 0,$$

then the general solution of the fractional equation (2.1) is as follows:

$$T(t) = c_1 h_\alpha(t, r_1) + c_2 h_\alpha(t, r_2) + \cdots + c_m h_\alpha(t, r_m), \tag{2.2}$$

where $\alpha = \frac{1}{n}$ is the order of the fractional derivative, [11] and [5].

3. Mathematical Statement of Problem

Consider the following initial-boundary value problem

$$\begin{cases} \frac{\partial^3}{\partial x^3} u(x,t) + a \frac{\partial^2}{\partial x^2} u(x,t) \\ \qquad + b \frac{\partial^{1+\alpha}}{\partial x \partial t^\alpha} u(x,t) + c \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = 0, & x \in (0,1), t > 0, \\ u(x,0) = \varphi(x), & x \in [0,1], \\ u(0,t) = u(1,t) = \left. \frac{\partial u(x,t)}{\partial x} \right|_{x=0} = 0, & t \geq 0, \end{cases} \tag{3.1}$$

where $\varphi(x)$ is a known and continuous function on the interval $[0,1]$ and $b \neq 0$, also $0 < \alpha < 1$. By using of the method of separation of variables, we have

$$u(x,t) = X(x)T(t),$$

now by substituting the prescribed solution, we conclude two ordinary differential equations

$$\begin{cases} X'''(x) + aX''(x) - b\lambda^2 X'(x) - c\lambda^2 X(x) = 0, \quad x \in (0,1), \\ X(0) = X(1) = X'(0) = 0, \end{cases} \tag{3.2}$$

$$T^{(\alpha)}(t) + \lambda^2 T(t) = 0, \quad t > 0. \tag{3.3}$$

The equation (3.2) with its boundary conditions is called the main spectral problem. Suppose that the proposed solution to equation (3.2) is $X(x) = e^{\rho x}$. Then its characteristic equation will be,

$$\rho^3 + a\rho^2 - b\lambda^2\rho - c\lambda^2 = 0. \tag{3.4}$$

Now, by using the change of variable $\rho = \sigma - \frac{a}{3}$ and replacing it in (3.4) we obtain

$$\sigma^3 - p\sigma + q = 0, \tag{3.5}$$

that

$$\begin{cases} p = \frac{a^2}{3} + b\lambda^2, \\ q = \frac{2a^3}{27} + \frac{ab\lambda^2}{3} - c\lambda^2. \end{cases} \tag{3.6}$$

Plugging $\sigma = u + v$ into (3.5), we obtain the following algebraic system

$$\begin{cases} u^3 + v^3 = -q, \\ u^3 v^3 = \frac{p^3}{27}, \end{cases} \tag{3.7}$$

the algebraic system of (3.7) helps us to rewrite it as a second-order algebraic equation as follows

$$t^2 + qt + \frac{p^3}{27} = 0. \tag{3.8}$$

Suppose $t_1 = u^3$ and $t_2 = v^3$, then the roots of (3.8) will be we obtain

$$\begin{cases} u^3 = -\frac{1}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}, \\ v^3 = -\frac{1}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}. \end{cases} \tag{3.9}$$

In the next part, we will find the asymptotic expansion of radical expressions.

4. Asymptotic Expansions of the Roots

We compute the inside radical term by plugging (3.6) into the inside radical term (3.9). We have

$$\frac{q^2}{4}-\frac{p^3}{27}=-\left(\frac{b}{3}\right)^3\lambda^6-\left(\frac{a^2b^2}{108}+\frac{abc}{6}-\frac{c^2}{4}\right)\lambda^4-\frac{a^3c}{27}\lambda^2.$$

The asymptotic expansion of a radical is given by:

(4.1)

$$\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}=\sqrt{-\frac{b^3}{27}\lambda^6-\left(\frac{a^2b^2}{108}+\frac{abc}{6}-\frac{c^2}{4}\right)\lambda^4-\frac{a^3c}{27}\lambda^2}$$
$$=\lambda i\left(\frac{b}{3}\sqrt{\frac{b}{3}}\lambda^2+\frac{\left(\frac{a^2b^2}{108}+\frac{abc}{6}-\frac{c^2}{4}\right)}{\frac{2b}{3}\sqrt{\frac{b}{3}}}+\frac{A}{\lambda^2}+\frac{B}{\lambda^4}+\cdots\right).$$

Now, by substituting (4.1) in (3.9) we have

$$\begin{cases} u=\sqrt{\frac{b}{3}}\lambda i+\frac{ab-3c}{6b}+\frac{A_1}{\lambda}+\frac{B_1}{\lambda^2}+\cdots, \\ v=-\sqrt{\frac{b}{3}}\lambda i+\frac{ab-3c}{6b}+\frac{A_2}{\lambda}+\frac{B_2}{\lambda^2}+\cdots. \end{cases} \tag{4.2}$$

We assume that w is the third root of the number M, so we have from complex analysis

$$w^3=M \quad\Rightarrow\quad w_k=\sqrt[3]{M}e^{\frac{2k\pi}{3}i},\quad k=0,1,2.$$

Suppose $\varepsilon=e^{\frac{2\pi}{3}i}$, then

$$\begin{cases} w_0=\sqrt[3]{M}, \\ w_1=\sqrt[3]{M}\varepsilon, \\ w_2=\sqrt[3]{M}\varepsilon^2, \end{cases} \tag{4.3}$$

so by using (4.2) and (3.9) we have

$$u_0=\sqrt{\frac{b}{3}}\lambda i+\frac{ab-3c}{6b}+\frac{A_1}{\lambda}+\frac{B_1}{\lambda^2}+\cdots,$$

and

$$v_0=-\sqrt{\frac{b}{3}}\lambda i+\frac{ab-3c}{6b}+\frac{A_2}{\lambda}+\frac{B_2}{\lambda^2}+\cdots.$$

So the eigenfunctions of the main spectral problem are given by

$$u_1=u\varepsilon,\qquad u_2=u\varepsilon^2, \tag{4.4}$$

and

$$v_1=v\varepsilon,\qquad v_2=v\varepsilon^2. \tag{4.5}$$

The eigenfunctions (4.4)-(4.5) should satisfy (3.7), that is

$$u_k v_l = \frac{p}{3}, \quad k, l = 0, 1, 2.$$

By algebraic computing we obtain

$$u_0 v_0 = \frac{p}{3}, \qquad u_1 v_2 = \frac{p}{3}, \qquad u_2 v_1 = \frac{p}{3}.$$

Now, from $\sigma = u + v$ we will have

$$\begin{cases} \sigma_1 = u_0 + v_0 = u + v, \\ \sigma_2 = u_1 + v_2 = u\varepsilon + v\varepsilon^2, \\ \sigma_3 = u_2 + v_1 = u\varepsilon^2 + v\varepsilon. \end{cases} \tag{4.6}$$

By using (4.2), (4.6) and $\rho_i = \sigma_i - \frac{a}{3}$ for $i = 0, 1, 2$, we obtain

$$\begin{cases} \sigma_1 = u_0 + v_0 = u + v = \frac{ab-3c}{3b} + \frac{A_1+A_2}{\lambda} + \frac{B_1+B_2}{\lambda} + \cdots, \\ \\ \rho_1 = \sigma_1 - \frac{a}{3} = -\frac{c}{b} + \frac{A_1+A_2}{\lambda} + \frac{B_1+B_2}{\lambda} + \cdots, \end{cases} \tag{4.7}$$

and

(4.8)

$$\begin{cases} \sigma_2 = u\varepsilon + v\varepsilon^2 \\ \quad = e^{\frac{2\pi}{3}i}\left[\sqrt{\frac{b}{3}}\lambda i\left(1 - e^2\right) + \left(\frac{ab-3c}{6b}\right)\left(1 + e^2\right) + \frac{A_2 e^2 + A_1}{\lambda} + \frac{B_2 e^2 + B_1}{\lambda^2} + \cdots\right], \\ \\ \rho_2 = \sigma_2 - \frac{a}{3}, \end{cases}$$

and

(4.9)

$$\begin{cases} \sigma_3 = u\varepsilon^2 + v\varepsilon \\ \quad = e^{\frac{2\pi}{3}i}\left[\sqrt{\frac{b}{3}}\lambda i \geq \left(e^2 - 1\right) + \left(\frac{ab-3c}{6b}\right)\left(e^2 + 1\right) + \frac{A_1 e^2 + A_2}{\lambda} + \frac{B_1 e^2 + B_2}{\lambda^2} + \cdots\right], \\ \\ \rho_3 = \sigma_3 - \frac{a}{3}. \end{cases}$$

The solution to equation (3.2) is given by

$$X(x) = \sum_{i=1}^{3} c_i e^{\rho_i x}, \qquad X(0) = X(1) = X'(0) = 0. \tag{4.10}$$

From (4.10) and its derivative, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0, \\ c_1 e^{\rho_1} + c_2 e^{\rho_2} + c_3 e^{\rho_3} = 0, \\ c_1 \rho_1 + c_2 \rho_2 + c_3 \rho_3 = 0, \end{cases} \tag{4.11}$$

now for this algebraic system, we have the following determinant

$$\Delta(\lambda) = \begin{vmatrix} 1 & 1 & 1 \\ e^{\rho_1} & e^{\rho_2} & e^{\rho_3} \\ \rho_1 & \rho_2 & \rho_1 \end{vmatrix} \tag{4.12}$$
$$= e^{\rho_1}(\rho_2 - \rho_3) + e^{\rho_2}(\rho_3 - \rho_1) + e^{\rho_3}(\rho_1 - \rho_2),$$

By using system (4.11) and $c_1 = -(c_2 + c_3)$ we have

$$\begin{cases} c_2 e^{\rho_2} + c_3 e^{\rho_3} = -c_1 e^{\rho_1}, \\ c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = -c_1\rho_1, \end{cases} \tag{4.13}$$

so

$$\begin{aligned} \Delta(\lambda) &= \begin{vmatrix} e^{\rho_2} & e^{\rho_3} \\ \rho_2 & \rho_3 \end{vmatrix} \\ &= \rho_3 e^{\rho_2} - \rho_2 e^{\rho_3} \\ &\neq 0. \end{aligned}$$

Now, for c_1 and c_2 we have

$$c_2 = \frac{c_1}{\Delta}\left(\rho_1 e^{\rho_3} - \rho_3 e^{\rho_1}\right), \qquad c_3 = \frac{c_1}{\Delta}\left(\rho_2 e^{\rho_1} - \rho_1 e^{\rho_2}\right),$$

so

$$\begin{aligned} X(x) &= \sum_{i=1}^{3} c_i e^{\rho_i x} \\ &= c_1\left[e^{\rho_1 x} + \frac{1}{\Delta}\left(\rho_1 e^{\rho_3} - \rho_3 e^{\rho_1}\right) e^{\rho_2 x} + \frac{1}{\Delta}\left(\rho_2 e^{\rho_1} - \rho_1 e^{\rho_2}\right) e^{\rho_3 x}\right]. \end{aligned}$$

From (4.7), (4.8) and (4.9) we obtain

$$\begin{cases} e^{\rho_1} = e^{-\frac{c}{b}}, \\ e^{\rho_2} = e^{\frac{\sqrt{b}}{2}\left(e^2-1\right)\left(\lambda+\frac{\sqrt{3}}{3}i\right)}, \\ e^{\rho_3} = e^{\frac{\sqrt{b}}{2}\left(1-e^2\right)\left(\lambda+\frac{\sqrt{3}}{3}i\right)}, \end{cases} \tag{4.14}$$

also we have

$$\begin{cases} \rho_1 - \rho_2 = -\frac{c}{b} - \frac{\sqrt{b}}{2}\left(e^2-1\right)\left(\lambda + \frac{\sqrt{3}}{3}i\right), \\ \rho_2 - \rho_3 = 2\sqrt{b}\left(e^2-1\right)\left(\lambda + \frac{\sqrt{3}}{3}i\right), \\ \rho_3 - \rho_1 = \frac{\sqrt{b}}{2}\left(1-e^2\right)\left(\lambda + \frac{\sqrt{3}}{3}i\right) + \frac{c}{a}. \end{cases} \tag{4.15}$$

By substituting the terms include λ coefficients in (4.12) we will have

$$\lambda_k = \frac{(2k+1)\pi}{\sqrt{b}\left(e^2-1\right)}i, \quad k \in \mathbb{Z},$$

therefore, the related eigenfunctions are given in the following form

$$(4.16)\qquad \begin{cases} X_n(x) = C\Bigg[e^{-\frac{c}{b}x} + \frac{1}{\Delta}\Big(-\frac{c}{b}e^{\frac{\sqrt{b}}{2}(1-e^2)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)} \\ -\frac{\sqrt{b}}{2}\left(1-e^2\right)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)e^{-\frac{c}{b}x}\Big)e^{\frac{\sqrt{b}}{2}(e^2-1)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)x} \\ +\frac{1}{\Delta}\Big(\frac{\sqrt{b}}{2}\left(e^2-1\right)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)e^{-\frac{c}{b}} \\ -\frac{c}{b}e^{\frac{\sqrt{b}}{2}(e^2-1)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)}\Big)e^{\frac{\sqrt{b}}{2}(1-e^2)\left(\lambda_n+\frac{\sqrt{3}}{3}i\right)x}\Bigg], \end{cases}$$

where

$$\Delta = \frac{\sqrt{b}}{2}\left(1-e^2\right)\left(\lambda+\frac{\sqrt{3}}{3}i\right)e^{\frac{\sqrt{b}}{2}(e^2-1)\left(\lambda+\frac{\sqrt{3}}{3}i\right)} - \frac{\sqrt{b}}{2}\left(e^2-1\right)\left(\lambda+\frac{\sqrt{3}}{3}i\right)e^{\frac{\sqrt{b}}{2}(1-e^2)\left(\lambda+\frac{\sqrt{3}}{3}i\right)}.$$

5. The Final Form of Solution and Numerical Examples

The final solution to the problem (3.1) is as follows:

$$u(x,t) = \sum_{n\in\mathbb{Z}} C_n X_n(x) T_n(t),$$

where $X_n(x)$ are the eigenfunctions of the main spectral problem and $T_n(t)$ are the Mittag-Leffeler functions or conformable fractional solutions. By imposing the given initial condition (3.1), we can calculate the unknown coefficients C_n as follows

$$u(x,0) = \sum_{n\in\mathbb{Z}} C_n X_n(x) T_n(0) = \varphi(x),$$

so we will have for C_n

$$(5.1)\qquad C_n = \frac{\langle \varphi(x), Z_n(x)\rangle}{\langle X_n(x), Z_n(x)\rangle\, T_n(0)},$$

where $Z_n(x)$ are eigenfunctions of the corresponding adjoint problem of (3.1). To find $Z_n(x)$, see [1] and [7]. For the conformable fractional derivative, we have $T_n(0) = e^{-\frac{\lambda^2}{\alpha}0^\alpha} = 1$. Now we could compute the solution form of the main problem (3.1) by asymptotic and spectral methods and so the following theorem holds.

Theorem 5.1. *Let the problem (3.1) be a fractional differential equation with the conformable derivative of order α then the solution will be as* *follows:*

$$u(x,t) = \sum_{n\in\mathbb{Z}} \frac{\langle \varphi(x), Z_n(x)\rangle}{\langle X_n(x), Z_n(x)\rangle} X_n(x) e^{-\frac{\lambda_n^2}{\alpha} t^\alpha},$$

where $X_n(x)$ is given by (4.16).

We now present some examples with their plots and absolute error for a few numerical solutions according to the distributions of eigenvalues and different values of α.

Example 5.2. Consider the fractional partial differential equation with $a = b = c = 1$ with conditions (2.2).

$$\frac{\partial^3}{\partial x^3} u(x,t) + \frac{\partial^2}{\partial x^2} u(x,t) + \frac{\partial^{1+\alpha}}{\partial x \partial t^\alpha} u(x,t) + \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = 0.$$

Figure 1 shows numerical solutions for $u(x,t)$ using (5.1) for $\alpha = 0.85, 0.95, 0.98, 1$.

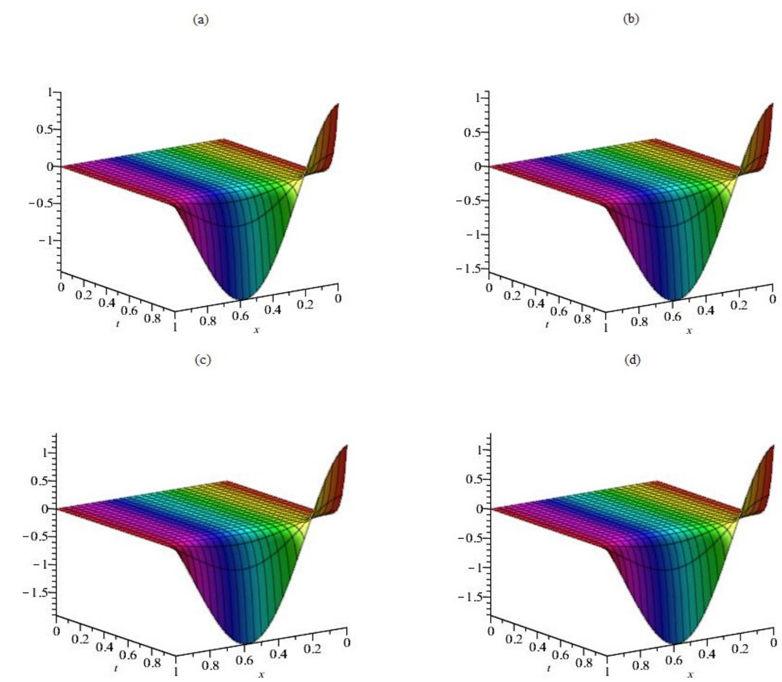

FIGURE 1. The graph of the approximate solution $u(x,t)$ for $\alpha = 0.85$(a),$\alpha = 0.95$(b), $\alpha = 0.98$(c) and $\alpha = 1$(d) for example (5.2).

Example 5.3. Consider the fractional partial differential equation with $a = c = 1$ and $b = 2$ with conditions (2.2).

$$\frac{\partial^3}{\partial x^3}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t) + 2\frac{\partial^{1+\alpha}}{\partial x \partial t^\alpha}u(x,t) + \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = 0.$$

Figure 2 shows numerical solutions for $u(x,t)$ using (5.1) for $\alpha = 0.85, 0.95, 0.98, 1$.

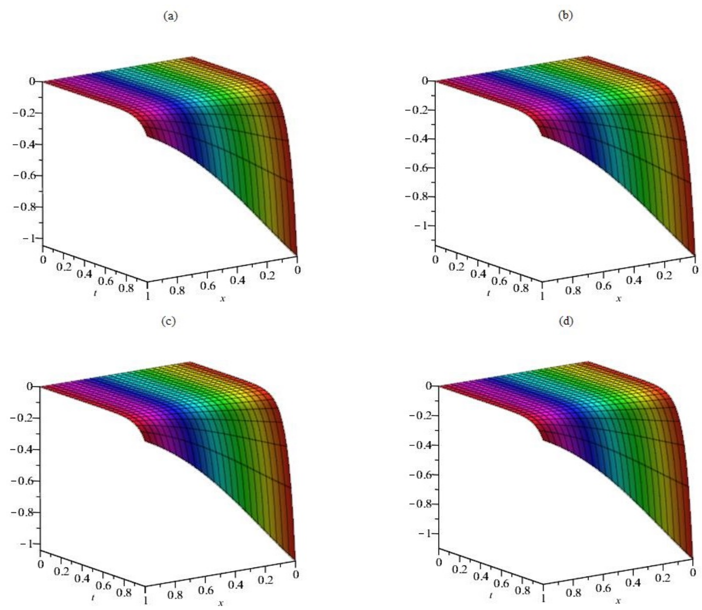

FIGURE 2. The graph of the approximate solution $u(x,t)$ for $\alpha = 0.85$(a),$\alpha = 0.95$(b), $\alpha = 0.98$(c) and $\alpha = 1$(d) for example (5.3).

The absolute error of the numerical solution of $u(x,t)$ for $\alpha = 0.95$ and $\alpha = 0.98$ versus $\alpha = 1$ is displayed in Tables (1 and 2) respectively.

TABLE 1. The absolute error of the numerical solution for $\alpha = 0.95$ vs $\alpha = 1$ for Example 5.2.

	x=0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
t=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.9	0.004	0.003	0.000	0.001	0.004	0.005	0.005	0.005	0.003	0.001

TABLE 2. The absolute error of the numerical solution for $\alpha = 0.98$ vs $\alpha = 1$ for Example 5.3.

	x=0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
t=0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.9	0.008	0.002	0.005	0.012	0.016	0.005	0.017	0.015	0.010	0.005

CONCLUSION

In this paper, we explained the spectral method and asymptotic methods and talked about the advantages of using these methods in solving partial fractional differential equations. First, the spectral problem was presented. Then the eigenvalues and eigenfunctions of main spectral problem were calculated. After that we obtained the form of approximate-analytical solutions to the given problem. The numerical results and absolute errors were presented in two examples. It has been shown that the approximate-analytical results were in good agreement with the exact solution.

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[1] Department of Mathematics, Faculty of Science, Azarbaijan Shahid Madani University, P.O.Box 53714-161, Tabriz, Iran.

Email address: jahanshahi@azaruniv.ac.ir

[2] Department of Mathematics, Faculty of Science, Azarbaijan Shahid Madani University, P.O.Box 53714-161, Tabriz, Iran.
Email address: reza.danaei@azaruniv.ac.ir